Logics

I. Decide whether the following formulas are valid or not (justify your answer).

1. $(B \to A) \to (A \vee B)$
2. $(A \wedge B \wedge \neg C) \vee ((A \wedge B) \to C)$

II. Give a sequent-calculus derivation of the following formula.

$$((\exists x.A[x]) \to B) \to (\forall x.A[x] \to B) \qquad \text{where } x \notin \mathrm{FV}(B)$$

III. Consider the implicative fragment of the propositional calculus, i.e., the fragment whose formulas are inductively defined as follows:

i. Every propositional variable is a formula.

ii. If α and β are formulas then $(\alpha \to \beta)$ is a formula.

In this fragment, the notion of derivability is defined by means of the corresponding sequent-calculus rules:

$$\Gamma, \alpha \vdash \alpha, \Delta \qquad \frac{\Gamma \vdash \alpha, \Delta \qquad \Gamma, \beta \vdash \Delta}{\Gamma, (\alpha \to \beta) \vdash \Delta} \qquad \frac{\Gamma, \alpha \vdash \beta, \Delta}{\Gamma \vdash (\alpha \to \beta), \Delta}$$

III.A. We say that two formulas α and β are equivalent (and we write $\alpha \equiv \beta$) if and only if both $\alpha \vdash \beta$ and $\beta \vdash \alpha$ are derivable. Establish the following properties:

1. *If* $\alpha \equiv \beta$ *then* $(\alpha \to \gamma) \equiv (\beta \to \gamma)$.
2. *If* $\alpha \equiv \beta$ *then* $(\gamma \to \alpha) \equiv (\gamma \to \beta)$.

III.B. We define a context to be a formula with a "hole", i.e., a formula with one occurrence of the special symbol "$\bullet$". More precisely, the notion of context is inductively defined as follows:

i. $\bullet$ is a context.

ii. If $\mathcal{C}$ is a context and α is a formula, then $(\alpha \to \mathcal{C})$ is a context.

iii. If $\mathcal{C}$ is a context and α is a formula, then $(\mathcal{C} \to \alpha)$ is a context.

Given a context $\mathcal{C}$ and a formula α, we write $\mathcal{C}[\alpha]$ for the formula obtained by replacing $\bullet$ by α in $\mathcal{C}$. For instance, if $\mathcal{C}$ is $((\beta \to \bullet) \to \gamma)$, then $\mathcal{C}[\alpha]$ is $((\beta \to \alpha) \to \gamma)$. Establish the following property:

For every context $\mathcal{C}$, and all formulas α and β, if $\alpha \equiv \beta$ then $\mathcal{C}[\alpha] \equiv \mathcal{C}[\beta]$.

Hint: proceed by induction on the definition of context. Use the properties established in III.A to handle the inductive cases.

III.C. From the property established in III.B, derive the following corollary:

For every context $\mathcal{C}$, and all formulas α and β such that $\alpha \equiv \beta$, if $\vdash \mathcal{C}[\alpha]$ is derivable then $\vdash \mathcal{C}[\beta]$ is derivable.