MPRI 2-27-1 Exam

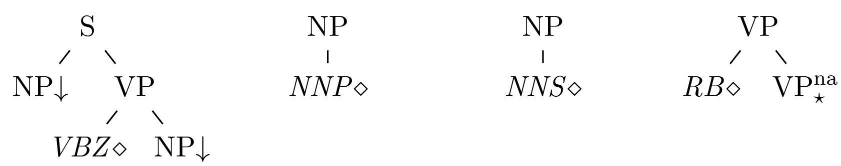
Duration: 3 hours
Paper documents are allowed. The numbers in front of questions are indicative of hardness or duration.

1 Two-level Syntax

Exercise 1 (Derivation trees)**.** In a tree adjoining grammar $\mathcal{G} = \langle N, \Sigma, T_\alpha, T_\beta, S\rangle$, the trees in $L_T(\mathcal{G})$ are called *derived* trees. We are interested here in another tree structure, called a *derivation* tree, for which we propose a formalisation here. Let us assume for simplicity that all the foot nodes of auxiliary trees have the '${}^{\mathrm{na}}$' null adjunction annotation.

For an elementary tree $\gamma \in T_\alpha \uplus T_\beta$, we define its *contents* $c(\gamma)$ to be a finite sequence over the alphabet $Q \stackrel{\text{def}}{=} \{q_A \mid A \in N \uplus N{\downarrow}\}$. Formally, we enumerate for this the labels in Q of its nodes in position order; the nodes labelled by $\Sigma \cup N^{\mathrm{na}}$ are ignored.

Consider for instance the TAG $\mathcal{G}_1$ with $N \stackrel{\text{def}}{=} \{\mathrm{S}, \mathrm{NP}, \mathrm{VP}\}$, $\Sigma \stackrel{\text{def}}{=} \{\mathit{VBZ}\diamond, \mathit{NNP}\diamond, \mathit{NNS}\diamond, \mathit{RB}\diamond\}$, $T_\alpha \stackrel{\text{def}}{=} \{\mathit{likes}, \mathit{Bill}, \mathit{mushrooms}\}$, $T_\beta \stackrel{\text{def}}{=} \{\mathit{possibly}\}$, and $S \stackrel{\text{def}}{=} \mathrm{S}$, where the elementary trees are shown below:



(*likes*) (*Bill*) (*mushrooms*) (*possibly*)

Then *likes* has contents $c(\mathit{likes}) = q_{\mathrm{S}}, q_{\mathrm{NP}\downarrow}, q_{\mathrm{VP}}, q_{\mathrm{NP}\downarrow}$, $c(\mathit{Bill}) = q_{\mathrm{NP}}$, $c(\mathit{mushrooms}) = q_{\mathrm{NP}}$, and $c(\mathit{possibly}) = q_{\mathrm{VP}}$.

We now define a finite ranked alphabet $\mathcal{F} \stackrel{\text{def}}{=} T_\alpha \uplus T_\beta \uplus \{\varepsilon^{(0)}\}$. For an elementary tree $\gamma \in T_\alpha \uplus T_\beta$, its *rank* is $r(\gamma) \stackrel{\text{def}}{=} |c(\gamma)|$ the length of its contents. For the symbol ε, its rank is $r(\varepsilon) \stackrel{\text{def}}{=} 0$. For a TAG $\mathcal{G} = \langle N, \Sigma, T_\alpha, T_\beta, S\rangle$, we construct a finite tree automaton $\mathcal{A}_{\mathcal{G}} \stackrel{\text{def}}{=} \langle Q, \mathcal{F}, \delta, q_{S\downarrow}\rangle$ where Q and $\mathcal{F}$ are defined as above and

$$\begin{aligned}\delta \stackrel{\text{def}}{=} &\{(q_{A\downarrow}, \alpha^{(r(\alpha))}, c(\alpha)) \mid A{\downarrow} \in N{\downarrow}, \alpha \in T_\alpha, \mathrm{rl}(\alpha) = A\}\\ \cup\ &\{(q_A, \beta^{(r(\beta))}, c(\beta)) \mid A \in N, \beta \in T_\beta, \mathrm{rl}(\beta) = A\}\\ \cup\ &\{(q_A, \varepsilon^{(0)}) \mid A \in N\}\end{aligned}$$

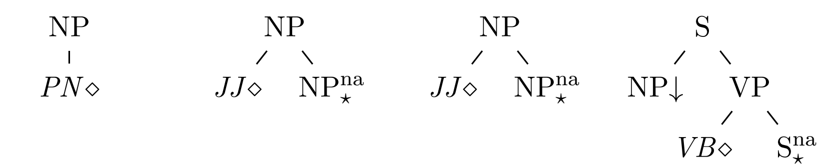
where 'rl' returns the root label of the tree.

[1] 1. Give the finite automaton $\mathcal{A}_{\mathcal{G}_1}$ associated with the example TAG $\mathcal{G}_1$.

Solution:

$$
\begin{aligned}
Q &= \{q_{\mathrm{S}\downarrow}, q_{\mathrm{NP}\downarrow}, q_{\mathrm{S}}, q_{\mathrm{VP}}, q_{\mathrm{NP}}\}\ ,\\
\mathcal{F} &= \{\mathit{likes}^{(4)}, \mathit{Bill}^{(1)}, \mathit{mushrooms}^{(1)}, \mathit{possibly}^{(1)}, \varepsilon^{(0)}\}\ ,\\
\delta &= \{(q_{\mathrm{S}\downarrow}, \mathit{likes}^{(4)}, q_{\mathrm{S}}, q_{\mathrm{NP}\downarrow}, q_{\mathrm{VP}}, q_{\mathrm{NP}\downarrow}),\\
&\quad (q_{\mathrm{NP}\downarrow}, \mathit{Bill}^{(1)}, q_{\mathrm{NP}}),\\
&\quad (q_{\mathrm{NP}\downarrow}, \mathit{mushrooms}^{(1)}, q_{\mathrm{NP}}),\\
&\quad (q_{\mathrm{S}}, \varepsilon^{(0)}),\\
&\quad (q_{\mathrm{VP}}, \mathit{possibly}^{(1)}, q_{\mathrm{VP}}),\\
&\quad (q_{\mathrm{VP}}, \varepsilon^{(0)}),\\
&\quad (q_{\mathrm{NP}}, \varepsilon^{(0)})\}
\end{aligned}
$$

[1] 2. Modify your automaton in order to also handle the trees *someone* $\in T_\alpha$ and *real*, *fake*, *thinks* $\in T_\beta$ shown below, where $PN\diamond$, $JJ\diamond$, $VB\diamond \in \Sigma$:



(*someone*) (*real*) (*fake*) (*thinks*)

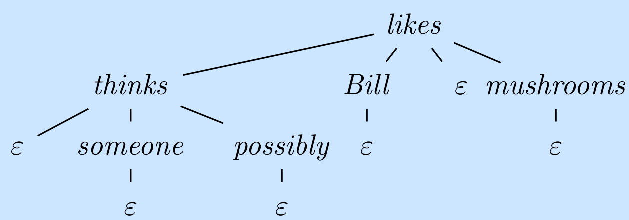
Solution: Add $\mathit{someone}^{(1)}$, $\mathit{real}^{(1)}$, $\mathit{fake}^{(1)}$, and $\mathit{thinks}^{(3)}$ to $\mathcal{F}$ and the rules

$$
\begin{gathered}
(q_{\mathrm{NP}\downarrow}, \mathit{someone}^{(1)}, q_{\mathrm{NP}})\\
(q_{\mathrm{NP}}, \mathit{real}^{(1)}, q_{\mathrm{NP}})\\
(q_{\mathrm{NP}}, \mathit{fake}^{(1)}, q_{\mathrm{NP}})\\
(q_{\mathrm{S}}, \mathit{thinks}^{(3)}, q_{\mathrm{S}}, q_{\mathrm{NP}\downarrow}, q_{\mathrm{VB}})
\end{gathered}
$$

to δ.

[1] 3. The intention that our finite automaton generates the *derivation* language $L_D(\mathcal{G}) \stackrel{\text{def}}{=} L(\mathcal{A}_{\mathcal{G}})$ of $\mathcal{G}$. Can you figure out what should be the derivation tree of '*Someone possibly thinks Bill likes mushrooms*'?

Solution:



[2] 4. Give a PDL node formula φ_1 such that $L(\mathcal{A}_{\mathcal{G}_1}) = \{t \in T(\mathcal{F}) \mid t, \text{root} \models \varphi_1\}$.

Solution:

$$
\begin{aligned}
\varphi_1 \stackrel{\text{def}}{=} \varphi_{\mathrm{S}\downarrow} \wedge [\downarrow^*]\Big(\quad & \mathit{likes} \implies \langle \downarrow; \mathsf{first}?; \varphi_{\mathrm{S}}?; \rightarrow; \varphi_{\mathrm{NP}\downarrow}?; \rightarrow; \varphi_{\mathrm{VP}}?; \rightarrow; \varphi_{\mathrm{NP}\downarrow}? \rangle \mathsf{last}\\
& \mathit{thinks} \implies \langle \downarrow; \mathsf{first}?; \varphi_{\mathrm{S}}?; \rightarrow; \varphi_{\mathrm{NP}\downarrow}?; \rightarrow; \varphi_{\mathrm{VP}}? \rangle \mathsf{last}\\
& \mathit{Bill} \implies \langle \downarrow; \mathsf{first}?; \varphi_{\mathrm{NP}}? \rangle \mathsf{last}\\
& \mathit{someone} \implies \langle \downarrow; \mathsf{first}?; \varphi_{\mathrm{NP}}? \rangle \mathsf{last}\\
& \mathit{real} \implies \langle \downarrow; \mathsf{first}?; \varphi_{\mathrm{NP}}? \rangle \mathsf{last}\\
& \mathit{fake} \implies \langle \downarrow; \mathsf{first}?; \varphi_{\mathrm{NP}}? \rangle \mathsf{last}\\
& \mathit{mushrooms} \implies \langle \downarrow; \mathsf{first}?; \varphi_{\mathrm{NP}}? \rangle \mathsf{last}\\
& \mathit{possibly} \implies \langle \downarrow; \mathsf{first}?; \varphi_{\mathrm{VP}}? \rangle \mathsf{last}\\
& \varepsilon \implies \mathsf{leaf} \quad \Big)
\end{aligned}
$$

where

$$
\begin{aligned}
&\varphi_{\mathrm{S}\downarrow} \stackrel{\text{def}}{=} \mathit{likes} && \varphi_{\mathrm{NP}\downarrow} \stackrel{\text{def}}{=} \mathit{Bill} \vee \mathit{mushrooms} \vee \mathit{someone} && \\
&\varphi_{\mathrm{S}} \stackrel{\text{def}}{=} \mathit{thinks} \vee \varepsilon && \varphi_{\mathrm{VP}} \stackrel{\text{def}}{=} \mathit{possibly} \vee \varepsilon && \varphi_{\mathrm{NP}} \stackrel{\text{def}}{=} \mathit{real} \vee \mathit{fake} \vee \varepsilon
\end{aligned}
$$

1.1 Macro Tree Transducers

Let $\mathcal{X}$ be a countable set of variables and $\mathcal{Y}$ a countable set of parameters; we assume $\mathcal{X}$ and $\mathcal{Y}$ to be disjoint. For Q a ranked alphabet with arities greater than zero, we abuse notations and write $Q(\mathcal{X})$ for the alphabet of pairs $(q, x) \in Q \times \mathcal{X}$ with $\mathit{arity}(q, x) \stackrel{\text{def}}{=} \mathit{arity}(q) - 1$. This is just for convenience, and $(q, x)(t_1, \ldots, t_n)$ is really the term $q(x, t_1, \ldots, t_n)$.

Syntax. A *macro tree transducer* (NMTT) is a tuple $\mathcal{M} = (Q, \mathcal{F}, \mathcal{F}', \Delta, I)$ where Q is a finite set of states, all of arity ≥ 1, $\mathcal{F}$ and $\mathcal{F}'$ are finite ranked alphabets, $I \subseteq Q_1$ is a set of root states of arity one, and Δ is a finite set of term rewriting rules of the form $q(f(x_1, \ldots, x_n), y_1, \ldots, y_p) \to e$ where $q \in Q_{1+p}$ for some $p \geq 0$, $f \in \mathcal{F}_n$ for some $n \in \mathbb{N}$, and $e \in T(\mathcal{F}' \cup Q(\mathcal{X}_n), \mathcal{Y}_p)$. Note that this imposes that any occurrence in e of a variable $x \in \mathcal{X}$ must be as the first argument of a state $q \in Q$.

Inside-Out Semantics. Given a NMTT, the *inside-out* rewriting relation over trees in $T(\mathcal{F} \cup \mathcal{F}' \cup Q)$ is defined by: $t \xrightarrow{\text{IO}} t'$ if there exist a rule $q(f(x_1, \ldots, x_n), y_1, \ldots, y_p) \to e$ in Δ, a context $C \in C(\mathcal{F} \cup \mathcal{F}' \cup Q)$, and two substitutions $\sigma \colon \mathcal{X} \to T(\mathcal{F})$ and $\rho \colon \mathcal{Y} \to T(\mathcal{F}')$ such that $t = C[q(f(x_1, \ldots, x_n), y_1, \ldots, y_p)\sigma\rho]$ and $t' = C[e\sigma\rho]$. In other words, in inside-out rewriting, when applying a rewriting rule $q(f(x_1, \ldots, x_n), y_1, \ldots, y_p) \to e$, the parameters $y_1, \ldots, y_p$ must be mapped to trees in $T(\mathcal{F}')$, with no remaining states from Q.

Similarly to context-free tree grammars, the *inside-out* transduction $[\![\mathcal{M}]\!]_{\text{IO}}$ realised by $\mathcal{M}$ is defined through inside-out rewriting semantics:

$$[\![\mathcal{M}]\!]_{\text{IO}} \stackrel{\text{def}}{=} \{(t, t') \in T(\mathcal{F}) \times T(\mathcal{F}') \mid \exists q \in I \,.\, q(t) \xrightarrow{\text{IO}}{}^{*} t'\} \,.$$

Example 1. Let $\mathcal{F} \stackrel{\text{def}}{=} \{a^{(1)}, \$^{(0)}\}$ and $\mathcal{F}' \stackrel{\text{def}}{=} \{f^{(3)}, a^{(1)}, b^{(1)}, \$^{(0)}\}$. Consider the NMTT $\mathcal{M} = (\{q^{(1)}, q'^{(3)}\}, \mathcal{F}, \mathcal{F}', \Delta, \{q\})$ with Δ the set of rules

$$\begin{aligned} q(a(x_1)) &\to q'(x_1, \$, \$) & q'(\$, y_1, y_2) &\to f(y_1, y_1, y_2) \\ q'(a(x_1), y_1, y_2) &\to q'(x_1, a(y_1), a(y_2)) & q'(a(x_1), y_1, y_2) &\to q'(x_1, a(y_1), b(y_2)) \\ q'(a(x_1), y_1, y_2) &\to q'(x_1, b(y_1), a(y_2)) & q'(a(x_1), y_1, y_2) &\to q'(x_1, b(y_1), b(y_2)) \end{aligned}$$

Then we have for instance the following derivation:

$$\begin{aligned} q(a(a(a(\$)))) &\xrightarrow{\text{IO}} q'(a(a(\$)), \$, \$) \\ &\xrightarrow{\text{IO}} q'(a(\$), b(\$), b(\$)) \\ &\xrightarrow{\text{IO}} q'(\$, a(b(\$)), b(b(\$))) \\ &\xrightarrow{\text{IO}} f(a(b(\$)), a(b(\$)), b(b(\$))) \end{aligned}$$

showing that $(\ a(a(a(\$))),\ f(a(b(\$)), a(b(\$)), b(b(\$)))\ ) \in [\![\mathcal{M}]\!]$.

Exercise 2 (Monadic trees)**.** An NMTT $\mathcal{M}$ is called *linear* and *non-deleting* if, in every rule $q(f(x_1, \ldots, x_n), y_1, \ldots, y_p) \to e$ in Δ, the term e is linear in $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_p\}$, i.e. each variable and each parameter occurs exactly once in the term e.

Let $\mathcal{F}' \stackrel{\text{def}}{=} \{a^{(1)}, b^{(1)}, \$^{(0)}\}$. Observe that trees in $T(\mathcal{F}')$ are in bijection with contexts in $C(\mathcal{F}')$ and words over $\{a, b\}^*$. For a context C from $C(\mathcal{F}')$, we write C^R for its *mirror context*, read from the leaf to the root. For instance, if $C = a(b(a(a(\Box))))$, then $C^R = a(a(b(a(\Box))))$. Formally, let $n \in \mathbb{N}$ be such that $\operatorname{dom} C = \{0^m \mid m \leq n\}$; then $C(0^n) = \Box$ and $C(0^m) \in \{a, b\}$ for $m < n$. Then C^R is defined by $\operatorname{dom} C^R \stackrel{\text{def}}{=} \operatorname{dom} C$, $C^R(0^n) \stackrel{\text{def}}{=} \Box$, and $C^R(0^m) \stackrel{\text{def}}{=} C^R(0^{n-m})$ for all $m < n$.

[2] 1. Give a linear and non-deleting NMTT $\mathcal{M}$ from $\mathcal{F}'$ to $\mathcal{F}'$ such that $[\![\mathcal{M}]\!]_{\text{IO}} = \{(C[\$], C[C^R[\$]]) \mid C \in C(\mathcal{F}')\}$. In terms of words over $\{a,b\}^*$, this transducer maps w to the palindrome ww^R. Is $[\![\mathcal{M}]\!]_{\text{IO}}(T(\mathcal{F}))$ a recognisable tree language?

Solution: Let $\mathcal{M} \stackrel{\text{def}}{=} (Q, \mathcal{F}', \mathcal{F}', \Delta, I)$ where $Q \stackrel{\text{def}}{=} \{q_i^{(1)}, q^{(2)}\}$, $I \stackrel{\text{def}}{=} \{q_i\}$, and Δ is the set of rules

$$\begin{array}{ccc} q_i(\$) \to \$ & q_i(a(x_1)) \to a(q(x_1, a(\$))) & q_i(b(x_1)) \to b(q(x_1, b(\$))) \\ q(\$, y_1) \to y_1 & q(a(x_1), y_1) \to a(q(x_1, a(y_1))) & q(b(x_1), y_1) \to b(q(x_1, b(y_1))) \; . \end{array}$$

We leave the proof of correctness to the reader.

This macro tree transducer is deterministic, and complete. Because a monadic tree language over $\mathcal{F}'$ is recognisable if and only if the corresponding word language over $\{a,b\}$ is recognisable, $[\![\mathcal{M}]\!]_{\text{IO}}(T(\mathcal{F}))$ is not a recognisable tree language. In turn, this shows that recognisable tree languages are not closed under linear non-deleting macro transductions, not even the complete deterministic ones.

Exercise 3 (From derivation to derived trees)**.** Consider again the tree adjoining grammar $\mathcal{G}_1$ from Exercise 1.

[3] 1. Give a linear non-deleting NMTT $\mathcal{M}_1$ that maps the derivation trees of $\mathcal{G}_1$ to its derived trees. Formally, we want $\text{dom}([\![\mathcal{M}_1]\!]_{\text{IO}}) = L_D(\mathcal{G}_1)$ and $[\![\mathcal{M}_1]\!]_{\text{IO}}(T(\mathcal{F})) = L_T(\mathcal{G}_1)$.

Solution: We set $\mathcal{F}' \stackrel{\text{def}}{=} N \uplus \Sigma$, $Q \stackrel{\text{def}}{=} \{q^{(1)}_{\text{S}\downarrow}, q^{(2)}_{\text{S}}, q^{(1)}_{\text{NP}\downarrow}, q^{(2)}_{\text{NP}}, q^{(2)}_{\text{VP}}\}$, $I \stackrel{\text{def}}{=} \{q^{(1)}_{\text{S}}\}$, and Δ:

$$q^{(1)}_{\text{S}\downarrow}(\mathit{likes}(x_1,x_2,x_3,x_4)) \to q^{(2)}_{\text{S}}\Big(x_1, \text{S}\big(q^{(1)}_{\text{NP}\downarrow}(x_2),\; q^{(2)}_{\text{VP}}\big(x_3, \text{VP}(\mathit{VBZ}\diamond,\; q^{(1)}_{\text{NP}\downarrow}(x_4))\big)\big)\Big)$$

$$q_{\mathrm{S}}^{(2)}(\mathit{thinks}(x_1, x_2, x_3), y_1) \to q_{\mathrm{S}}^{(2)}\big(x_1, \mathrm{S}\big(q_{\mathrm{NP}\downarrow}^{(1)}(x_2), q_{\mathrm{VP}}^{(2)}(x_3, \mathrm{VP}(\mathit{VB}\diamond, y_1))\big)\big)$$

$$q_{\mathrm{S}}^{(2)}(\varepsilon, y_1) \to y_1$$

$$q_{\mathrm{NP}\downarrow}^{(1)}(\mathit{Bill}(x_1)) \to q_{\mathrm{NP}}^{(2)}\big(x_1, \mathrm{NP}(\mathit{NNP}\diamond)\big)$$

$$q_{\mathrm{NP}\downarrow}^{(1)}(\mathit{mushrooms}(x_1)) \to q_{\mathrm{NP}}^{(2)}\big(x_1, \mathrm{NP}(\mathit{NNS}\diamond)\big)$$

$$q_{\mathrm{NP}\downarrow}^{(1)}(\mathit{someone}(x_1)) \to q_{\mathrm{NP}}^{(2)}\big(x_1, \mathrm{NP}(\mathit{PN}\diamond)\big)$$

$$q_{\mathrm{NP}}^{(2)}(\mathit{real}(x_1), y_1) \to q_{\mathrm{NP}}^{(2)}\big(x_1, \mathrm{NP}(\mathit{JJ}\diamond, y_1)\big)$$

$$q_{\mathrm{NP}}^{(2)}(\mathit{fake}(x_1), y_1) \to q_{\mathrm{NP}}^{(2)}\big(x_1, \mathrm{NP}(\mathit{JJ}\diamond, y_1)\big)$$

$$q_{\mathrm{NP}}^{(2)}(\varepsilon, y_1) \to y_1$$

$$q_{\mathrm{VP}}^{(2)}(\mathit{possibly}(x_1), y_1) \to q_{\mathrm{VP}}^{(2)}\big(x_1, \mathrm{VP}(\mathit{RB}\diamond, y_1)\big)$$

$$q_{\mathrm{VP}}^{(2)}(\varepsilon, y_1) \to y_1$$

Exercise 4 (Context-free tree grammar)**.** Let $\mathcal{M} = (Q, \mathcal{F}, \mathcal{F}', \Delta, I)$ be an NMTT and $\mathcal{A} = (Q', \mathcal{F}, \delta, I')$ be an NFTA.

[5] 1. Show that $L \stackrel{\text{def}}{=} [\![\mathcal{M}]\!]_{\text{IO}}(L(\mathcal{A})) = \{t' \in T(\mathcal{F}') \mid \exists t \in L(\mathcal{A}) \,.\, (t, t') \in [\![\mathcal{M}]\!]_{\text{IO}}\}$ is an inside-out context-free tree language, i.e., show how to construct a CFTG $\mathcal{G} = (N, \mathcal{F}', S, R)$ such that $L_{\text{IO}}(\mathcal{G}) = L$.

Solution: Let

$$N \stackrel{\text{def}}{=} (Q \times Q') \uplus \{S\}$$

where each pair $(q^{(1+p)}, q')$ from $Q \times Q'$ has arity p, and

$$\begin{aligned} R \stackrel{\text{def}}{=} &\{S \to (q, q')^{(0)} \mid q \in I, q' \in I'\} \\ \cup\ &\{(q, q')^{(p)}(y_1, \ldots, y_p) \to e[q'_i/x_i]_i \quad \mid \exists n \,.\, \exists f \in \mathcal{F}_n \,.\, q^{(1+p)}(f(x_1, \ldots, x_n), y_1, \ldots, y_n) \to e \in \Delta \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{and } (q', f, q'_1, \ldots, q'_n) \in \delta\} \end{aligned}$$

where we abuse notation as indicated at the beginning of the section. For a tree $e \in T(N \cup \mathcal{F}')$, we let $N(e) = \{(q_1, q'_1), \ldots, (q_n, q'_n)\}$ be the set of symbols from N occurring inside e.

Let us show that, for all $k \in \mathbb{N}$, for all $e \in T(N \cup \mathcal{F}')$ with $N(e) = \{(q_1, q'_1), \ldots, (q_n, q'_n)\}$ and for all $t' \in T(\mathcal{F}')$, $e \stackrel{\text{IO}}{\Rightarrow}{}^{k}_{\mathcal{G}}\, t'$ if and only if $\exists t_1, \ldots, t_n \in T(\mathcal{F})$ such that $e[t_i/q'_i]_{1 \le i \le n} \stackrel{\text{IO}}{\Rightarrow}{}^{k}_{\mathcal{M}}$ t' and for all $1 \le i \le n$, $t_i \stackrel{\delta_B}{\Rightarrow}{}^{*}_{\mathcal{A}}\, q'_i$.

We prove the statement by induction, first over k the number of rewriting steps in $\mathcal{G}$ and $\mathcal{M}$, and second over the term e. We only prove the 'if' direction, as the 'only if' one is similar.

If Assume $e \stackrel{\text{IO}}{\Rightarrow}{}^{k}_{\mathcal{G}}\, t'$.

If $e = f(e_1, \ldots, e_m)$ for some $m \in \mathbb{N}$ and $f \in \mathcal{F}'_m$, then this rewrite can be decomposed as

$$e = f(e_1, \ldots, e_m) \stackrel{\text{IO}}{\Rightarrow}{}^{k}_{\mathcal{G}}\, f(t'_1, \ldots, t'_m) = t'$$

where for all $1 \le j \le m$, $t'_j \in T(\mathcal{F}')$ is such that

$$e_j \stackrel{\text{IO}}{\Rightarrow}{}^{k_j}_{\mathcal{G}}\, t'_j$$

and

$$k = \sum_{1 \le j \le m} k_j \,.$$

Let $N(e_j) = \{(q_{j,1}, q'_{j,1}), \ldots, (q_{j,n_j}, q'_{j,n_j})\}$; then $N(e) = \bigcup_{1\leq j\leq m} N(e_j)$.

For each $1 \leq j \leq m$, by induction hypothesis on the subterms e_j since $k_j \leq k$, there exist $t_{j,1}, \ldots, t_{j,n_j} \in T(\mathcal{F})$ such that

$$e_j[t_{j,i}/q'_{j,i}]_{1\leq i\leq n_j} \overset{\text{IO}}{\Rightarrow}{}^{k_j}_{\mathcal{M}}\, t'_j$$

and

$$t_{j,i} \overset{\delta_B}{\Rightarrow}{}^{*}_{\mathcal{A}}\, q'_{j,i}$$

for all $1 \leq i \leq n_j$. Thus

$$f(e_1, \ldots, e_m)[t_{j,i}/q'_{j,i}]_{1\leq j\leq m, 1\leq i\leq n_j} \overset{\text{IO}}{\Rightarrow}{}^{k}_{\mathcal{M}}\, f(t'_1, \ldots, t'_m) = t'$$

as desired.

If $\boldsymbol{e = (q, q')^{(p)}(e_1, \ldots, e_p)}$ for some $p \in \mathbb{N}$ and $(q, q')^{(p)} \in Q \times Q'$, then this rewrite can be decomposed as

$$\begin{aligned} e = (q, q')^{(p)}(e_1, \ldots, e_p) &\overset{\text{IO}}{\Rightarrow}{}^{k'}_{\mathcal{G}}\, (q, q')^{(p)}(t'_1, \ldots, t'_p) \\ &\overset{\text{IO}}{\Rightarrow}_{\mathcal{G}}\, e'[q'_i/x_i]_{1\leq i\leq m}[t'_j/y_j]_{1\leq j\leq p} \\ &\overset{\text{IO}}{\Rightarrow}{}^{k''}_{\mathcal{G}}\, t' \end{aligned}$$

where for all $1 \leq j \leq m$, $t'_j \in T(\mathcal{F}')$ is such that

$$e_j \overset{\text{IO}}{\Rightarrow}{}^{k_j}_{\mathcal{G}}\, t'_j$$

and $k' = \sum_{1\leq j\leq m} k_j$ and $k = 1+k'+k''$; also $N(e) = \{(q, q')\}\cup\bigcup_{1\leq j\leq p} N(e_j)$ where $N(e_j) = \{(q_{j,1}, q'_{j,1}), \ldots, (q_{j,n_j}, q'_{j,n_j})\}$. Such a rule application relies on the existence of $m \in \mathbb{N}$ and $f \in \mathcal{F}_m$ such that there are rules $q^{(1+p)}(f(x_1, \ldots, x_m), y_1, \ldots, y_p) \to e' \in \Delta$ and $(q', f, q'_1, \ldots, q'_m) \in \delta$.

By induction hypothesis on $k_j < k$ for each $1 \leq j \leq p$, there exist $t_{j,1}, \ldots, t_{j,n_j} \in T(\mathcal{F})$ such that

$$e_j[t_{j,i}/q'_{j,i}]_{1\leq i\leq n_j} \overset{\text{IO}}{\Rightarrow}{}^{k_j}_{\mathcal{M}}\, t'_j$$

and

$$t_{j,i} \overset{\delta_B}{\Rightarrow}{}^{*}_{\mathcal{A}}\, q'_{j,i}$$

for all $1 \leq i \leq n_j$.

Furthermore, $N(e'[t'_j/y_j]_{1\leq j\leq p}[q'_i/x_i]_{1\leq i\leq m}) = \{(q_1, q'_1), \ldots, (q_m, q'_m)\}$ and by induction hypothesis over $k'' < k$, there exist $t_1, \ldots, t_m \in T(\mathcal{F})$ such that

$$e'[t'_j/y_j]_{1\leq j\leq p}[t_i/x_i]_{1\leq i\leq m} \overset{\text{IO}}{\Rightarrow}{}^{k''}_{\mathcal{M}} t'$$

and

$$t_i \overset{\delta_B}{\Rightarrow}{}^{*}_{\mathcal{A}} q'_i$$

for all $1 \leq i \leq m$. Note that, because $(q', f, q'_1, \ldots, q'_m) \in \delta$, the latter imply

$$f(t_1, \ldots, t_m) \overset{\delta_B}{\Rightarrow}{}^{*}_{\mathcal{A}} f(q'_1, \ldots, q'_m) \overset{\delta_B}{\Rightarrow}_{\mathcal{A}} q' \ .$$

Thus, in $\mathcal{M}$, we have the rewrite

$$\begin{aligned}
&e[f(t_1, \ldots, t_m)/q][t'_{j,i}/q'_{j,i}]_{1\leq j\leq m, 1\leq i\leq n_i}\\
&= q^{(1+p)}(f(t_1, \ldots, t_m), e_1[t'_{1,i}/q'_{1,i}]_{1\leq i\leq n_1}, \ldots, e_m[t'_{m,i}/q'_{m,i}]_{1\leq i\leq n_m})\\
&= q^{(1+p)}(f(x_1, \ldots, x_m), e_1[t'_{1,i}/q'_{1,i}]_{1\leq i\leq n_1}, \ldots, e_m[t'_{m,i}/q'_{m,i}]_{1\leq i\leq n_m})[t_1/x_1, \ldots, t_m/x_m]\\
&\overset{\text{IO}}{\Rightarrow}{}^{k'}_{\mathcal{M}} q^{(1+p)}(f(x_1, \ldots, x_m), t'_1, \ldots, t'_p)[t_1/x_1, \ldots, t_m/x_m]\\
&\overset{\text{IO}}{\Rightarrow}_{\mathcal{M}} e'[t_i/x_i]_{1\leq i\leq m}[t'_j/y_j]_{1\leq j\leq p}\\
&\overset{\text{IO}}{\Rightarrow}{}^{k''}_{\mathcal{M}} t'
\end{aligned}$$

as desired.

2 Scope ambiguities and propositional attitudes

Exercise 5. One considers the two following signatures:

$$(\Sigma_{\text{ABS}})\qquad \begin{aligned}
\textsc{suzy} &: NP\\
\textsc{bill} &: NP\\
\textsc{mushroom} &: N\\
\textsc{a} &: N \to (NP \to S) \to S\\
\textsc{a}_{\mathit{inf}} &: N \to (NP \to S_{\mathit{inf}}) \to S_{\mathit{inf}}\\
\textsc{eat} &: NP \to NP \to S_{\mathit{inf}}\\
\textsc{to} &: (NP \to S_{\mathit{inf}}) \to VP\\
\textsc{want} &: VP \to NP \to S
\end{aligned}$$

$$(\Sigma_{\text{S-FORM}}) \qquad \begin{aligned} \textit{\textbf{Suzy}} &: \textit{string} \\ \textit{\textbf{Bill}} &: \textit{string} \\ \textit{\textbf{mushroom}} &: \textit{string} \\ \textit{\textbf{a}} &: \textit{string} \\ \textit{\textbf{eat}} &: \textit{string} \\ \textit{\textbf{to}} &: \textit{string} \\ \textit{\textbf{wants}} &: \textit{string} \end{aligned}$$

where, as usual, *string* is defined to be $o \to o$ for some atomic type o.

One then defines a morphism ($\mathcal{L}_{\text{SYNT}} : \Sigma_{\text{ABS}} \to \Sigma_{\text{S-FORM}}$) as follows:

$$(\mathcal{L}_{\text{SYNT}}) \qquad \begin{aligned} NP &:= \textit{string} \\ N &:= \textit{string} \\ S &:= \textit{string} \\ S_{\textit{inf}} &:= \textit{string} \\ VP &:= \textit{string} \\[4pt] \text{SUZY} &:= \textit{\textbf{Suzy}} \\ \text{BILL} &:= \textit{\textbf{Bill}} \\ \text{MUSHROOM} &:= \textit{\textbf{mushroom}} \\ \text{A} &:= \lambda xy.\, y\,(\textit{\textbf{a}} + x) \\ \text{A}_{\textit{inf}} &:= \lambda xy.\, y\,(\textit{\textbf{a}} + x) \\ \text{EAT} &:= \lambda xy.\, y + \textit{\textbf{eat}} + x \\ \text{TO} &:= \lambda x.\, \textit{\textbf{to}} + (x\,\epsilon) \\ \text{WANT} &:= \lambda xy.\, y + \textit{\textbf{wants}} + x \end{aligned}$$

where, as usual, the concatenation operator $(+)$ is defined as functional composition, and the empty word (ϵ) as the identity function.

[1] 1. Give two different terms, say t_0 and t_1, such that:

$$\mathcal{L}_{\text{SYNT}}(t_0) = \mathcal{L}_{\text{SYNT}}(t_1) = \textit{\textbf{Bill}} + \textit{\textbf{wants}} + \textit{\textbf{to}} + \textit{\textbf{eat}} + \textit{\textbf{a}} + \textit{\textbf{mushroom}}$$

Solution:

$$\begin{aligned} t_0 &= \text{WANT}\,(\text{TO}\,(\lambda x.\, \text{A}_{\textit{inf}}\, \text{MUSHROOM}\,(\lambda y.\, \text{EAT}\, y\, x)))\, \text{BILL} \\ t_1 &= \text{A}\, \text{MUSHROOM}\,(\lambda y.\, \text{WANT}\,(\text{TO}\,(\lambda x.\, \text{EAT}\, y\, x))\, \text{BILL}) \end{aligned}$$

Exercise 6. One considers a third signature :

$$(\Sigma_{\text{L-FORM}}) \qquad \begin{aligned} \mathbf{suzy} &: \mathsf{ind} \\ \mathbf{bill} &: \mathsf{ind} \\ \mathbf{mushroom} &: \mathsf{ind} \to \mathsf{prop} \\ \mathbf{eat} &: \mathsf{ind} \to \mathsf{ind} \to \mathsf{prop} \\ \mathbf{want} &: \mathsf{ind} \to \mathsf{prop} \to \mathsf{prop} \end{aligned}$$

One then defines a morphism ($\mathcal{L}_{\text{SEM}} : \Sigma_{\text{ABS}} \to \Sigma_{\text{L-FORM}}$) as follows:

$$(\mathcal{L}_{\text{SEM}}) \qquad \begin{aligned} NP &:= \mathsf{ind} \\ N &:= \mathsf{ind} \to \mathsf{prop} \\ S &:= \mathsf{prop} \\ S_{inf} &:= \mathsf{prop} \\ VP &:= \mathsf{ind} \to \mathsf{prop} \\[1ex] \text{SUZY} &:= \mathbf{suzy} \\ \text{BILL} &:= \mathbf{bill} \\ \text{MUSHROOM} &:= \mathbf{mushroom} \\ \text{A} &:= \lambda xy.\, \exists z.\, (x\, z) \land (y\, z) \\ \text{A}_{inf} &:= \lambda xy.\, \exists z.\, (x\, z) \land (y\, z) \\ \text{EAT} &:= \lambda xy.\, \mathbf{eat}\, y\, x \\ \text{TO} &:= \lambda x.\, x \\ \text{WANT} &:= \lambda xy.\, \mathbf{want}\, y\, (x\, y) \end{aligned}$$

[1] 1. Compute the different semantic interpretations of the sentence *Bill wants to eat a mushroom*, i.e., compute $\mathcal{L}_{\text{SEM}}(t_0)$ and $\mathcal{L}_{\text{SEM}}(t_1)$.

Solution:

$$\begin{aligned} \mathcal{L}_{\text{SEM}}(t_0) &= \mathbf{want}\,\mathbf{bill}\,(\exists z.\,(\mathbf{mushroom}\, z) \land (\mathbf{eat}\,\mathbf{bill}\, z)) \\ \mathcal{L}_{\text{SEM}}(t_1) &= \exists z.\,(\mathbf{mushroom}\, z) \land (\mathbf{want}\,\mathbf{bill}\,(\mathbf{eat}\,\mathbf{bill}\, z)) \end{aligned}$$

Exercise 7. One extends Σ_{ABS} and $\mathcal{L}_{\text{SYNT}}$, respectively, as follows:

$$\begin{aligned} &(\Sigma_{\text{ABS}}) && \text{WANT2} : \ VP \to NP \to S \\ &(\mathcal{L}_{\text{SYNT}}) && \text{WANT2} := \lambda xyz.\, z + \boldsymbol{wants} + x + y \end{aligned}$$

[1] 1. Extend $\mathcal{L}_{\text{SEM}}$ accordingly in order to allow for the analysis of a sentence such as *Bill wants Suzy to eat a mushroom.*

Solution:

$$(\mathcal{L}_{\text{SEM}}) \quad \text{WANT2} := \lambda xyz.\, \mathbf{want}\, z\, (y\, x)$$

Exercise 8. One extends Σ_{ABS} as follows:

$$\begin{array}{ll}(\Sigma_{\text{ABS}}) & \text{EVERYONE} : (NP \to S) \to S \\ & \quad \text{THINK} : S \to NP \to S\end{array}$$

in order to allow for the analysis of the following sentence:

(1) *everyone thinks Bill wants to eat a mushroom.*

[3] 1. Extend $\Sigma_{\text{S-FORM}}$, $\mathcal{L}_{\text{SYNT}}$, $\Sigma_{\text{L-FORM}}$, and $\mathcal{L}_{\text{SEM}}$ accordingly.

Solution:

$$\begin{array}{ll}
(\Sigma_{\text{S-FORM}}) & \boldsymbol{everyone} : \ \mathit{string} \\
& \boldsymbol{thinks} : \ \mathit{string} \\
(\mathcal{L}_{\text{SYNT}}) & \text{EVERYONE} := \lambda x.\, x\, \boldsymbol{everyone} \\
& \text{THINK} := \lambda xy.\, y + \boldsymbol{thinks} + x \\
(\Sigma_{\text{L-FORM}}) & \mathbf{human} : \ \mathsf{ind} \to \mathsf{prop} \\
& \mathbf{think} : \ \mathsf{ind} \to \mathsf{prop} \to \mathsf{prop} \\
(\mathcal{L}_{\text{SEM}}) & \text{EVERYONE} := \lambda x.\, \forall y.\, (\mathbf{human}\, y) \to (x\, y) \\
& \text{THINK} := \lambda xy.\, \mathbf{think}\, y\, x
\end{array}$$

[2] 2. Give the several λ-terms that correspond to the different parsings of sentence (1).

Solution: There are four such terms:

$$\text{EVERYONE}\, (\lambda x.\, \text{THINK}\, (\text{WANT}\, (\text{TO}\, (\lambda z.\, \text{A}_{\mathit{inf}}\, \text{MUSHROOM}\, (\lambda y.\, \text{EAT}\, y\, z)))\, \text{BILL})\, x)$$
$$\text{EVERYONE}\, (\lambda x.\, \text{THINK}\, (\text{A MUSHROOM}\, (\lambda y.\, \text{WANT}\, (\text{TO}\, (\lambda z.\, \text{EAT}\, y\, z))\, \text{BILL}))\, x)$$
$$\text{EVERYONE}\, (\lambda x.\, \text{A MUSHROOM}\, (\lambda y.\, \text{THINK}\, (\text{WANT}\, (\text{TO}\, (\lambda z.\, \text{EAT}\, y\, z))\, \text{BILL})\, x))$$
$$\text{A MUSHROOM}\, (\lambda y.\, \text{EVERYONE}\, (\lambda x.\, \text{THINK}\, (\text{WANT}\, (\text{TO}\, (\lambda z.\, \text{EAT}\, y\, z))\, \text{BILL})\, x))$$