Logics

I. Decide whether the following formula is valid or not (justify your answer).

$$(((a \to b) \to a) \land (a \to c)) \to c$$

II. Prove that the following formula is derivable.

$$\neg(\alpha \lor \beta) \to (\neg\alpha \land \neg\beta)$$

III. Consider a first-order language without functional symbol, and whose only relational symbol is R. Then, consider the formula ϕ, which is defined as follows:

$$\phi = \phi_0 \to (\phi_1 \to \phi_2)$$

where:

$$\phi_0 = \forall x.\, (\exists y.\, R(x, y))$$
$$\phi_1 = \forall x.\, (\forall y.\, (\forall z.\, R(x, y) \to (R(y, z) \to R(x, z))))$$
$$\phi_2 = \exists x.\, R(x, x)$$

Define a model $\mathscr{M} = \langle D, I \rangle$ as follows:

$$D = \mathbb{N}$$
$$I(R) = \{(m, n) \in \mathbb{N} \times \mathbb{N} \,|\, m < n\}$$

Show that $\mathscr{M}$ is a counter-model of ϕ.

IV. Let $\mathscr{F}$ be a ranked alphabet of function symbols, and let $\mathscr{X}$ be an alphabet of variables. The set of terms $\mathscr{T}$, built over $\mathscr{F}$ and $\mathscr{X}$, is inductively defined as follows:

1. If $x \in \mathscr{X}$, then $x \in \mathscr{T}$;
2. If $c \in \mathscr{F}$ is of arity 0, then $c \in \mathscr{T}$;
3. If $f \in \mathscr{F}$ is of arity n, and $t_1, \ldots, t_n \in \mathscr{T}$, then $f(t_1, \ldots, t_n) \in \mathscr{T}$.

Let $\mathscr{M} = \langle D, I \rangle$ be a model, and $\rho : D^{\mathscr{X}}$ be a valuation. The interpretation of a term t is inductively defined as follows:

1. $[\![x]\!]\rho = \rho(x)$, for $x \in \mathscr{X}$;

2. $[\![c]\!]\rho = I(c)$, for $c \in \mathscr{F}$ of arity 0;

3. $[\![f(t_1, \ldots, t_n)]\!]\rho = I(f)([\![t_1]\!]\rho, \ldots, [\![t_n]\!]\rho)$, for $f \in \mathscr{F}$ of arity n.

Let $\rho \in D^{\mathscr{X}}$ be a valuation, and let $a \in D$ and $x \in \mathscr{X}$. The valuation $\rho[x{:=}a]$ is defined as follows:

$$\rho[x{:=}a](y) = \begin{cases} a & \text{if } x = y \\ \rho(y) & \text{if } x \neq y \end{cases}$$

Finally, let t and u be terms, and x be a variable. The substitution of x by u in t, in notation $t[u/x]$, is inductively defined as follows:

1. $x[u/x] = u$;

2. $y[u/x] = y$, for $y \in \mathscr{X}$ and $y \neq x$;

3. $c[u/x] = c$, for $c \in \mathscr{F}$ of arity 0;

4. $f(t_1, \ldots, t_n)[u/x] = f(t_1[u/x], \ldots, t_n[u/x])$, for $f \in \mathscr{F}$ of arity n.

Let x be a variable, u be a term, and ρ be a valuation. Show that for every term t:

$$[\![t[u/x]]\!]\rho = [\![t]\!]\rho[x{:=}[\![u]\!]\rho]$$

Hint: proceed by induction on the structure of t.