Elliptic curves, number theory and cryptography

Week 2, Lecture 2

Aurore Guillevic

Aarhus University

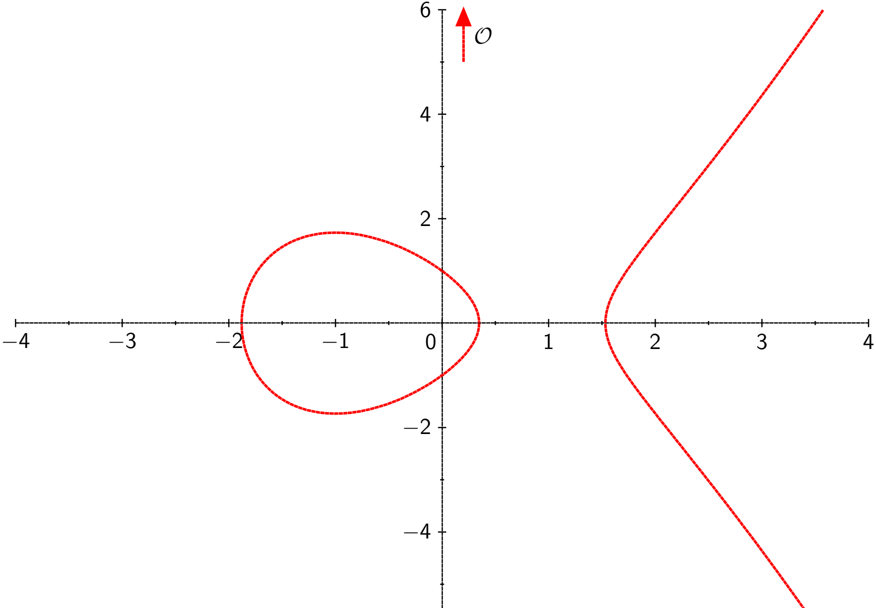
Spring semester, 2022

These slides at

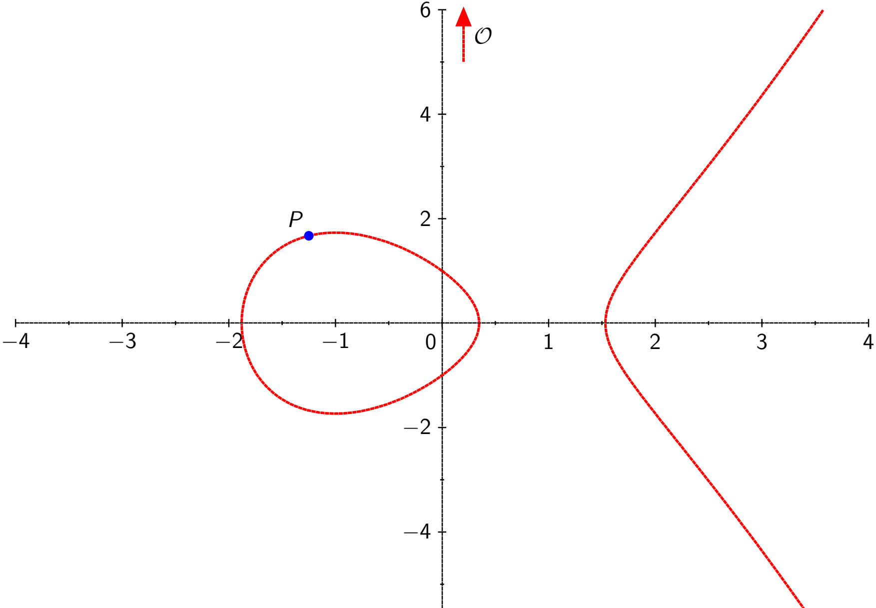
`https://members.loria.fr/AGuillevic/files/Enseignements/AU/lectures/lecture02.pdf`

Outline

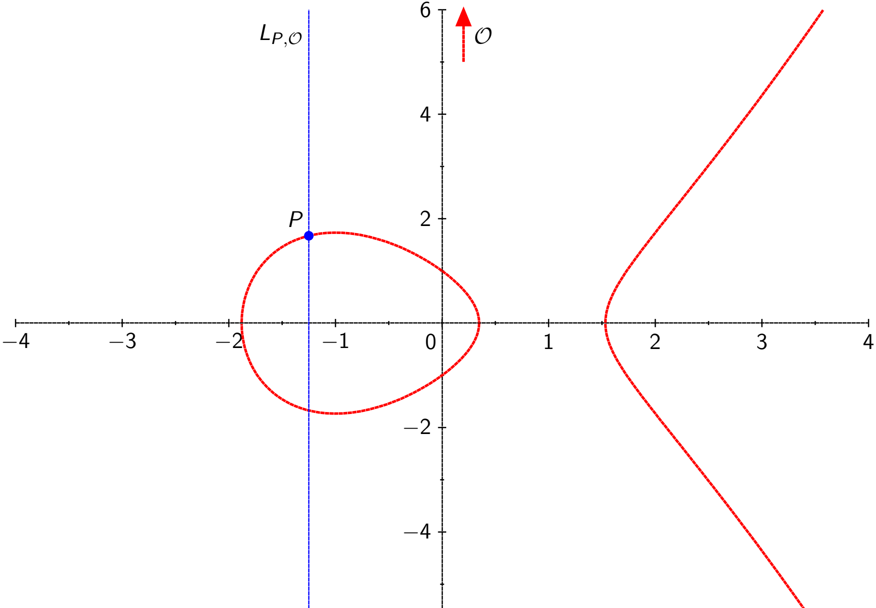
$P + (-P)$



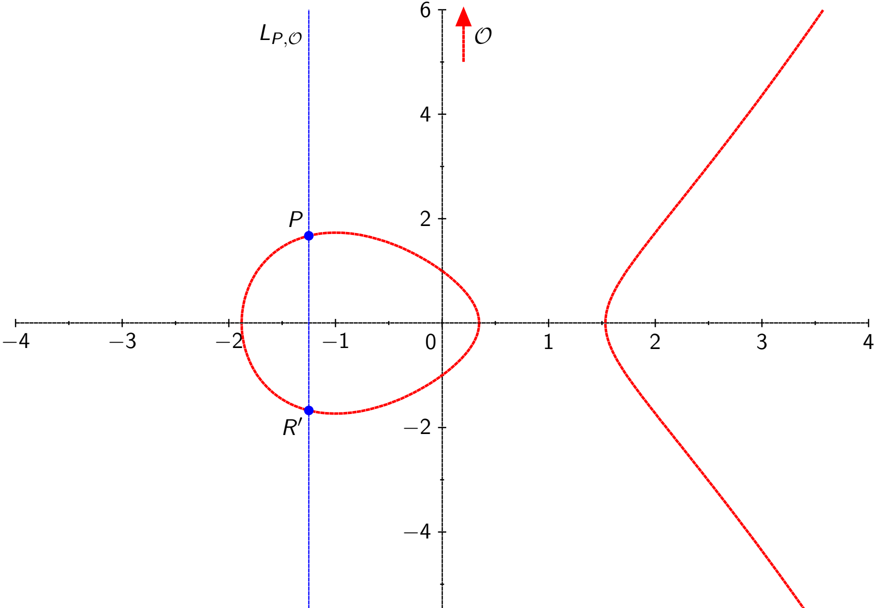
$P + (-P)$



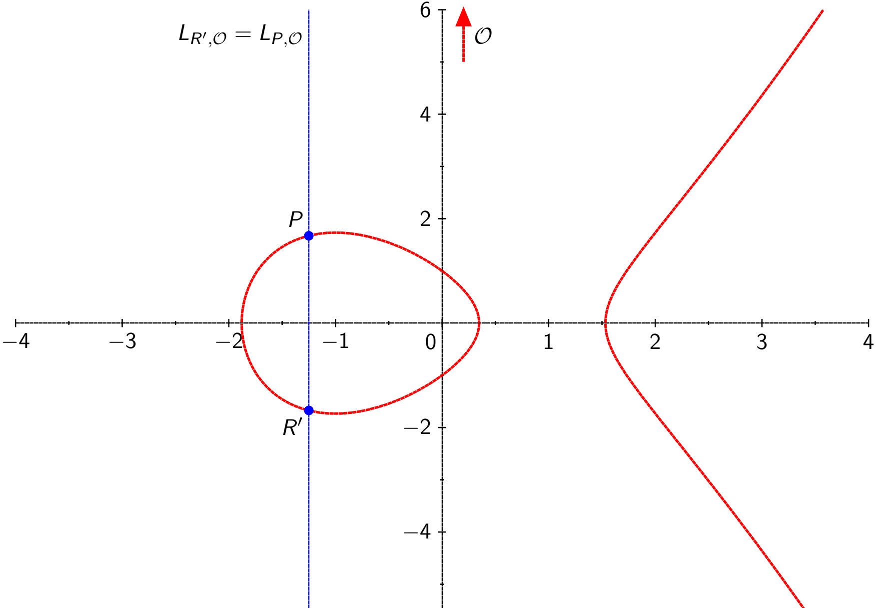
$P + (-P)$



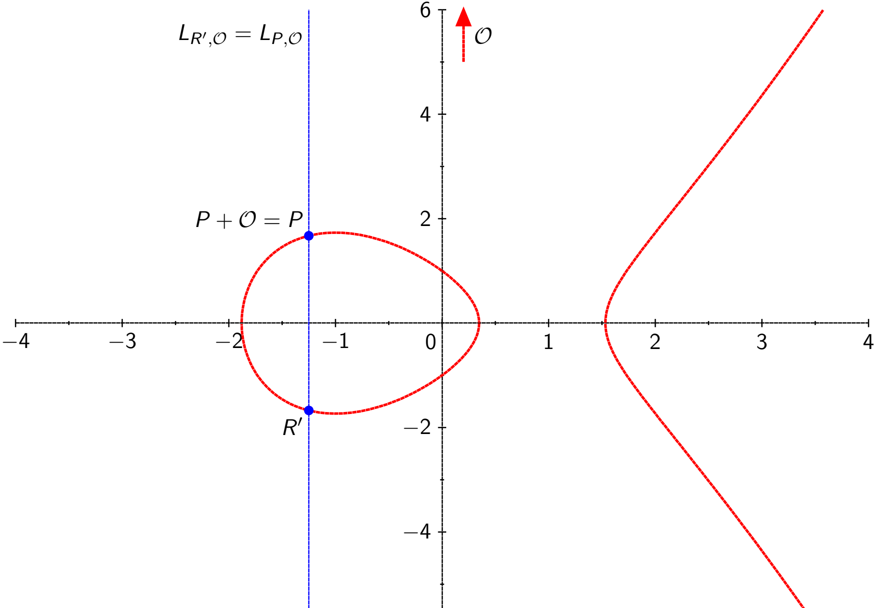
$P + (-P)$



$P + (-P)$



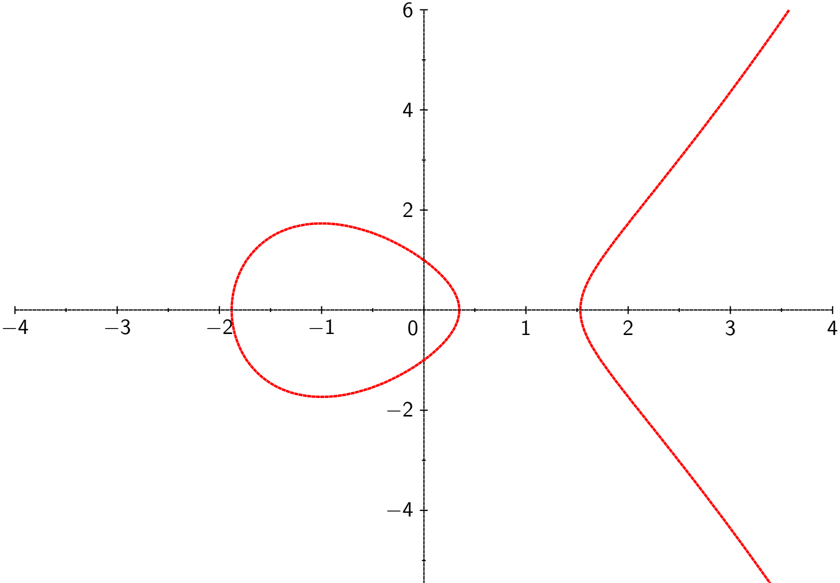
$P + (-P)$



$L_{R',\mathcal{O}} = L_{P,\mathcal{O}}$
$\mathcal{O}$
$P + \mathcal{O} = P$
R'
6
4
2
−2
−4
−4
−3
−2
−1
0
1
2
3
4

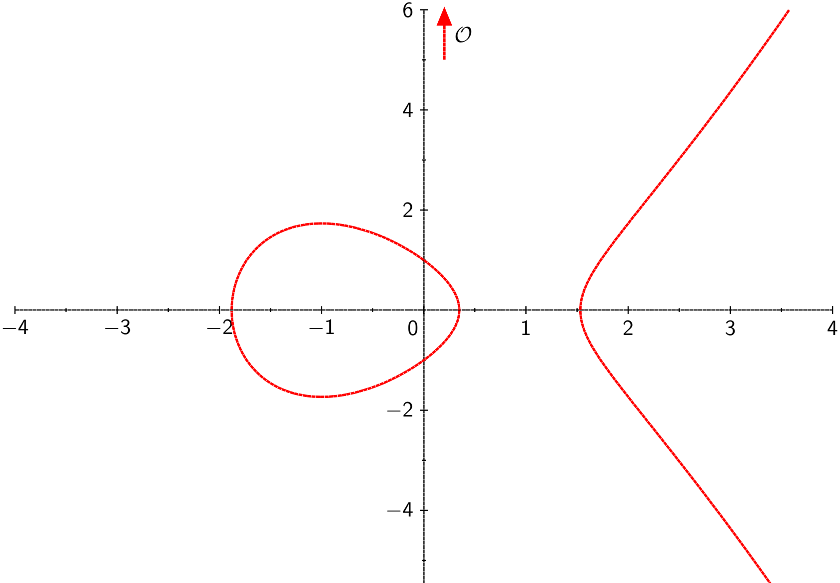
Projective space and point at infinity

$E/\mathbb{R} : y^2 = x^3 - 3x + 1$



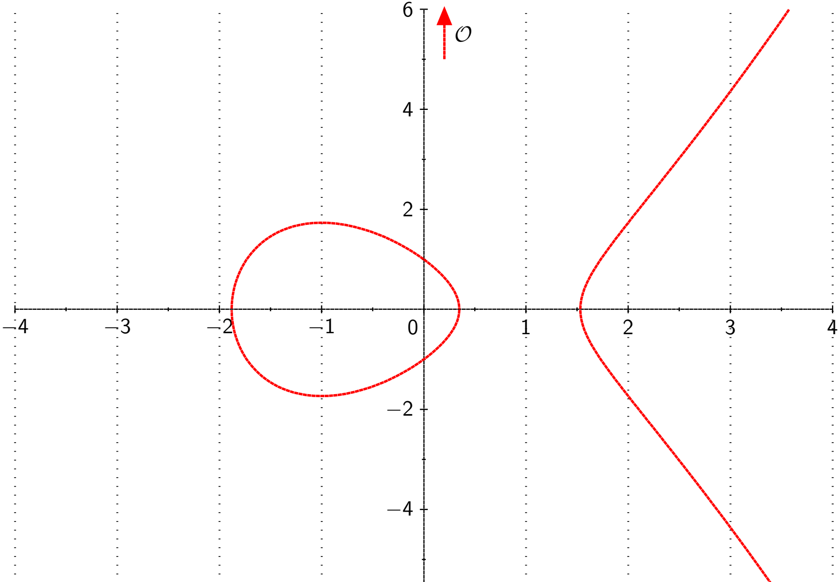
Projective space and point at infinity

$E/\mathbb{R} : y^2 = x^3 - 3x + 1$



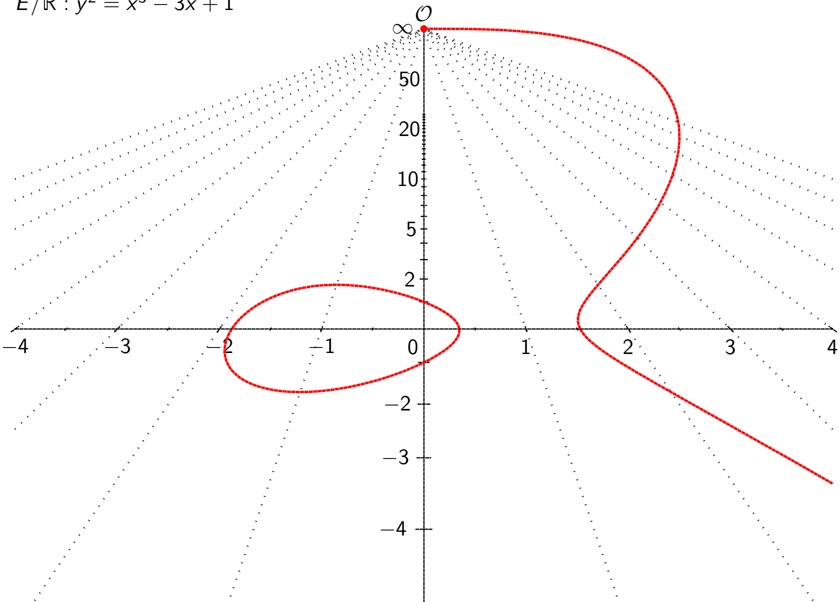
Projective space and point at infinity

$E/\mathbb{R} : y^2 = x^3 - 3x + 1$



Projective space and point at infinity

$E/\mathbb{R} : y^2 = x^3 - 3x + 1$



Projective space and point at infinity

$$E/K\colon y^2 = x^3 + Ax + B \qquad \mathrm{Char}(K) \neq 2, 3$$

Affine plane (Euclidean plane) over a field K

$$\mathbb{A}^2(K) = \{(x, y)\colon x, y \in K\}$$

Group of points of E on K

The set of rational points on the curve E/K is

$$E(K) = \left\{(x, y) \in \mathbb{A}^2(K) \mid (x, y) \text{ satisfies } E\right\} \cup \{P_\infty\}$$

where P_∞ is the *point at infinity*.

We cannot represent the point at infinity P_∞ in the affine space $\mathbb{A}$: there is no (∞, ∞).
Intuition: store the denominator z of the coordinates (x, y) in a 3rd coord.

Projective space and point at infinity

Elliptic curves are **projective plane (smooth) curves**

Projective plane

The **projective plane** of dimension 2 defined over a field K, denoted $\mathbb{P}^2(K)$ is

$$\mathbb{P}^2(K) = \left\{(X, Y, Z) \in K^3 \mid (X, Y, Z) \neq (0, 0, 0)\right\} / \sim$$

with the equivalence relation $(X, Y, Z) \sim (X', Y', Z') \iff$
there exists $\lambda \neq 0 \in K$ such that $(X, Y, Z) = (\lambda X', \lambda Y', \lambda Z')$.

The **equivalence class** w.r.t. $\sim$ is denoted $(X : Y : Z)$
with colons instead of commas.

Projective space and point at infinity

Projective space

The **projective space** of dimension n defined over a field K, denoted $\mathbb{P}^n(K)$ is

$$\mathbb{P}^n(K) = \left\{(X_0, X_1, \ldots, X_n) \in K^{n+1} \mid (X_0, X_1, \ldots, X_n) \neq \mathbf{0} = (0, 0, \ldots, 0)\right\} / \sim$$

with the equivalence relation $(X_0, X_1, \ldots, X_n) \sim (X_0', X_1', \ldots, X_n') \iff$
there exists $\lambda \neq 0 \in K$ such that $(X_0, X_1, \ldots, X_n) = (\lambda X_0', \lambda X_1', \lambda \ldots, X_n')$.

The **equivalence class** w.r.t. $\sim$ is denoted $(X_0 : X_1 : \ldots : X_n)$
with colons instead of commas.

Outline

Homogenization

A polynomial $f \in K[x, y]$ defines a plane curve $\mathcal{C}_0$ in $\mathbb{A}^2(K)$
$\rightarrow$ a **homogeneous polynomial** $F \in K[X, Y, Z]$ defines
a projective plane curve $\mathcal{C}$ in $\mathbb{P}^2(K)$

Degree of a bivariate polynomial

Let the degree $d = \deg f$ to be the largest value $i + j$ of the (non-zero) monomials $x^i y^j$ of f:

$$f = \sum_{i,j:\ a_{ij} \neq 0} a_{ij} x^i y^j, \quad d = \max_{i,j:\ a_{ij} \neq 0} i + j \ .$$

Homogenization

Homogenization of a polynomial

The **homogenization** of $f(x,y) = \sum_{i,j:\ a_{ij}\neq 0} a_{ij} x^i y^j \in K[x,y]$ is

$$F(X,Y,Z) = \sum_{i,j:\ a_{ij}\neq 0} a_{ij} X^i Y^j Z^{d-i-j}\ ,\ \text{where } d = \deg(f)\ .$$

Equivalently (Washington's book 2.3 page 19),

$$F(X,Y,Z) = Z^d f\left(\frac{X}{Z}, \frac{Y}{Z}\right)\ ,\ \text{where } d = \deg(f)\ .$$

From this definition we have

- F is homogeneous of degree d;
- $F(x,y,1) = f(x,y)$;
- $F(x,y,0) \neq 0$, and
- $F(X,Y,Z) = 0$ does not contain the line at infinity

Why homogenization?

(slide added to answer a question)
In the projective space, a point $P(X_0, Y_0, Z_0)$ has many possible representations:

$$P = (\lambda X_0, \lambda Y_0, \lambda Z_0) \text{ for any scalar } \lambda \neq 0$$

$P \in \mathcal{C}$ a curve of $\mathbb{P}^2 \implies P$ is a zero of a polynomial $F(X, Y, Z)$.

But then we require $F(\lambda X_0, \lambda Y_0, \lambda Z_0) = 0$ for all $\lambda \neq 0$.
Thanks to homogenization, we have

$$F(\lambda X_0, \lambda Y_0, \lambda Z_0) = \lambda^d F(X_0, Y_0, Z_0)$$

hence

$$P \in \mathcal{C} \iff F(X_0, Y_0, Z_0) = 0 \iff F(\lambda X_0, \lambda Y_0, \lambda Z_0) = 0 \ \forall \lambda \neq 0$$

A projective plane curve is smooth

Let $E\colon F(X,Y,Z)=0$ over a field K, where F is a homogeneous polynomial.
There is no singular point (X_0,Y_0,Z_0) such that

$$\begin{cases} \dfrac{\partial F}{\partial X}(X_0,Y_0,Z_0)=0 \\[2ex] \dfrac{\partial F}{\partial Y}(X_0,Y_0,Z_0)=0 \\[2ex] \dfrac{\partial F}{\partial Z}(X_0,Y_0,Z_0)=0 \end{cases}$$

where $\partial F/\partial X$, $\partial F/\partial Y$, $\partial F/\partial Z$ are the partial derivatives.

A line in $\mathbb{P}^2(K)$

Affine plane (Euclidean plane) over a field K

$$\mathbb{A}^2(K) = \{(x, y) \colon x, y \in K\}$$

A line in the affine plane $\mathbb{A}^2(K)$ is defined by an equation of the form

$$\mathcal{L} \colon ax + by + c = 0 \ , \ \text{with } (a, b, c) \neq (0, 0, 0).$$

Applying the homogenization formula, one has:

Projective Line

A **projective line** in $\mathbb{P}^2(K)$ has an equation of the form

$$\mathcal{L} \colon aX + bY + cZ = 0 \ , \ \text{with } (a, b, c) \neq (0, 0, 0).$$

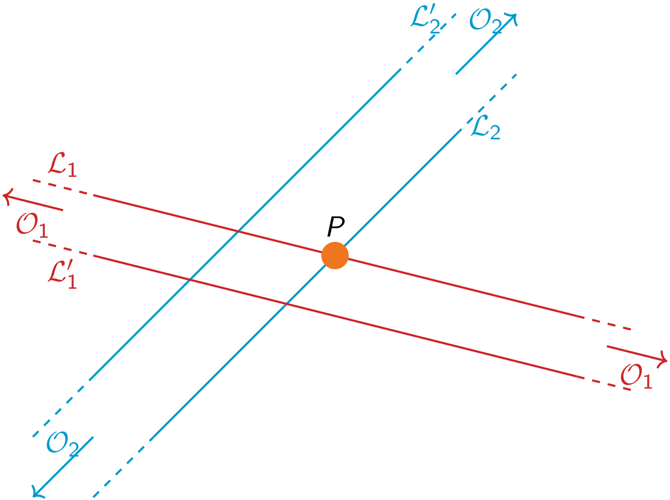
- Two distinct points of $\mathbb{A}^2$ determine a line in $\mathbb{A}^2$
- two lines of $\mathbb{A}^2$ determine one point in $\mathbb{A}^2$ unless they are parallel.

The projective plane will contain the intersection point of parallel lines at infinity.

Two parallel lines meet at infinity

At infinity is not a single point

Distinct pairs of parallel lines do not meet at the same point at infinity.
$\mathcal{L}_1 \cap \mathcal{L}_2 = \{P\}$ in $\mathbb{A}^2$ so $\mathcal{L}_1, \mathcal{L}_2$ cannot share a 2nd point $\mathcal{O}$



Points at infinity

The **Points at infinity** in the projective plane $\mathbb{P}^2(K)$ correspond to **directions** of parallel lines in $\mathbb{A}^2(K)$

$$\mathbb{P}^2 = \mathbb{A}^2 \cup \{\text{the directions in } \mathbb{A}^2\}$$

where *direction* is not oriented, like the slope of a line.
The set of directions in $\mathbb{A}^2$ is

$$\{(x, y) \in K^2\}/\sim$$

where $(x, y) \sim (x', y') \iff \exists\lambda \neq 0 \in K$, $(x, y) = (\lambda x, \lambda y)$.

We have

$$\mathbb{P}^2(K) = \mathbb{A}^2(K) \cup \mathbb{P}^1(K)$$

Correspondence of $\mathbb{A}^2 \cup \mathbb{P}^1$ and $\mathbb{P}^2$

$$\mathbb{P}^2(K) = \left\{(X, Y, Z) \in K^3,\ (X, Y, Z) \neq (0, 0, 0)\right\} / \sim$$

$$\begin{aligned}
\mathbb{P}^2(K) \quad &\longleftrightarrow \quad \mathbb{A}^2(K) \cup \mathbb{P}^1(K) \\
(X, Y, Z) \quad &\mapsto \quad \begin{cases} \left(\dfrac{X}{Z}, \dfrac{Y}{Z}\right) \in \mathbb{A}^2(K) & \text{if } Z \neq 0 \\ (X, Y) \in \mathbb{P}^1(K) & \text{if } Z = 0 \end{cases} \\
(x, y, 1) \quad &\leftarrow\!\shortmid \quad (x, y) \in \mathbb{A}^2(K) \\
(X, Y, 0) \quad &\leftarrow\!\shortmid \quad (X, Y) \in \mathbb{P}^1(K)
\end{aligned}$$

Projective plane smooth curve

A projective plane cubic curve $\mathcal{C}$ in $\mathbb{P}^2(K)$ is given by an equation

$$\mathcal{C}\colon F(X, Y, Z) = 0$$

where F is a homogeneous polynomial of degree 3.

An elliptic curve in $\mathbb{P}^2(K)$ is given by an equation

$$\mathcal{E}\colon Y^2Z = X^3 + aXZ^2 + bZ^3,\ 4a^3 + 27b^2 \neq 0$$

and the group of points on $\mathcal{E}$ is

$$\mathcal{E}(K) = \{(X, Y, Z) \in \mathbb{P}^2(K)\colon F_{\mathcal{E}}(X, Y, Z) = 0\}$$

Point at infinity in the Projective Plane

$$\mathcal{E}\colon\ Y^2 Z = X^3 + aXZ^2 + bZ^3,\ 4a^3 + 27b^2 \neq 0$$

$$Z = 0 \implies \mathcal{E}\colon 0 = X^3$$

The **Point at infinity** is

$$(X, Y, Z = 0) \in \mathcal{E}(K) \implies X = 0$$

There is no condition on Y except $Y \neq 0$ because $(0, 0, 0) \notin \mathbb{P}^2$.
Then $(0, \lambda, 0)$ for any $\lambda \neq 0$ is the direction of a vertical line in $\mathbb{A}^2$.

Point at infinity on $\mathcal{E}$

The equivalence class of the point at infinity on $\mathcal{E}$ is $\mathcal{O} = (0 : 1 : 0)$.

Projective coordinates

Washington's book section 2.6.1

Addition and doubling can be done without special treatment of points of order 2

$P(x, 0) \in \mathbb{A}^2 \mapsto (X, 0, 1) \in \mathbb{P}^2$

$P(X_1, Y_1, Z_1) + Q(X_2, Y_2, Z_2)$

Suppose that none is $\mathcal{O}$, then $Z_1 \neq 0$, $Z_2 \neq 0$.

Their affine part is $P(x_1, y_1) = (X_1/Z_1, Y_1/Z_1)$ and $Q(x_2, y_2) = (X_2/Z_2, Y_2/Z_2)$.

$\mathcal{L}$ through P and Q has slope $\lambda = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{Y_2/Z_2 - Y_1/Z_1}{X_2/Z_2 - X_1/Z_1} = \dfrac{Y_2 Z_1 - Y_1 Z_2}{X_2 Z_1 - X_1 Z_2}$

If $P = Q$ then $\lambda = \dfrac{3x_1^2 + a}{2y_1} = \dfrac{3X_1^2/Z_1^2 + a}{2Y_1/Z_1} = \dfrac{3X_1^2 + aZ_1^2}{2Y_1 Z_1}$

Addition law in projective coordinates (in $\mathbb{P}^2(K)$)

See the Elliptic Curve Formula Database (EFD) by Tanja Lange:
`www.hyperelliptic.org/EFD/g1p/auto-shortw-projective.html`
Let $P_1 = (X_1, Y_1, Z_1)$ and $P_2 = (X_2, Y_2, Z_2)$ be two points on

$$E\colon Y^2Z = X^3 + aXZ^2 + bZ^3 \ .$$

Adapting directly the formula $\lambda = (y_2 - y_1)/(x_2 - x_1)$, resp. $\lambda = (3x_1^2 + a)/(2y_1)$ to projective coordinates with $x_i = X_i/Z_i$, $y_i = Y_i/Z_i$, the slope of the line (P_1, P_2) is given by

$$\lambda = \begin{cases} \dfrac{Y_2Z_1 - Y_1Z_2}{X_2Z_1 - X_1Z_2} & \text{if } P_1 \neq \pm P_2 \\[2ex] \dfrac{3X_1^2 + aZ_1^2}{2Y_1Z_1} & \text{if } P_1 = P_2 \text{ and } Y_1 \neq 0 \end{cases}$$

Addition law in projective coordinates in $\mathbb{P}^2(K)$

Cohen, Miyaji and Ono published at Asiacrypt'1998 the formulas

$$
\begin{aligned}
u &= Y_2 \cdot Z_1 - Y_1 \cdot Z_2 \\
v &= X_2 \cdot Z_1 - X_1 \cdot Z_2 \\
A &= u^2 \cdot Z_1 \cdot Z_2 - v^3 - 2v^2 \cdot X_1 Z_2 \\
X_3 &= v \cdot A \\
Y_3 &= u \cdot (v^2 X_1 Z_2 - A) - v^3 \cdot Y_1 Z_2 \\
Z_3 &= v^3 \cdot Z_1 Z_2
\end{aligned}
$$

this costs 11 Mult., the squares u^2, v^2, then $v^3 = v^2 \cdot v$, hence 12 Mult. $+$ 2 Squares and negligible additions and subtractions.

Addition law in projective coordinates in $\mathbb{P}^2(K)$

For doubling, Cohen, Miyaji and Ono have

$$\begin{aligned} w &= aZ_1^2 + 3X_1^2 \\ s &= Y_1 \cdot Z_1 \\ B &= X_1 \cdot Y_1 \cdot s \\ h &= w^2 - 8B \\ X_3 &= 2h \cdot s \\ Y_3 &= w \cdot (4B - h) - 8 \cdot (Y_1 s)^2 \\ Z_3 &= 8s^3 \end{aligned}$$

this costs 6 Mult., 5 Squares and $w^3 = w^2 \cdot w$, hence
7 Mult. + 5 Squares and negligible additions, subtractions and a multiplication by a.

Corner cases of addition law in projective coordinates in $\mathbb{P}^2(K)$

If $P(X_1, Y_1, Z_1)$ and $Q = -P_1 = (X_1, -Y_1, Z_1)$ with $Y_1 \neq 0$
then the addition formula computes
$(X_3, Y_3, Z_3) = (0, Y_3, 0)$ and $Y_3 = 8Y_1^3 Z_1^5 \neq 0$
This is the point at infinity $\mathcal{O}$, without division by 0.

If $P_1(X_1, 0, Z_1)$ has order 2, the doubling formula computes
$(0, Y_3, 0) = \mathcal{O}$ without a division by 0.

Other coordinate systems and forms of elliptic curves

There are many other coordinate systems:

- affine (x, y)
- projective $(X, Y, Z) \mapsto (X/Z, Y/Z)$
- Jacobian $(X, Y, Z) \mapsto (X/Z^2, Y/Z^3)$
- extended Jacobian $(X, Y, Z, Z^2) \mapsto (X/Z^2, Y/Z^3)$
- ...

that can be combined with different **forms of curves**:

- Short Weierstrass with $a = -3$, $a = 1$, $a = 0$, $b = 0$, etc
- Specificities: points of order 2 or 4 available
- Montgomery form
- Edwards, twisted Edwards form
- Jacobi Quartic
- Huff form
- ...

$\rightarrow$ EFD contains almost all of them.

Outline

Étienne Bézout

French mathematician (1730 – 1783)
Scientist in the Navy

You can read about Bézout's theorem on Wikipedia at this link:
https://en.wikipedia.org/wiki/B%C3%A9zout%27s_theorem

https://mathshistory.st-andrews.ac.uk/Biographies/Bezout/pictdisplay/

Multiplicity of intersection

Let $\mathcal{C}$ and $\mathcal{C}'$ be two projective plane curves with no common component, that is they are defined by homogeneous polynomials F and G with no common factor.
the **Multiplicity of intersection** of $\mathcal{C}$ and $\mathcal{C}'$ at $P \in \mathbb{P}^2$ is the unique integer $I_P(\mathcal{C}, \mathcal{C}') \geq 0$ such that

1. $I_P(\mathcal{C}, \mathcal{C}') = 0 \iff P \notin \mathcal{C} \cap \mathcal{C}'$
2. If $P \in \mathcal{C}_1 \cap \mathcal{C}_2$, if P is a non-singular point of $\mathcal{C}_1$ and $\mathcal{C}_2$, and if $\mathcal{C}_1$ and $\mathcal{C}_2$ have different tangent directions at P, then $I_P(\mathcal{C}_1, \mathcal{C}_2) = 1$
   One often says in this case that $\mathcal{C}_1$ and $\mathcal{C}_2$ intersect *transversally* at P.
3. If $P \in \mathcal{C}_1 \cap \mathcal{C}_2$ and if $\mathcal{C}_1$ and $\mathcal{C}_2$ do not intersect transversally at P, then $I_P(\mathcal{C}_1, \mathcal{C}_2) \geq 2$.

Bézout's theorem

Silverman–Tate book appendix A.
Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be projective curves with no common component. Then

$$\sum_{P \in \mathcal{C}_1 \cap \mathcal{C}_2} I_P(\mathcal{C}_1, \mathcal{C}_2) = (\deg \mathcal{C}_1)(\deg \mathcal{C}_2) \ ,$$

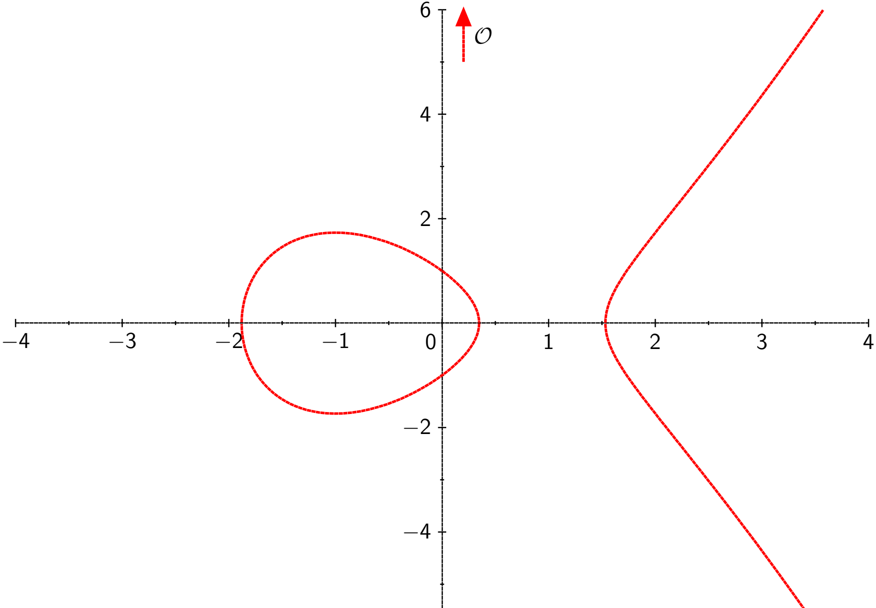
where the sum is over all points of $\mathcal{C}_1 \cap \mathcal{C}_2$ in the algebraically closed field K (e.g. $\mathbb{C}$ or $\overline{\mathbb{F}_p}$).

In particular, if $\mathcal{C}_1$ and $\mathcal{C}_2$ are smooth curves with only transversal intersections, then $\#\mathcal{C}_1 \cap \mathcal{C}_2 = (\deg \mathcal{C}_1)(\deg \mathcal{C}_2)$;
and in all cases there is an inequality

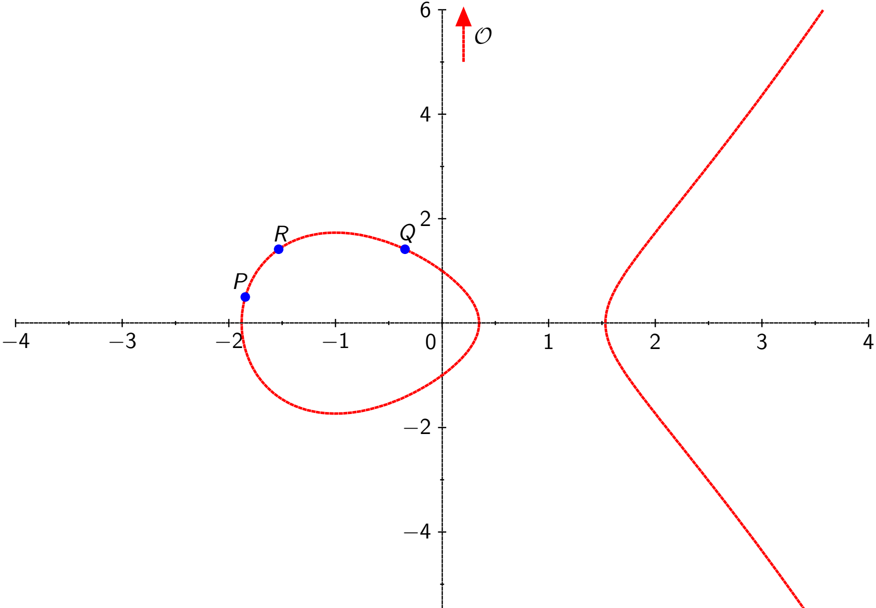
$$\#(\mathcal{C}_1 \cap \mathcal{C}_2) \leq (\deg \mathcal{C}_1)(\deg \mathcal{C}_2)$$

Outline

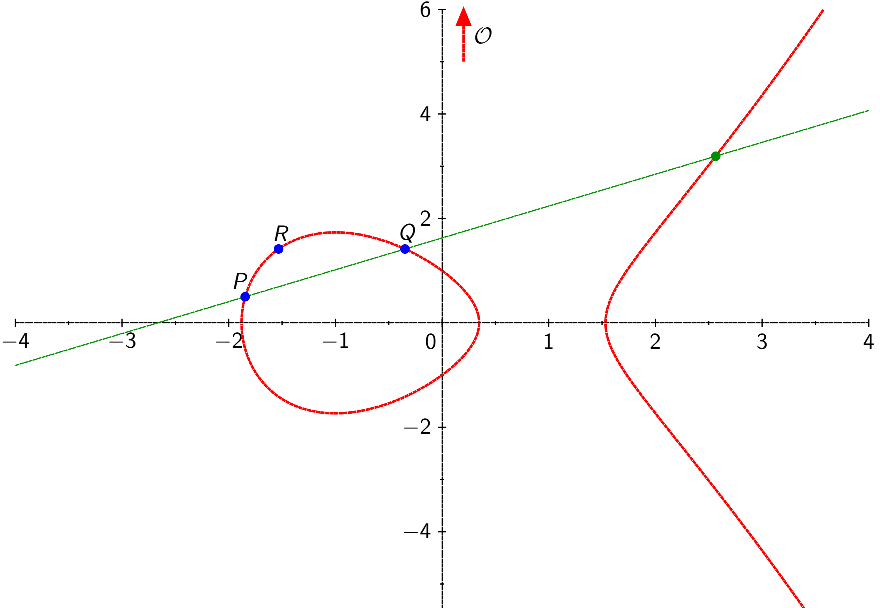
Associativity: $(P+Q)+R = P+(Q+R)$



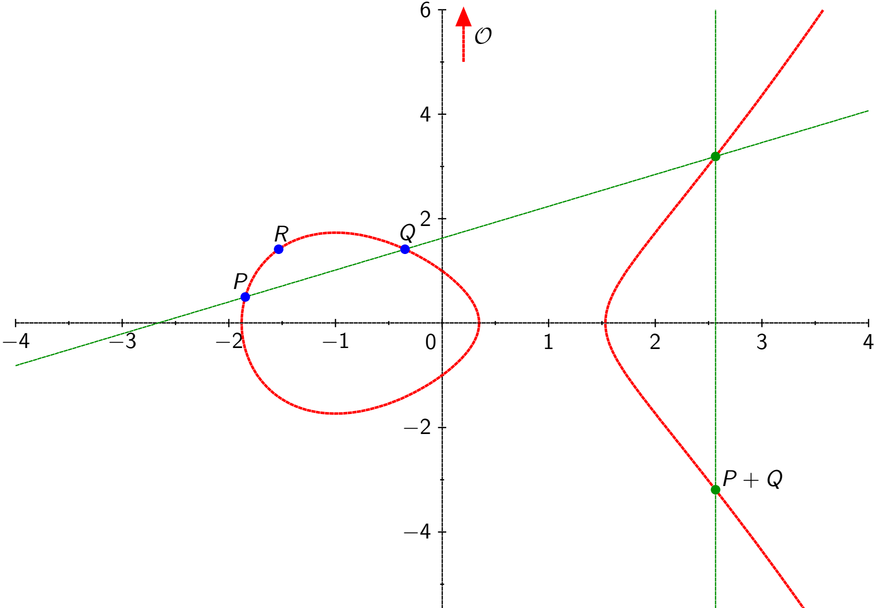
Associativity: $(P + Q) + R = P + (Q + R)$



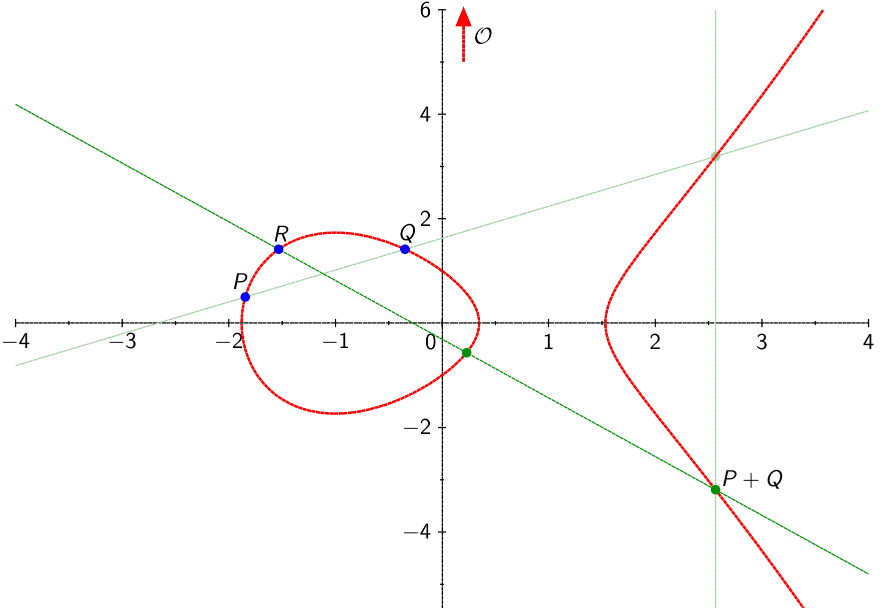
Associativity: $(P + Q) + R = P + (Q + R)$



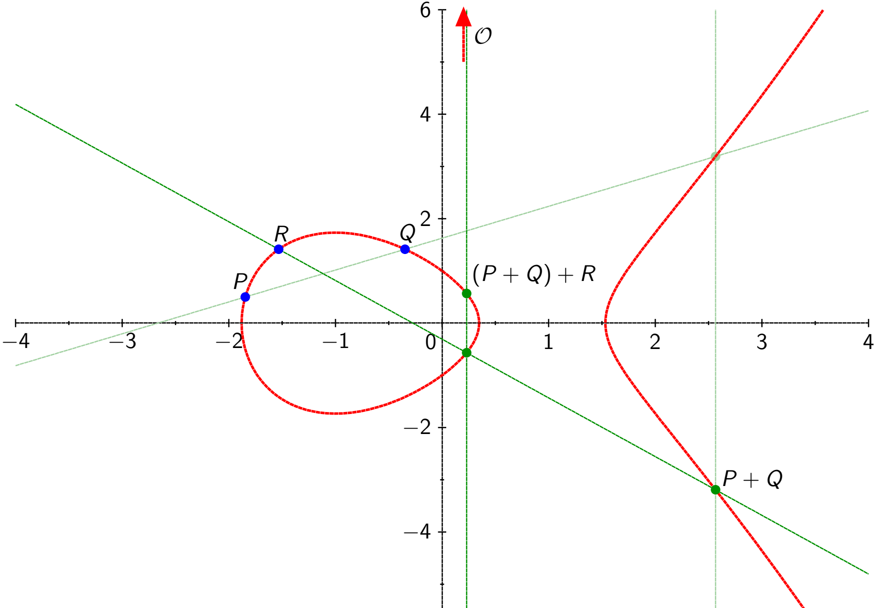
Associativity: $(P + Q) + R = P + (Q + R)$



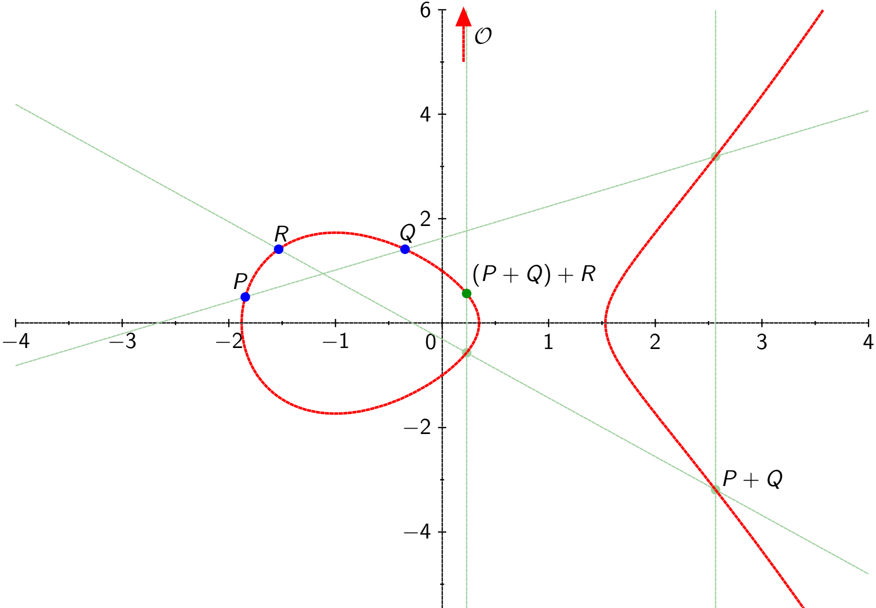
Associativity: $(P + Q) + R = P + (Q + R)$



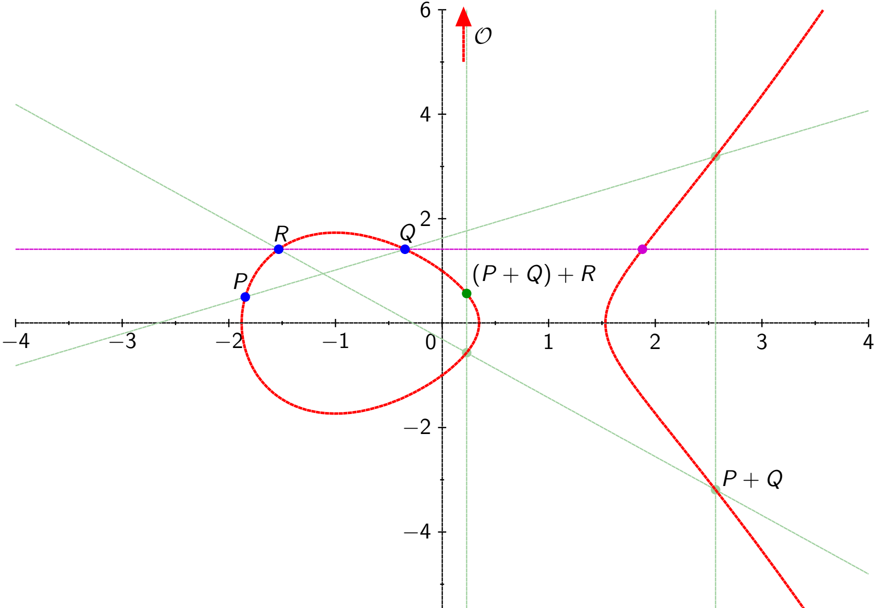
Associativity: $(P + Q) + R = P + (Q + R)$



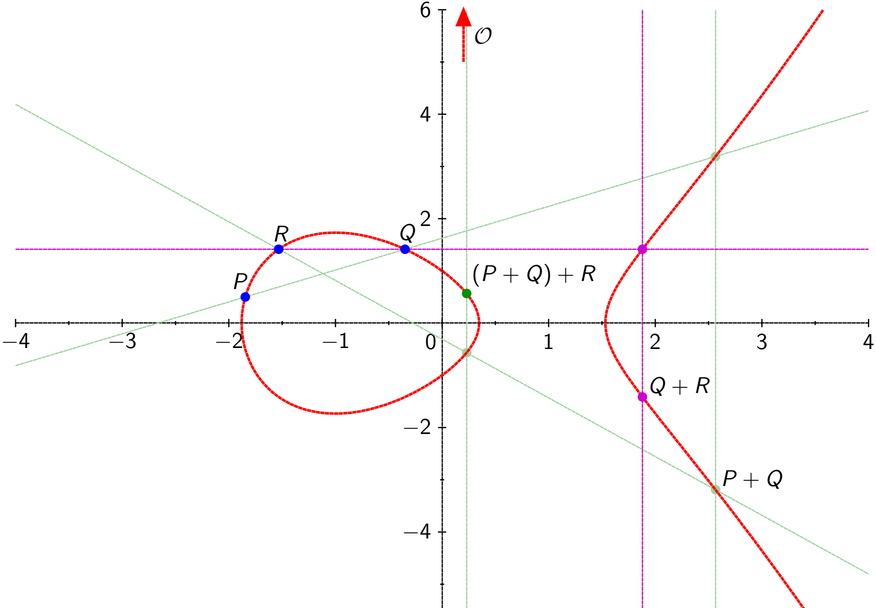
Associativity: $(P + Q) + R = P + (Q + R)$



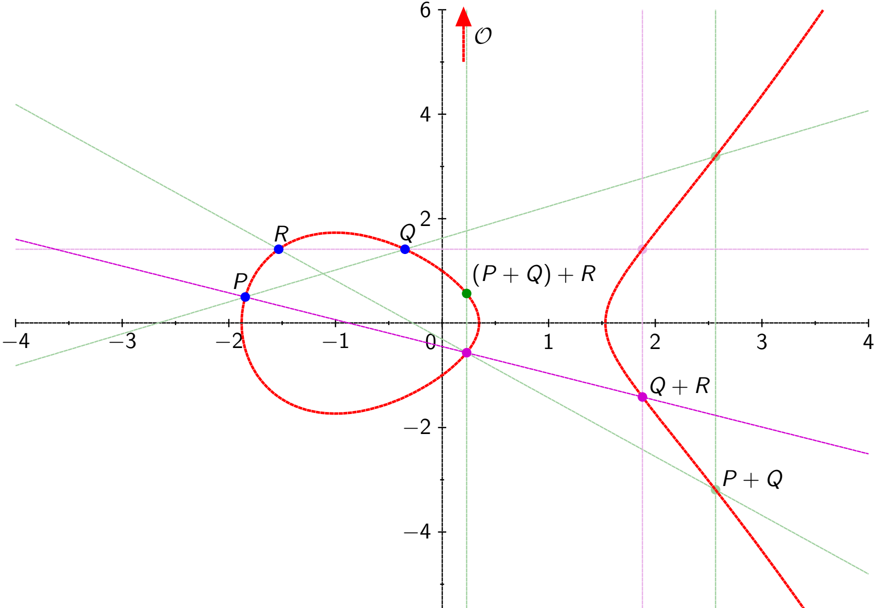
Associativity: $(P + Q) + R = P + (Q + R)$



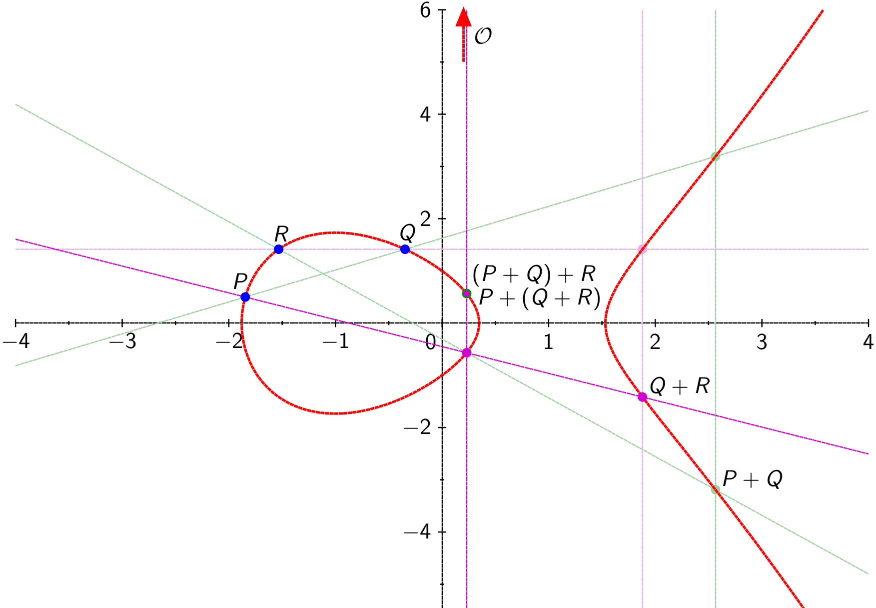
Associativity: $(P + Q) + R = P + (Q + R)$



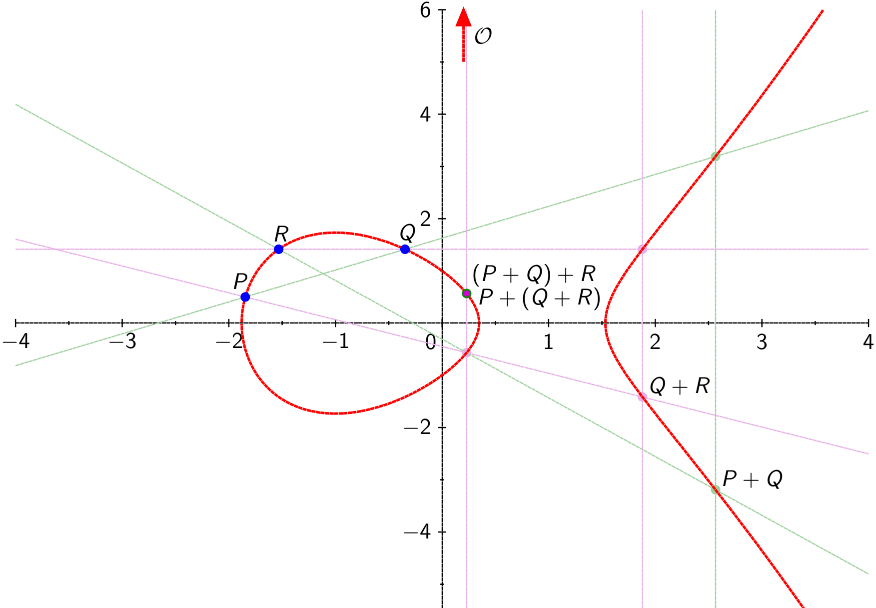
Associativity: $(P + Q) + R = P + (Q + R)$



Associativity: $(P+Q)+R = P+(Q+R)$



Associativity: $(P+Q)+R = P+(Q+R)$



Idea of the proof using Bézout's theorem

This will NOT be in the exam

Silverman–Tate book pages 16–21 and 238–240.

From Bézout's theorem, two distinct cubic projective plane curves without a common component have exactly 9 intersection points.

Theorem A

Let $\mathcal{C}$, $\mathcal{C}_1$ and $\mathcal{C}_2$ be three cubic curves. Suppose $\mathcal{C}$ goes through eight of the nine intersection points of $\mathcal{C}_1$ and $\mathcal{C}_2$. Then $\mathcal{C}$ goes through the ninth intersection point.

Idea of the proof using Bézout's theorem

Let's consider an elliptic curve $\mathcal{C}$ and the eight points

$$P, Q, R, \mathcal{O}, -(P+Q), P+Q, -(Q+R), (Q+R) \in \mathcal{C} \ .$$

To show associativity, we need to show that there is a unique ninth point:

$$-((P+Q)+R) = -(P+(Q+R)) \ .$$

Idea of the proof using Bézout's theorem

Let $\mathcal{C}_1$ be defined by the equations of the three lines through the nine distinct points $P, Q, -(P+Q) \in \ell_{P,Q}$, the vertical $-(Q+R), Q+R, \mathcal{O} \in v_{Q+R}$, and $R, (P+Q), -((P+Q)+R) \in \ell_{P+Q,R}$ multiplied together:

$$\mathcal{C}_1 \colon F_1(X,Y,Z) = \ell_{P,Q} \cdot v_{Q+R} \cdot \ell_{P+Q,R} = 0$$

Idea of the proof using Bézout's theorem

Let $\mathcal{C}_1$ be defined by the equations of the three lines through the nine distinct points $P, Q, -(P+Q) \in \ell_{P,Q}$, the vertical $-(Q+R), Q+R, \mathcal{O} \in v_{Q+R}$, and $R, (P+Q), -((P+Q)+R) \in \ell_{P+Q,R}$ multiplied together:

$$\mathcal{C}_1\colon F_1(X,Y,Z) = \ell_{P,Q} \cdot v_{Q+R} \cdot \ell_{P+Q,R} = 0$$

Let $\mathcal{C}_2$ be defined by the equations of the three lines through the nine distinct points $Q, R, -(Q+R) \in \ell_{Q,R}$, the vertical $P+Q, -(P+Q), \mathcal{O} \in v_{P+Q}$, and $P, Q+R, -(P+(Q+R)) \in \ell_{P,Q+R}$ multiplied together:

$$\mathcal{C}_2\colon F_2(X,Y,Z) = \ell_{Q,R} \cdot v_{P+Q} \cdot \ell_{P,Q+R} = 0$$

Idea of the proof using Bézout's theorem

Let $\mathcal{C}_1$ be defined by the equations of the three lines through the nine distinct points $P, Q, -(P+Q) \in \ell_{P,Q}$, the vertical $-(Q+R), Q+R, \mathcal{O} \in v_{Q+R}$, and $R, (P+Q), -((P+Q)+R) \in \ell_{P+Q,R}$ multiplied together:

$$\mathcal{C}_1 \colon F_1(X,Y,Z) = \ell_{P,Q} \cdot v_{Q+R} \cdot \ell_{P+Q,R} = 0$$

Let $\mathcal{C}_2$ be defined by the equations of the three lines through the nine distinct points $Q, R, -(Q+R) \in \ell_{Q,R}$, the vertical $P+Q, -(P+Q), \mathcal{O} \in v_{P+Q}$, and $P, Q+R, -(P+(Q+R)) \in \ell_{P,Q+R}$ multiplied together:

$$\mathcal{C}_2 \colon F_2(X,Y,Z) = \ell_{Q,R} \cdot v_{P+Q} \cdot \ell_{P,Q+R} = 0$$

Then $\mathcal{C}_1$ and $\mathcal{C}_2$ are two cubic curves of $\mathbb{P}^2$ that intersect at nine distinct points, namely the known

$$P, Q, R, \mathcal{O}, -(P+Q), P+Q, -(Q+R), (Q+R) \in \mathcal{C}_1 \cap \mathcal{C}_2$$

and a ninth intersection point $P_9 \in \mathcal{C}_1 \cap \mathcal{C}_2$.

Idea of the proof using Bézout's theorem

Now $\mathcal{C}$ is a curve that goes to the first eight points

$$P, Q, R, \mathcal{O}, -(P+Q), P+Q, -(Q+R), (Q+R) \in \mathcal{C}$$

Hence by Theorem A it also goes through the 9-th point of $\mathcal{C}_1 \cap \mathcal{C}_2$.
Thus the ninth intersection point of $\mathcal{C}_1$ and $\mathcal{C}_2$ lies on $\mathcal{C}$: $P_9 \in \mathcal{C}_1 \cap \mathcal{C}_2$, $P_9 \in \mathcal{C}$.

Idea of the proof using Bézout's theorem

Now $\mathcal{C}$ is a curve that goes to the first eight points

$$P, Q, R, \mathcal{O}, -(P+Q), P+Q, -(Q+R), (Q+R) \in \mathcal{C}$$

Hence by Theorem A it also goes through the 9-th point of $\mathcal{C}_1 \cap \mathcal{C}_2$.
Thus the ninth intersection point of $\mathcal{C}_1$ and $\mathcal{C}_2$ lies on $\mathcal{C}$: $P_9 \in \mathcal{C}_1 \cap \mathcal{C}_2$, $P_9 \in \mathcal{C}$.

Both $-((P+Q)+R) \in \mathcal{C}_1$ and $-(P+(Q+R)) \in \mathcal{C}_2$ also lies on $\mathcal{C}$ by construction.
Hence $-((P+Q)+R), P_9 \in \mathcal{C} \cap \mathcal{C}_1$ and $-(P+(Q+R)), P_9 \in \mathcal{C} \cap \mathcal{C}_2$

Idea of the proof using Bézout's theorem

Now $\mathcal{C}$ is a curve that goes to the first eight points

$$P, Q, R, \mathcal{O}, -(P+Q), P+Q, -(Q+R), (Q+R) \in \mathcal{C}$$

Hence by Theorem A it also goes through the 9-th point of $\mathcal{C}_1 \cap \mathcal{C}_2$.
Thus the ninth intersection point of $\mathcal{C}_1$ and $\mathcal{C}_2$ lies on $\mathcal{C}$: $P_9 \in \mathcal{C}_1 \cap \mathcal{C}_2$, $P_9 \in \mathcal{C}$.

Both $-((P+Q)+R) \in \mathcal{C}_1$ and $-(P+(Q+R)) \in \mathcal{C}_2$ also lies on $\mathcal{C}$ by construction.
Hence $-((P+Q)+R), P_9 \in \mathcal{C} \cap \mathcal{C}_1$ and $-(P+(Q+R)), P_9 \in \mathcal{C} \cap \mathcal{C}_2$

But by Bézout's theorem, $\#(\mathcal{C} \cap \mathcal{C}_1) \leq 9$ and $\#(\mathcal{C} \cap \mathcal{C}_2) \leq 9$ as cubic curves,

Idea of the proof using Bézout's theorem

Now $\mathcal{C}$ is a curve that goes to the first eight points

$$P, Q, R, \mathcal{O}, -(P+Q), P+Q, -(Q+R), (Q+R) \in \mathcal{C}$$

Hence by Theorem A it also goes through the 9-th point of $\mathcal{C}_1 \cap \mathcal{C}_2$.
Thus the ninth intersection point of $\mathcal{C}_1$ and $\mathcal{C}_2$ lies on $\mathcal{C}$: $P_9 \in \mathcal{C}_1 \cap \mathcal{C}_2$, $P_9 \in \mathcal{C}$.

Both $-((P+Q)+R) \in \mathcal{C}_1$ and $-(P+(Q+R)) \in \mathcal{C}_2$ also lies on $\mathcal{C}$ by construction.
Hence $-((P+Q)+R), P_9 \in \mathcal{C} \cap \mathcal{C}_1$ and $-(P+(Q+R)), P_9 \in \mathcal{C} \cap \mathcal{C}_2$

But by Bézout's theorem, $\#(\mathcal{C} \cap \mathcal{C}_1) \leq 9$ and $\#(\mathcal{C} \cap \mathcal{C}_2) \leq 9$ as cubic curves,
so finally

$$P_9 = -(P+(Q+R)) = -((P+Q)+R)\ .$$

Proof of Theorem A

Theorem A

Let $\mathcal{C}$, $\mathcal{C}_1$ and $\mathcal{C}_2$ be three cubic curves. Suppose $\mathcal{C}$ goes through eight of the nine intersection points of $\mathcal{C}_1$ and $\mathcal{C}_2$. Then $\mathcal{C}$ goes through the ninth intersection point.

This will NOT be in the exam

Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two distinct cubic smooth plane curves without a common component.
By Bézout's theorem, $\mathcal{C}_1$ and $\mathcal{C}_2$ intersect at exactly 9 points $P_1, \ldots, P_9$.
Consider the 9 distinct points $P_1, \ldots, P_9$ in $\mathbb{P}^2(K)$.

Let $\mathcal{C}'$ be another cubic smooth plane curve going through the first eight points $P_1, \ldots, P_8$.
We will show that $\mathcal{C}'$ also goes through P_9.

Proof of Theorem A

Consider a generic cubic projective plane curve $\mathcal{C}\colon F(X, Y, Z) = 0$ given by a homogeneous irreducible degree 3 polynomial

$$F = a_0 + a_1 XZ^2 + a_2 X^2 Z + a_3 X^3 + a_4 YZ^2 + a_5 Y^2 Z + a_6 Y^3 + a_7 XYZ + a_8 X^2 Y + a_9 XY^2$$

with 10 parameters $\{a_i\}_{0 \le i \le 9}$.

Proof of Theorem A

Consider a generic cubic projective plane curve $\mathcal{C}\colon F(X, Y, Z) = 0$ given by a homogeneous irreducible degree 3 polynomial

$$F = a_0 + a_1 XZ^2 + a_2 X^2 Z + a_3 X^3 + a_4 YZ^2 + a_5 Y^2 Z + a_6 Y^3 + a_7 XYZ + a_8 X^2 Y + a_9 XY^2$$

with 10 parameters $\{a_i\}_{0 \leq i \leq 9}$.

$P_1 \in \mathcal{C} \implies$ an equation $F(X_1, Y_1, Z_1)$ forces a condition on the a_is.

Proof of Theorem A

Consider a generic cubic projective plane curve $\mathcal{C}\colon F(X,Y,Z)=0$ given by a homogeneous irreducible degree 3 polynomial

$$F = a_0 + a_1XZ^2 + a_2X^2Z + a_3X^3 + a_4YZ^2 + a_5Y^2Z + a_6Y^3 + a_7XYZ + a_8X^2Y + a_9XY^2$$

with 10 parameters $\{a_i\}_{0\leq i\leq 9}$.

$P_1 \in \mathcal{C} \implies$ an equation $F(X_1, Y_1, Z_1)$ forces a condition on the a_is.
Going through the 8 points $P_1, \ldots, P_8$ forces 8 conditions on the a_is.

Proof of Theorem A

Consider a generic cubic projective plane curve $\mathcal{C}\colon F(X,Y,Z)=0$ given by a homogeneous irreducible degree 3 polynomial

$$F = a_0 + a_1XZ^2 + a_2X^2Z + a_3X^3 + a_4YZ^2 + a_5Y^2Z + a_6Y^3 + a_7XYZ + a_8X^2Y + a_9XY^2$$

with 10 parameters $\{a_i\}_{0\leq i\leq 9}$.

$P_1 \in \mathcal{C} \implies$ an equation $F(X_1, Y_1, Z_1)$ forces a condition on the a_is.
Going through the 8 points $P_1, \ldots, P_8$ forces 8 conditions on the a_is.

The set of $\{a_i\}_{0\leq i\leq 9}$ is a K-vector space of dimension 10,
and the 8 conditions $P_i \in \mathcal{C} \iff F(X_i, Y_i, Z_i) = 0$ make it a K-vector space of dim 2.

Proof of Theorem A

Let (F_λ, F_μ) a basis of this 2-dimensional vector space.
F_λ, F_μ are homogeneous polynomials of degree 3 and linearly independents.
They define curves $\mathcal{F}_\lambda$ and $\mathcal{F}_\mu$.

Proof of Theorem A

Let (F_λ, F_μ) a basis of this 2-dimensional vector space.
F_λ, F_μ are homogeneous polynomials of degree 3 and linearly independents.
They define curves $\mathcal{F}_\lambda$ and $\mathcal{F}_\mu$.

The former generic cubic curve $\mathcal{C}'$ defined by $F'(X, Y, Z)$ goes through $P_1, \ldots, P_8$.
We have $F'(X_i, Y_i, Z_i) = 0$ for all $1 \leq i \leq 8$.
We also have $F' = \lambda F_\lambda + \mu F_\mu$ for a choice of $\lambda, \mu \in K$ as F_λ, F_μ form a basis.

Proof of Theorem A

Let (F_λ, F_μ) a basis of this 2-dimensional vector space.
F_λ, F_μ are homogeneous polynomials of degree 3 and linearly independents.
They define curves $\mathcal{F}_\lambda$ and $\mathcal{F}_\mu$.

The former generic cubic curve $\mathcal{C}'$ defined by $F'(X, Y, Z)$ goes through $P_1, \dots, P_8$.
We have $F'(X_i, Y_i, Z_i) = 0$ for all $1 \le i \le 8$.
We also have $F' = \lambda F_\lambda + \mu F_\mu$ for a choice of $\lambda, \mu \in K$ as F_λ, F_μ form a basis.

By Bézout's theorem, $\mathcal{F}_\lambda$ and $\mathcal{F}_\mu$ being two general cubic curves, they have $(\deg \mathcal{F}_\lambda)(\deg \mathcal{F}_\mu) = 9$ points of intersection, counting multiplicities.

Proof of Theorem A

But actually we know explicitly a basis for this 2-dim vector space:
$\mathcal{C}_1$ and $\mathcal{C}_2$ that are distinct and go to $P_1, \dots, P_8$.
So a basis is actually F_1, F_2 and $F = \nu_1 F_1 + \nu_2 F_2$ with
$\mathcal{C}_1 \colon F_1(X, Y, Z) = 0$ and $\mathcal{C}_2 \colon F_2(X, Y, Z) = 0$.

Proof of Theorem A

But actually we know explicitly a basis for this 2-dim vector space:
$\mathcal{C}_1$ and $\mathcal{C}_2$ that are distinct and go to $P_1, \ldots, P_8$.
So a basis is actually F_1, F_2 and $F = \nu_1 F_1 + \nu_2 F_2$ with
$\mathcal{C}_1 \colon F_1(X, Y, Z) = 0$ and $\mathcal{C}_2 \colon F_2(X, Y, Z) = 0$.

And moreover $P_9 \in \mathcal{C}_1 \cap \mathcal{C}_2 \implies F_1(P_9) = 0 = F_2(P_9)$
Because $\mathcal{C}'$ is defined by $F' = \nu_1 F_1 + \nu_2 F_2$, then evaluating at P_9, we get
$F'(P_9) = 0$ and $\mathcal{C}'$ also goes through P_9.

Other approaches

In Washington's book Section 2.4,
looking carefully at polynomials and again intersection multiplicities.
Alternatively: with *resultants* of polynomials.

Further optional reading on the topic:

- Washington's book Section 2.4 pages 20 to 32;
- Silverman–Tate book Appendix A.

Outline

Scalar multiplication

With an addition law on E, the points on the curve form a group $E(K)$.

Scalar multiplication (exponentiation)

The **multiplication-by-m** map, or **scalar multiplication** is

$$\begin{array}{rcl} [m]\colon E & \to & E \\ P & \mapsto & \underbrace{P + \ldots + P}_{m \text{ copies of } P} \end{array}$$

for any $m \in \mathbb{Z}$, with $[-m]P = [m](-P)$ and $[0]P = \mathcal{O}$.

- a key-ingredient operation in public-key cryptography
- given $m > 0$, computing $[m]P$ as $P + P + \ldots P$ with $m - 1$ additions is **exponential** in the size of m: $m = e^{\ln m}$
- we can compute $[m]P$ in $O(\log m)$ operations on E.

Naive Scalar multiplication: Double-and-Add

Input: E defined over a field K, $m > 0$, $P \in E(K)$
Output: $[m]P \in E$

if $m = 0$ **then** **return** $\mathcal{O}$
Write m in binary expansion $m = \sum_{i=0}^{n-1} b_i 2^i$ where $b_i \in \{0, 1\}$
$R \leftarrow P$
for $i = n - 2$ *dowto* 0 **do** — loop invariant: $R = [\lfloor m/2^i \rfloor]P$
    $R \leftarrow [2]R$
    if $b_i = 1$ **then**
        $R \leftarrow R + P$
return R

Question: What are the best- and worst-case costs of the algorithm?
Question: Why is this algorithm dangerous if m is secret?

Naive Scalar multiplication: Double-and-Add

msb = most significant bits (highest powers)
lsb = least significant bits (units)
Pervious slide: **Most Significant Bits First** algorithm.

In Washington's book, §2.2 INTEGER TIMES A POINT p.18,
the LSB-first algorithm is given, disadvantage: one extra temporary variable.

Outline

Outline

Projective space and the point at infinity

Projective space $\mathbb{P}^2$ as $\mathbb{A}^2 \times \mathbb{P}^1$

Multiplicity of intersection and Bézout theorem

Associativity of the addition law

Scalar multiplication on elliptic curves

Recap on complexity

The Discrete Log Problem in cryptography

Public-key cryptography

Introduced in 1976 (Diffie–Hellman, DH) and 1977 (Rivest–Shamir–Adleman, RSA)
Asymmetric means distinct public and private keys

- encryption with a public key
- decryption with a private key
- deducing the private key from the public key is a very hard problem

Two hard problems:

- Integer factorization (for RSA)
- Discrete logarithm computation in a finite group (for Diffie–Hellman)

Discrete logarithm problem

$\mathbf{G}$ multiplicative group of order r
g generator, $\mathbf{G} = \{1, g, g^2, g^3, \ldots, g^{r-2}, g^{r-1}\}$

Given $h \in \mathbf{G}$, find integer $x \in \{0, 1, \ldots, r-1\}$ such that $h = g^x$.
Exponentiation easy: $(g, x) \mapsto g^x$
Discrete logarithm hard in well-chosen groups $\mathbf{G}$

Choice of group

Prime finite field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ where p is a prime integer
Multiplicative group: $\mathbb{F}_p^* = \{1, 2, \ldots, p-1\}$
Multiplication *modulo p*

Finite field $\mathbb{F}_{2^n} = \mathrm{GF}(2^n)$, $\mathbb{F}_{3^m} = \mathrm{GF}(3^m)$ for efficient arithmetic, now broken

Elliptic curves $E\colon y^2 = x^3 + ax + b/\mathbb{F}_p$

Diffie-Hellman key exchange

Alice **Bob**

Diffie-Hellman key exchange

Alice		**Bob**
$(\mathbf{G}, \cdot), g, r = \#\mathbf{G}$	public parameters	$(\mathbf{G}, \cdot), g, r = \#\mathbf{G}$

Diffie-Hellman key exchange

Alice

$(\mathbf{G}, \cdot), g, r = \#\mathbf{G}$

secret key $\mathsf{sk}_A = a \leftarrow (\mathbb{Z}/r\mathbb{Z})^*$

public value $\mathsf{PK}_A = g^a$

Bob

$(\mathbf{G}, \cdot), g, r = \#\mathbf{G}$

secret key $\mathsf{sk}_B = b \leftarrow (\mathbb{Z}/r\mathbb{Z})^*$

public value $\mathsf{PK}_B = g^b$

Diffie-Hellman key exchange

Alice

$(\mathbf{G}, \cdot), g, r = \#\mathbf{G}$

secret key $\mathsf{sk}_A = a \leftarrow (\mathbb{Z}/r\mathbb{Z})^*$

public value $\mathsf{PK}_A = g^a$

Bob

$(\mathbf{G}, \cdot), g, r = \#\mathbf{G}$

secret key $\mathsf{sk}_B = b \leftarrow (\mathbb{Z}/r\mathbb{Z})^*$

public value $\mathsf{PK}_B = g^b$

$$\xleftarrow[\mathsf{PK}_A]{\mathsf{PK}_B} \quad \xrightarrow{}$$

Alice ← PK_B — Bob ← PK_A

Diffie-Hellman key exchange

Alice

$(\mathbf{G}, \cdot), g, r = \#\mathbf{G}$

secret key $\mathsf{sk}_A = a \leftarrow (\mathbb{Z}/r\mathbb{Z})^*$

public value $\mathsf{PK}_A = g^a$

Bob

$(\mathbf{G}, \cdot), g, r = \#\mathbf{G}$

secret key $\mathsf{sk}_B = b \leftarrow (\mathbb{Z}/r\mathbb{Z})^*$

public value $\mathsf{PK}_B = g^b$

PK_B

$\longleftrightarrow$

PK_A

gets Bob's public key PK_B

$sk = {\mathsf{PK}_B}^a = g^{ab}$

gets Alice's public key PK_A

$sk = {\mathsf{PK}_A}^b = g^{ab}$

Asymmetric cryptography

Factorization (RSA cryptosystem)

Discrete logarithm problem (use in Diffie-Hellman, etc)

Given a finite cyclic group $(\mathbf{G}, \cdot)$, a generator g and $h \in \mathbf{G}$, compute x s.t. $h = g^x$.
$\rightarrow$ can we invert the exponentiation function $(g, x) \mapsto g^x$?

Common choice of $\mathbf{G}$:

- prime finite field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ (1976)
- characteristic 2 field $\mathbb{F}_{2^n}$ ($\approx$ 1979)
- elliptic curve $E(\mathbb{F}_p)$ (1985)

Discrete log problem

How fast can we invert the exponentiation function $(g, x) \mapsto g^x$?

- $g \in G$ generator, $\exists$ always a preimage $x \in \{1, \ldots, \#G\}$
- naive search, try them all: $\#G$ tests
- $O(\sqrt{\#G})$ generic algorithms

Discrete log problem

How fast can we invert the exponentiation function $(g, x) \mapsto g^x$?

- $g \in G$ generator, $\exists$ always a preimage $x \in \{1, \ldots, \#G\}$
- naive search, try them all: $\#G$ tests
- $O(\sqrt{\#G})$ generic algorithms
  - Shanks baby-step-giant-step (BSGS): $O(\sqrt{\#G})$, deterministic
  - random walk in G, cycle path finding algorithm in a connected graph (Floyd) $\rightarrow$ Pollard: $O(\sqrt{\#G})$, probabilistic
    (the cycle path encodes the answer)
  - parallel search (parallel Pollard, Kangarous)

Discrete log problem

How fast can we invert the exponentiation function $(g, x) \mapsto g^x$?

- $g \in G$ generator, $\exists$ always a preimage $x \in \{1, \ldots, \#G\}$
- naive search, try them all: $\#G$ tests
- $O(\sqrt{\#G})$ generic algorithms
  - Shanks baby-step-giant-step (BSGS): $O(\sqrt{\#G})$, deterministic
  - random walk in G, cycle path finding algorithm in a connected graph (Floyd) $\rightarrow$ Pollard: $O(\sqrt{\#G})$, probabilistic
    (the cycle path encodes the answer)
  - parallel search (parallel Pollard, Kangarous)
- independent search in each distinct subgroup
  $+$ Chinese remainder theorem (Pohlig-Hellman)

Discrete log problem

How fast can we invert the exponentiation function $(g, x) \mapsto g^x$?

$\rightarrow$ choose G of large prime order (no subgroup)

$\rightarrow$ complexity of inverting exponentiation in $O(\sqrt{\#G})$

$\rightarrow$ security level 128 bits means $\sqrt{\#G} \geq 2^{128}$
take $\#G = 2^{256}$
analogy with symmetric crypto, keylength 128 bits (16 bytes)

Discrete log problem

How fast can we invert the exponentiation function $(g, x) \mapsto g^x$?

$\rightarrow$ choose G of large prime order (no subgroup)

$\rightarrow$ complexity of inverting exponentiation in $O(\sqrt{\#G})$

$\rightarrow$ security level 128 bits means $\sqrt{\#G} \geq 2^{128}$
take $\#G = 2^{256}$
analogy with symmetric crypto, keylength 128 bits (16 bytes)

Use additional structure of G if any.
$\Longrightarrow$ Number Field Sieve algorithms.

Credits

- Rémi Clarisse PhD thesis at tel-03506116
- Jérémie Detrey summer school lecture at ARCHI'2017 summer school