Higher-Order Matching in the Linear λ-calculus with Pairing

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Abstract. We prove that higher-order matching in the linear λ-calculus with pairing is decidable. We also establish its NP-completeness under the assumption that the right-hand side of the equation to be solved is given in normal form.

1 Introduction

The decidability of higher-order matching (which consists in determining whether a simply typed λ-term is an instance of another one, modulo the conversion rules of the λ-calculus), has been intensively studied in the literature. In particular, second-order matching [12], third-order matching [5], and fourth-order matching [16] have been shown to be decidable (both modulo β and $\beta\eta$). On the other hand, it has been proved that, starting from the sixth order, higher-order matching modulo β is undecidable [14] (for $\beta\eta$, the problem is still open).

In two recent papers [9] and [10], we studied the decidability and the complexity of a quite restricted form of higher-order matching, namely, higher-order matching in the linear λ-calculus. This calculus corresponds, through the Curry-Howard isomorphism, to the implicative fragment of Girard's linear logic [7], and may be naturally extended by taking into account the other connectives of linear logic. We follow this line of research in the present paper by considering the linear λ-calculus with pairing, i.e., the calculus corresponding to the negative fragment of multiplicative additive linear logic.

The paper is organized as follows. Section 2 presents the necessary mathematical notions and notations that we use in the sequel. In section 3, we show that deciding whether a linear λ-term with pairs may be reduced to a given normal form may be done in polynomial time. Finally, section 4 shows that every term may be turned into another term that has the same behaviour with respect to reductions, and whose length is bounded in terms of the redices it contains and the size of its normal form. This technical result allows us to conclude that higher-order matching in the linear λ-calculus with pairing is decidable. We also obtain that the problem is NP-complete when the right member of the equation is given in normal form.

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2 Mathematical Background

Definition 1. *Let $\mathscr{A}$ be a finite set of atomic types. The set $\mathscr{F}$ of types is defined according to the following grammar:*

$$\mathscr{F} \ ::= \ \mathscr{A} \mid (\mathscr{F} \multimap \mathscr{F}) \mid (\mathscr{F} \mathbin{\&} \mathscr{F}).$$

Definition 2. *Let $(\Sigma_\alpha)_{\alpha\in\mathscr{F}}$ be a family of pairwise disjoint finite sets indexed by $\mathscr{F}$, whose almost every member is empty. Let $(\mathscr{X}_\alpha)_{\alpha\in\mathscr{F}}$ and $(\mathscr{Y}_\alpha)_{\alpha\in\mathscr{F}}$ be two families of pairwise disjoint countably infinite sets indexed by $\mathscr{F}$, such that $(\bigcup_{\alpha\in\mathscr{F}} \mathscr{X}_\alpha) \cap (\bigcup_{\alpha\in\mathscr{F}} \mathscr{Y}_\alpha) = \emptyset$. The set $\mathscr{T}$ of raw λ-terms is defined according to the following grammar:*

$$\mathscr{T} \ ::= \ \Sigma \mid \mathscr{X} \mid \mathscr{Y} \mid \lambda\mathscr{X}.\mathscr{T} \mid (\mathscr{T}\,\mathscr{T}) \mid \langle\mathscr{T},\mathscr{T}\rangle \mid (\pi_1\mathscr{T}) \mid (\pi_2\mathscr{T}),$$

where $\Sigma = \bigcup_{\alpha\in\mathscr{F}} \Sigma_\alpha$, $\mathscr{X} = \bigcup_{\alpha\in\mathscr{F}} \mathscr{X}_\alpha$, and $\mathscr{Y} = \bigcup_{\alpha\in\mathscr{F}} \mathscr{Y}_\alpha$.

In the above definition, the elements of Σ correspond to constants, and the elements of $\mathscr{X}$ are the λ-variables. The elements of $\mathscr{Y}$ are called the *unknowns*, and will be denoted by uppercase bold letters ($\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \ldots$).

The notions of free and bound occurrences of a λ-variable are defined as usual, and we write $\mathrm{FV}(t)$ for the set of λ-variables that occur free in a λ-term t. A λ-term that does not contain any subterm of the form $\langle t, u\rangle$ is called a *purely applicative λ-term*. A λ-term that does not contain any unknown is called a *pure λ-term*.

The notion of linear λ-term is then defined as follows.

Definition 3. *The family $(\mathscr{T}_\alpha)_{\alpha\in\mathscr{F}}$ of sets of linear λ-terms is inductively defined as follows:*

1. *if* $\mathrm{a} \in \Sigma_\alpha$ *then* $\mathrm{a} \in \mathscr{T}_\alpha$;
2. *if* $\mathbf{X} \in \mathscr{Y}_\alpha$ *then* $\mathbf{X} \in \mathscr{T}_\alpha$;
3. *if* $x \in \mathscr{X}_\alpha$ *then* $x \in \mathscr{T}_\alpha$;
4. *if* $x \in \mathscr{X}_\alpha$, $t \in \mathscr{T}_\beta$, *and* $x \in \mathrm{FV}(t)$, *then* $\lambda x.\,t \in \mathscr{T}_{(\alpha\multimap\beta)}$;
5. *if* $t \in \mathscr{T}_{(\alpha\multimap\beta)}$, $u \in \mathscr{T}_\alpha$, *and* $\mathrm{FV}(t)\cap\mathrm{FV}(u) = \emptyset$, *then* $(t\,u) \in \mathscr{T}_\beta$;
6. *if* $t \in \mathscr{T}_\alpha$, $u \in \mathscr{T}_\beta$, *and* $\mathrm{FV}(t) = \mathrm{FV}(u)$ *then* $\langle t, u\rangle \in \mathscr{T}_{\alpha\&\beta}$;
7. *if* $t \in \mathscr{T}_{\alpha\&\beta}$ *then* $(\pi_1 t) \in \mathscr{T}_\alpha$;
8. *if* $t \in \mathscr{T}_{\alpha\&\beta}$ *then* $(\pi_2 t) \in \mathscr{T}_\beta$.

The conditions on the free variables in clauses 4, 5, and 6 correspond to the linearity conditions. They constraint the way λ-variables may occur in a term.

We define $\mathscr{T}$ to be $\bigcup_{\alpha\in\mathscr{F}} \mathscr{T}_\alpha$ (which is a proper subset of the set of raw λ-terms). It is easy to prove that the sets $(\mathscr{T}_\alpha)_{\alpha\in\mathscr{F}}$ are pairwise disjoint. Consequently, we may define the type of a linear λ-term t to be the unique linear type α such that $t \in \mathscr{T}_\alpha$.

We let $t[x{:=}u]$ denote the usual capture-avoiding substitution of a λ-variable by a λ-term, and $t[x_1{:=}u_1, \ldots, x_n{:=}u_n]$ denote the usual notion of parallel substitution. If σ denotes such a parallel substitution $[x_1{:=}u_1,\ \ldots,\ x_n{:=}u_n]$, we write $t.\sigma$ for $t[x_1{:=}u_1, \ldots, x_n{:=}u_n]$, $\sigma(x_i)$ for u_i, and we define $\mathrm{dom}(\sigma)$ to be the finite set of variables $\{x_1, \ldots, x_n\}$. We use the same notations to denote the substitutions of unknowns by λ-terms. The substitution the domain of which is empty is the identity and is noted Id.

We take for granted the usual notions of β reduction, left projection, and right projection:

$$(\lambda x.\, t)\, u \to t[x{:=}u], \qquad \pi_1\langle t, u\rangle \to t, \qquad \pi_2\langle t, u\rangle \to u.$$

The union of these three notions of reduction induces the relation of one step reduction ($\to$), the relation of at most one step reduction ($\xrightarrow{=}$), the relations of n steps reduction ($\xrightarrow{n}$), and the relations of many steps reduction ($\xrightarrow{*}$). The equality between linear λ-terms ($=$) is defined to be the reflexive, symmetric, transitive closure of the relation of reduction and we write $\equiv$ for syntatic equality. The linear λ-calculus with pairing is strongly normalizable, the equality ($=$) is then decidable and every term has a unique normal form.

We now give a precise definition of the matching problem with which this paper is concerned.

Definition 4. *A matching problem (in the linear λ-calculus with pairing) consists of a pair of linear λ-terms (t, u) of the same type such that u does not contain any unknown.*

Such a problem admits a solution if and only if there exists a substitution $[\mathbf{X}_1{:=}t_1,\ \ldots,\ \mathbf{X}_n{:=}t_n]$ such that $t[\mathbf{X}_1{:=}t_1, \ldots, \mathbf{X}_n{:=}t_n] = u$

We end this section with two remarks about the previous definition:

1. In the substitution $[\mathbf{X}_1{:=}t_1, \ldots, \mathbf{X}_n{:=}t_n]$ we do not require $t_1, \ldots, t_n$ to be pure terms. This is mandatory. Consider, for instance, the following matching problem: $\pi_1\mathbf{X} = c$, where c is a constant of type α, and $\mathbf{X}$ an unknown of type $\alpha \,\&\, \beta$. This problem admits the solution $[\mathbf{X}{:=}\langle c, \mathbf{Y}\rangle]$, where $\mathbf{Y}$ is an unknown of type β. Now, if we would require the solution to be made of pure terms, we would face the problem of constructing a closed term of type β, which is undecidable.
2. In defining the notion of equality, we did not take into account η-reduction and surjective pairing. In fact, all the results we obtain in this paper also hold for this stronger notion of equality.

3 A Polynomially Bounded Reduction Strategy

One of the key properties in establishing the NP-completeness of higher-order matching in the linear λ-calculus [9] is that any linear λ-term (without pairs) may be reduced to its normal form in polynomial (actually, linear) time. This is a direct consequence of the fact that the length of the linear λ-terms strictly decreases under β-reduction.

This property does not hold in the presence of pairing. Indeed, in a linear λ-term of the form $\lambda x.\, \langle t, u\rangle$, there are at least two occurrences of x that are bound by the abstraction. Consequently, the length of a redex such as $(\lambda x.\, \langle t, u\rangle)\, v$ may be stricly less than the length of its contractum $\langle t, u\rangle[x{:=}v]$. In fact, it is even not the case (modulo $P \neq co\text{-}NP$) that a linear λ-term with pairs may be reduced to its normal form in polynomial time [15]. Consequently, in this section we establish the following weaker property: if $t \xrightarrow{*} u$ then there exists a reduction strategy $t \xrightarrow{n} u$ such that n is polynomially bounded with respect to the length of t and u.

In order to establish this property, we first define two notions of complexity.

Definition 5. *The complexity $\rho(\alpha)$ of a linear type α is defined to be the number of connectives it contains:*

1. $\rho(\mathrm{a}) = 0$*, for* a *atomic;*
2. $\rho(\alpha \multimap \beta) = \rho(\alpha) + \rho(\beta) + 1$*;*
3. $\rho(\alpha \mathbin{\&} \beta) = \rho(\alpha) + \rho(\beta) + 1$*.*

The complexity $\rho(t)$ of a linear λ-term t is defined to be the complexity of its type.

Definition 6. *The norm $\mu(t)$ of a linear λ-term t is inductively defined as follows:*

1. $\mu(c) = 0$*, for c being a constant, a λ-variable, or an unknown.*
2. $\mu(\lambda x.\, t_1) = \mu(t_1)$
3. $\mu(t_1\, t_2) = \begin{cases} \mu(t_1) + \mu(t_2) + \rho(t_1), & \textit{if } t_1\, t_2 \textit{ is a redex.} \\ \mu(t_1) + \mu(t_2), & \textit{otherwise.} \end{cases}$
4. $\mu(\langle t_1, t_2\rangle) = \max(\mu(t_1), \mu(t_2))$
5. $\mu(\pi_1 t_1) = \begin{cases} \mu(t_1) + \rho(t_1), & \textit{if } \pi_1 t_1 \textit{ is a redex.} \\ \mu(t_1), & \textit{otherwise.} \end{cases}$
6. $\mu(\pi_2 t_1) = \begin{cases} \mu(t_1) + \rho(t_1), & \textit{if } \pi_2 t_1 \textit{ is a redex.} \\ \mu(t_1), & \textit{otherwise.} \end{cases}$

The above norm does not strictly decrease when reducing a term. This is due to clause 4. Indeed, in case $t_1 \to t_2$ with $\mu(t_1) \leq \mu(u)$, we have that $\langle t_1, u\rangle \to \langle t_2, u\rangle$ while $\mu(\langle t_1, u\rangle) = \mu(\langle t_2, u\rangle)$. In fact, this is the only problematic case, and we will prove that the norm strictly decreases under reduction if there is no reduction step that takes place within a pair. To this end, we introduce the following notion of external reduction.

Definition 7. *The relation of external reduction ($\triangleright$) is defined by means of the following formal system.*

$$(\lambda x.\, t)\, u \triangleright t[x{:=}u] \qquad \pi_1\langle t, u\rangle \triangleright t \qquad \pi_2\langle t, u\rangle \triangleright u \qquad \frac{t \triangleright u}{\lambda x.\, t \triangleright \lambda x.\, u}$$

$$\frac{t \triangleright u}{t\, v \triangleright u\, v} \qquad \frac{t \triangleright u}{v\, t \triangleright v\, u} \qquad \frac{t \triangleright u}{\pi_1 t \triangleright \pi_1 u} \qquad \frac{t \triangleright u}{\pi_2 t \triangleright \pi_2 u}$$

We state two technical lemmas that will be useful in the sequel. Their proofs, which are not difficult, are left to the reader.

Lemma 1. *Let t and $\lambda x.\, u$ be two linear λ-terms such that $t \xrightarrow{*} \lambda x.\, u$. Then, there exists a linear λ-term $\lambda x.\, v$ such that $t \overset{*}{\rhd} \lambda x.\, v \xrightarrow{*} \lambda x.\, u$.* □

Lemma 2. *Let t and $\langle u_1, u_2\rangle$ be two linear λ-terms such that $t \xrightarrow{*} \langle u_1, u_2\rangle$. Then, there exists a linear λ-term $\langle v_1, v_2\rangle$ such that $t \overset{*}{\rhd} \langle v_1, v_2\rangle \xrightarrow{*} \langle u_1, u_2\rangle$.* □

We are now in a position of proving that the norm of a term strictly decreases under external reduction. The keystone of the proof is the following substitution lemma.

Lemma 3. *Let $t \in \mathscr{T}$, and $u, x \in \mathscr{T}_\alpha$ be such that $x \in \mathrm{FV}(t)$. Then, we have that $\mu(t[x{:=}u]) \leq \mu(t) + \mu(u) + \rho(u)$.*

Proof. The proof proceeds by induction on the structure of t.

1. $t \equiv x$.
$$\begin{aligned}\mu(t[x{:=}u]) &= \mu(x[x{:=}u])\\ &= \mu(u)\\ &\leq \mu(u) + \rho(u)\\ &= \mu(t) + \mu(u) + \rho(u)\end{aligned}$$
2. $t \equiv \lambda y.\, t_1$.
$$\begin{aligned}\mu(t[x{:=}u]) &= \mu(\lambda y.\, t_1[x{:=}u])\\ &= \mu(t_1[x{:=}u])\\ &\leq \mu(t_1) + \mu(u) + \rho(u)\\ &\qquad\text{(by induction hypothesis)}\\ &= \mu(t) + \mu(u) + \rho(u)\end{aligned}$$
3. $t \equiv t_1\, t_2$. We distinguish between two cases:
   (a) $x \in \mathrm{FV}(t_1)$. Because of the linearity of t, we have that $x \notin \mathrm{FV}(t_2)$. Consequently, $t[x{:=}u] = t_1[x{:=}u]\, t_2$. Then, there are three subcases:
      i. $t_1 \equiv x$ and $u \equiv \lambda y.\, u_1$.
$$\begin{aligned}\mu(t[x{:=}u]) &= \mu(x\, t_2[x{:=}u])\\ &= \mu(u\, t_2)\\ &= \mu(t_2) + \mu(u) + \rho(u)\\ &= \mu(x) + \mu(t_2) + \mu(u) + \rho(u)\\ &= \mu(t) + \mu(u) + \rho(u)\end{aligned}$$
      ii. $t_1 \equiv \lambda y.\, t_{11}$.
$$\begin{aligned}\mu(t[x{:=}u]) &= \mu(t_1[x{:=}u]\, t_2)\\ &= \mu(t_1[x{:=}u]) + \mu(t_2) + \rho(t_1[x{:=}u])\\ &= \mu(t_1[x{:=}u]) + \mu(t_2) + \rho(t_1)\\ &\qquad\text{(by stability of typing under substitution)}\\ &\leq \mu(t_1) + \mu(t_2) + \rho(t_1) + \mu(u) + \rho(u)\\ &\qquad\text{(by induction hypothesis)}\\ &= \mu(t) + \mu(u) + \rho(u)\end{aligned}$$

iii. Otherwise.

$$\begin{aligned}
\mu(t[x{:=}u]) &= \mu(t_1[x{:=}u]\, t_2) \\
&= \mu(t_1[x{:=}u]) + \mu(t_2) \\
&\leq \mu(t_1) + \mu(t_2) + \mu(u) + \rho(u) \\
&\qquad\qquad \text{(by induction hypothesis)} \\
&= \mu(t) + \mu(u) + \rho(u)
\end{aligned}$$

(b) $x \in \mathrm{FV}(t_2)$. There are two subcases, which are similar to Subcases ii and iii of Case (a).

4. $t \equiv \langle t_1, t_2 \rangle$.

$$\begin{aligned}
\mu(t[x{:=}u]) &= \mu(\langle t_1[x{:=}u], t_2[x{:=}u] \rangle) \\
&= \max(\mu(t_1[x{:=}u]), \mu(t_2[x{:=}u])) \\
&\leq \max(\mu(t_1) + \mu(u) + \rho(u), \mu(t_2) + \mu(u) + \rho(u)) \\
&\qquad\qquad\qquad\qquad\qquad \text{(by induction hypothesis)} \\
&= \max(\mu(t_1), \mu(t_2)) + \mu(u) + \rho(u) \\
&= \mu(t) + \mu(u) + \rho(u)
\end{aligned}$$

5. $t \equiv \pi_1 t_1$. We distinguish between three cases:

(a) $t_1 \equiv x$ and $u \equiv \langle u_1, u_2 \rangle$.

$$\begin{aligned}
\mu(t[x{:=}u]) &= \mu(\pi_1 u) \\
&= \mu(u) + \rho(u) \\
&= \mu(\pi_1 x) + \mu(u) + \rho(u) \\
&= \mu(t) + \mu(u) + \rho(u)
\end{aligned}$$

(b) $t_1 \equiv \langle t_{11}, t_{12} \rangle$.

$$\begin{aligned}
\mu(t[x{:=}u]) &= \mu(\pi_1(t_1[x{:=}u])) \\
&= \mu(t_1[x{:=}u]) + \rho(t_1[x{:=}u]) \\
&= \mu(t_1[x{:=}u]) + \rho(t_1) \\
&\qquad\qquad \text{(by stability of typing under substitution)} \\
&\leq \mu(t_1) + \rho(t_1) + \mu(u) + \rho(u) \\
&\qquad\qquad\qquad\qquad \text{(by induction hypothesis)} \\
&= \mu(t) + \mu(u) + \rho(u)
\end{aligned}$$

(c) Otherwise.

$$\begin{aligned}
\mu(t[x{:=}u]) &= \mu(\pi_1(t_1[x{:=}u])) \\
&= \mu(t_1[x{:=}u]) \\
&\leq \mu(t_1) + \mu(u) + \rho(u) \\
&\qquad\qquad \text{(by induction hypothesis)} \\
&= \mu(t) + \mu(u) + \rho(u)
\end{aligned}$$

6. $t \equiv \pi_2 t_1$. This case is similar to the previous one. □

Proposition 1. *Let t and u be two linear λ-terms such that $t \triangleright u$. Then $\mu(u) < \mu(t)$.*

Proof. The proof proceeds by induction on the derivation of $t \triangleright u$. We only give the base cases, the induction steps are straightforward.

1. $t \equiv (\lambda x.\, t_1)\, t_2$ and $u \equiv t_1[x{:=}t_2]$.

$$\begin{aligned}\mu(u) &= \mu(t_1[x{:=}t_2]) \\ &\leq \mu(t_1) + \mu(t_2) + \rho(t_2) \quad \text{(by Lemma 3)} \\ &< \mu(t_1) + \mu(t_2) + \rho(t_2) + \rho(t_1) + 1 \\ &= \mu(t_1) + \mu(t_2) + \rho(\lambda x.\, t_1) \\ &= \mu(t)\end{aligned}$$

2. $t \equiv \pi_1\langle t_1, t_2\rangle$, and $u \equiv t_1$.

$$\begin{aligned}\mu(u) &= \mu(t_1) \\ &\leq \max(\mu(t_1), \mu(t_2)) \\ &< \max(\mu(t_1), \mu(t_2)) + \rho(\langle t_1, t_2\rangle) \\ &= \mu(\pi_1\langle t_1, t_2\rangle) \\ &= \mu(t)\end{aligned}$$

3. $t \equiv \pi_2\langle t_1, t_2\rangle$, and $u \equiv t_2$. This case is similar to the previous one. □

Corollary 1. *Let t and u be two linear λ-terms such that $t \overset{n}{\triangleright} u$. Then $n \leq \mu(t) - \mu(u)$.*

Proof. By iterating Proposition 1. □

As we explained at the beginning of this section, we intend to establish that whenever $t \to u$, there exists a reduction strategy that is polynomially bounded by the size of both t and u. The idea is to use $\mu(t)$. This is not sufficient because it only works for external reduction. Now, a reduction step that takes place within one of the two components of a pair is useless if the residual of this component eventually disappears because of a subsequent projection. However, if there is no subsequent projection the residual of the pair will occur in u. These observations, which suggest that we must take into account the number of pair components that occur in u, motivate the next definition.

Definition 8. *The number of slices $\#(t)$ of a linear λ-term t is defined as follows:*

1. *if t is a purely applicative term then $\#(t) = 1$,*
2. *otherwise, the definition $\#(t)$ obeys the following equations:*
   (a) $\#(\lambda x.\, t_1) = \#(t_1)$
   (b) $\#(t_1\, t_2) = \begin{cases} \#(t_1), & \textit{if } t_2 \textit{ is a purely applicative term,} \\ \#(t_2), & \textit{if } t_1 \textit{ is a purely applicative term,} \\ \#(t_1) + \#(t_2), & \textit{otherwise.} \end{cases}$
   (c) $\#(\langle t_1, t_2\rangle) = \#(t_1) + \#(t_2)$

(d) $\#(\pi_1 t_1) = \#(t_1)$
(e) $\#(\pi_2 t_1) = \#(t_1)$

We now state and prove the main proposition of this section.

Proposition 2. *Let t and u be two linear λ-terms such that $t \xrightarrow{*} u$. Then, there exists $n \in \mathbb{N}$ such that $t \xrightarrow{n} u$ and $n \le \mu(t) \times \#(u)$*

Proof. The proof proceeds by induction on the subterm/reduction relation.

1. $t \equiv x$. We must have $u \equiv x$, and consequently $n = 0 = \mu(x)$.
2. $t \equiv \lambda x.\, t_1$. We must have $u \equiv \lambda x.\, u_1$, with $t_1 \xrightarrow{*} u_1$. Hence, the property holds by induction hypothesis.
3. $t \equiv t_1\, t_2$. If $u \equiv u_1\, u_2$, with $t_1 \xrightarrow{*} u_1$ and $t_2 \xrightarrow{*} u_2$, the induction is straightforward. Otherwise, there exist t'_{11} and t'_2 such that:

$$t_1\, t_2 \xrightarrow{*} (\lambda x.\, t'_{11})\, t'_2 \rightarrow t'_{11}[x{:=}t'_2] \xrightarrow{*} u,$$

where $t_1 \xrightarrow{*} \lambda x.\, t'_{11}$ and $t_2 \xrightarrow{*} t'_2$. Then, by Lemma 1, there exists t_{11} such that $t_1 \overset{*}{\triangleright} \lambda x.\, t_{11} \xrightarrow{*} \lambda x.\, t'_{11}$. Therefore there exists $n_1, n_2 \in \mathbb{N}$ such that $t_1\, t_2 \overset{n_1}{\triangleright} (\lambda x.\, t_{11})\, t_2 \triangleright t_{11}[x{:=}t_2] \xrightarrow{n_2} u$, because $t_{11}[x{:=}t_2] \xrightarrow{*} t'_{11}[x{:=}t'_2]$. Hence, by Corollary 1, we have:

$$n_1 + 1 \le \mu(t_1\, t_2) - \mu(t_{11}[x{:=}t_2]),$$

which implies $n_1 + 1 \le (\mu(t_1\, t_2) - \mu(t_{11}[x{:=}t_2])) \times \#(u)$, since $\#(u) > 0$. On the other hand, by induction hypothesis, we have $n_2 \le \mu(t_{11}[x{:=}t_2])) \times \#(u)$. Consequently, we have that $n_1 + n_2 + 1 \le \mu(t_1\, t_2) \times \#(u)$. Then, we take $n = n_1 + n_2 + 1$.
4. $t \equiv \langle t_1, t_2 \rangle$. We must have that $u \equiv \langle u_1, u_2 \rangle$, with $t_1 \xrightarrow{*} u_1$ and $t_2 \xrightarrow{*} u_2$. Hence, by induction hypothesis, there exists $n_1, n_2 \in \mathbb{N}$ such that: $t_1 \xrightarrow{n_1} u_1$ with $n_1 \le \mu(t_1) \times \#(u_1)$, and $t_2 \xrightarrow{n_2} u_1$ with $n_2 \le \mu(t_2) \times \#(u_2)$. Consequently, we have that $\langle t_1, t_2 \rangle \xrightarrow{n_1} \langle u_1, t_2 \rangle \xrightarrow{n_2} \langle u_1, u_2 \rangle$. Then, we may take $n = n_1 + n_2$ because the following inequalities hold:

$$\begin{aligned} n_1 + n_2 &\le \mu(t_1) \times \#(u_1) + \mu(t_2) \times \#(u_2) \\ &\le \max(\mu(t_1), \mu(t_2)) \times \#(u_1) + \max(\mu(t_1), \mu(t_2)) \times \#(u_2) \\ &= \max(\mu(t_1), \mu(t_2)) \times (\#(u_1) + \#(u_2)) \\ &= \mu(\langle t_1, t_2 \rangle) \times \#(\langle u_1, u_2 \rangle) \end{aligned}$$

5. $t \equiv \pi_1 t_1$. If $u \equiv \pi_1 u_1$, with $t_1 \xrightarrow{*} u_1$, the induction is straightforward. Otherwise, there exist t'_{11} and t'_{12} such that $\pi_1 t_1 \xrightarrow{*} \pi_1 \langle t'_{11}, t'_{12} \rangle \rightarrow t'_{11} \xrightarrow{*} u$, where $t_1 \xrightarrow{*} \langle t'_{11}, t'_{12} \rangle$. Then, by Lemma 2, there exist t_{11} and t_{12} such that $t_1 \overset{*}{\triangleright} \langle t_{11}, t_{12} \rangle \xrightarrow{*} \langle t'_{11}, t'_{12} \rangle$. Consequently, there exists $n_1, n_2 \in \mathbb{N}$ such that: $\pi_1 t_1 \overset{n_1}{\triangleright} \pi_1 \langle t_{11}, t_{12} \rangle \triangleright t_{11} \xrightarrow{n_2} u$, because $t_{11} \xrightarrow{*} t'_{11}$. Then, by Corollary 1, we have $n_1 + 1 \le \mu(\pi_1 t_1) - \mu(t_{11})$, which implies $n_1 + 1 \le (\mu(\pi_1 t_1) - \mu(t_{11})) \times \#(u)$. By induction hypothesis, we also have that $n_2 \le \mu(t_{11}) \times \#(u)$. Hence, we have that $n_1 + n_2 + 1 \le \mu(\pi_1 t_1) \times \#(u)$, and we take $n = n_1 + n_2 + 1$.
6. $t \equiv \pi_1 t_2$. This case is similar to the previous one. □

4 Decidability and NP-Completeness

In the presence of pairs, linear λ-terms may contain subterms which are bound to disappear during the reduction because of projections. Those subterms may be arbitrarily huge and contain many redices. The previous section showed how to cope with them in order not to reduce useless redices and to have a polynomial reduction. The purpose of this one is to prove that if $t \xrightarrow{n} u$ then there exists some t' obtained by deleting useless subterms from t and the size of which is polynomial with respect to n and the size of u. Together with the results of the previous section this property will help us to obtain decidability and complexity insights about the matching problem.

In order to model deletion in terms, we add a special constant ($\diamondsuit$) to the calculus. This constant may be used to replace any term of any type in the formation rules of Definition 3 (not taking into account the side condition on free variables in the case of the formation of a pair).

Within this new notion of term, we ditinguish those obtained by adding the following term formation rules to the formation rules of Definition 3:

$$\text{if } t \in \mathscr{T}_\alpha \text{ then } \langle t, \diamondsuit\rangle \in \mathscr{T}_{\alpha\&\beta} \text{ and } \langle \diamondsuit, t\rangle \in \mathscr{T}_{\beta\&\alpha}.$$

those terms are called *hollow terms.* A substitution is hollow if for all x (*resp.* $\mathbf{X}$) $\sigma(x)$ (*resp.* $\sigma(\mathbf{X})$) is hollow.

The fact that a certain term is obtained from another one by deleting one of its subterm induces a reflexive and transitive relation ($\sqsubseteq$) on terms defined by the following formal system:

$$\diamondsuit \sqsubseteq t \qquad \frac{t \sqsubseteq u}{\lambda x.\, t \sqsubseteq \lambda x.\, u} \qquad \frac{t_1 \sqsubseteq u_1 \quad t_2 \sqsubseteq u_2}{t_1\, t_2 \sqsubseteq u_1\, u_2}$$

$$\frac{t_1 \sqsubseteq u_1 \quad t_2 \sqsubseteq u_2}{\langle t_1, t_2\rangle \sqsubseteq \langle u_1, u_2\rangle} \qquad \frac{t \sqsubseteq u}{\pi_1 t \sqsubseteq \pi_1 u} \qquad \frac{t \sqsubseteq u}{\pi_2 t \sqsubseteq \pi_2 u}$$

This relation is naturally extended to the substitutions:

Definition 9. *Given two substitutions σ_1 and σ_2, we write $\sigma_1 \sqsubseteq \sigma_2$ if for all x (resp. for all $\mathbf{X}$) $\sigma_1(x) \sqsubseteq \sigma_2(x)$ (resp. $\sigma_1(\mathbf{X}) \sqsubseteq \sigma_2(\mathbf{X})$)*

Definition 10. *We define the length of term $|t|$ as follows:*

1. $|\diamondsuit| = 1$
2. $|h| = 1$ *if h is an atomic term*
3. $|\lambda x.t| = |t| + 1$
4. $|t_1 t_2| = |t_1| + |t_2|$
5. $|\langle t_1, t_2\rangle| = |t_1| + |t_2|$
6. $|\pi_1(t)| = |t| + 1$ *and* $|\pi_2(t)| = |t| + 1$

Then we define the length of a substitution to be $|\sigma| = \sum_{x\in\mathrm{dom}(\sigma)} |\sigma(x)| + \sum_{\mathbf{X}\in\mathrm{dom}(\sigma)} |\sigma(\mathbf{X})|$

Lemma 4. *If t is a hollow term then there is a linear λ-term u such that $FV(u) = FV(t)$, $t \sqsubseteq u$, $|u| \leq |t|^2$.*

Proof. The proof is an induction on the structure of t. We only present the case where $t \equiv \langle \diamondsuit, t' \rangle$, the other ones being either similar to this one or straightforward.

If $t \equiv \langle \diamondsuit, t' \rangle$ the by induction hypothesis we have the existence of a linear term u' such that $FV(u') = FV(t')$, $t' \sqsubseteq u'$ and $|u'| \leq |t'|^2$. Let $FV(u') = \{x_1; \ldots; x_n\}$ and $\mathbf{X}$ be an unknown with a type so that $u \equiv \langle \mathbf{X} x_1 \ldots x_n, u' \rangle$ is a linear λ-term with the same type as t. Obviously we have $FV(u) = FV(t)$ and $t \sqsubseteq u$, it remains to show that $|u| \leq |t|$. As $FV(t') = \{x_1; \ldots; x_n\}$, $n \leq |t'|$ and :

$$|u| = |u'| + n + 1 \leq |t'|^2 + n + 1 \leq |t'|^2 + |t'| + 1 \leq (|t'| + 1)^2 \leq |t|^2$$

□

Lemma 5. *If $u_1 \sqsubseteq u_2$ and $\sigma_1 \sqsubseteq \sigma_2$ then $u_1.\sigma_1 \sqsubseteq u_2.\sigma_2$.*

Proof. By induction on the structure of u_1 :

1. If $u_1 \equiv \diamondsuit$ then $u_1.\sigma_1 \equiv \diamondsuit$ and obviously $u_1.\sigma_1 \sqsubseteq u_2.\sigma_2$.
2. If $u_1 \equiv x$ then $u_2 \equiv x$ and $u_i.\sigma_i \equiv \sigma_i(x)$. As $\sigma_1 \sqsubseteq \sigma_2$, we have $\sigma_1(x) \sqsubseteq \sigma_2(x)$ and as a consequence we have $u_1.\sigma_1 \sqsubseteq u_2.\sigma_2$.
3. If $u_1 \equiv h$ where h is a constant, then $u_2 \equiv h$ and $u_i.\sigma_i \equiv h$. So $u_1.\sigma_1 \sqsubseteq u_2.\sigma_2$.
4. The other cases are direct consequences of the induction hypothesis. □

With a specific strategy, the relation $\sqsubseteq$ can be preserved through reduction.

Lemma 6. *If $v_1 \sqsubseteq v_2$ and $v_1 \rightarrow w_1$ then there exists w_2 such that $v_2 \rightarrow w_2$ and $w_1 \sqsubseteq w_2$.*

Proof. By induction on the structure of v_1:

1. If $v_1 \equiv (\lambda x.t_1)t_2$ and $w_1 \equiv t_1[x := t_2]$ then $v_2 \equiv (\lambda x.t'_1)t'_2$ where $t_i \sqsubseteq t'_i$. Then from lemma 5 if we let $w_2 \equiv t'_1[x := t'_2]$ then $w_1 \sqsubseteq w_2$.
2. The other cases are straightforward. □

Lemma 7. *If $v_1 \sqsubseteq v_2$ and $v_2 \rightarrow w_2$ then there exists w_1 such that $v_1 \stackrel{=}{\rightarrow} w_1$ and $w_1 \sqsubseteq w_2$.*

Proof. This lemma can be proved by induction on the structure of v_2 in a way similar to the previous one. The only difference appears in the case where $v_1 \equiv \diamondsuit$. In that case $w_1 \equiv \diamondsuit$ and obviously $w_1 \sqsubseteq w_2$. □

As a consequence, under certain conditions, the relation $\sqsubseteq$ preserves equality between terms.

Lemma 8. *If v is the normal form of v_1, v does not contain any occurence of $\diamondsuit$ and $v_1 \sqsubseteq v_2$ then $v_1 = v_2$.*

Proof. The lemma can be proved by iterating Lemma 6 and remarking that as $\diamondsuit$ has no occurence in v if $v \sqsubseteq w$ then $v \equiv w$. □

Lemma 9. *If* $v_1 = v_3$ *and* $v_1 \sqsubseteq v_2 \sqsubseteq v_3$ *then* $v_1 = v_2$.

Proof. If v is the common normal form of v_1 and v_3, then there exists n such that $v_1 \xrightarrow{n} v$. In order to prove the lemma we use an induction on n.

In case $n = 0$, then v_1 is in normal form. But v_3 is not necessarily in normal form. We are going to prove by induction on v_1 that the normal form of v_2 is v_1.

1. $v_1 \equiv \diamondsuit$: we then have to prove that if $v_3 = \diamondsuit$, the fact that $v_2 \sqsubseteq v_3$ implies that $v_2 = \diamondsuit$.
   We proceed by induction on the length of the reduction $v_3 \xrightarrow{p} \diamondsuit$. In case $p = 0$, $v_3 \equiv \diamondsuit$ and then $v_2 \sqsubseteq \diamondsuit$ so $v_2 \equiv \diamondsuit$. If $p > 0$ then $v_3 \to v_3' \xrightarrow{p-1} \diamondsuit$. By Lemma 7 there is v_2' such that $v_2 \xrightarrow{=} v_2'$ and $v_2' \sqsubseteq v_3'$. Then, by induction hypothesis, $v_2' = \diamondsuit$ and $v_2 = \diamondsuit$.
2. The other cases are simple consequences of induction.

Now if $n > 0$ then $v_1 \to v_1' \xrightarrow{n-1} v$. By Lemma 6 there exists v_2' such that $v_2 \to v_2'$ and $v_1' \sqsubseteq v_2'$. Still from Lemma 6 there exists v_3' such that $v_3 \to v_3'$ and $v_2' \sqsubseteq v_3'$. Thus we have $v_1' \sqsubseteq v_2' \sqsubseteq v_3'$ and $v_1' \xrightarrow{n-1} v$, the induction hypothesis gives that $v_2' = v_1'$ which allows us to conclude that $v_1 = v_2$. □

When two terms are dominated (with respect to $\sqsubseteq$) by another one, they share a common syntactic structure but each of them can have specific subterms. The following lemma proves the existence of a term which possesses both their common and specific features.

Lemma 10. *If there are three hollow terms* v_1, v_2 *and* v *such that* $v_1 \sqsubseteq v$ *and* $v_2 \sqsubseteq v$ *then there is* v_3 *such that :*

1. v_3 *is a hollow term*
2. $v_3 \sqsubseteq v$, $v_1 \sqsubseteq v_3$ *and* $v_2 \sqsubseteq v_3$
3. $|v_3| \leq |v_1| + |v_2| - 1$

Proof. We proceed by induction on the structure of v. The only interesting case consists in having $v \equiv \langle u_1, u_2 \rangle$, $v_1 \equiv \langle w_1, \diamondsuit \rangle$ and $v_2 \equiv \langle \diamondsuit, w_2 \rangle$ the other cases are straightforward. In that case, it suffices to take $v_3 \equiv \langle w_1, w_2 \rangle$ to respect the conditions of the lemma. □

The next lemma is the generalisation of the previous one to the substitutions.

Lemma 11. *If* σ , σ_1 *and* σ_2 *are hollow substitutions,* $\mathrm{dom}(\sigma) \neq \emptyset$, $\sigma_1 \sqsubseteq \sigma$ *and* $\sigma_2 \sqsubseteq \sigma$ *then there is* σ_3 *such that:*

1. $\sigma_3 \sqsubseteq \sigma$, $\sigma_1 \sqsubseteq \sigma_3$ *and* $\sigma_2 \sqsubseteq \sigma_3$.
2. $|\sigma_3| \leq |\sigma_1| + |\sigma_2| - 1$.

Proof. The proof of this lemma uses an induction on the size of $\mathrm{dom}(\sigma)$, the initial case is simply proved using the previous lemma and if $x \notin \mathrm{dom}(\sigma)$ we set $\sigma_3(x)$ to be equal to x. □

If a term t is obtained from a term v by deleting some of its subterms (*i.e.* $t \sqsubseteq v$) then t and v still share a main global syntactic structure. In particular one can expect that if v is the result of a substitution σ applied to a term u then t is also *somehow* the result of applying a substitution to a certain term. The next lemma explicits precisely this fact.

Lemma 12. *If t and u are hollow terms, σ is a hollow substitution and $t \sqsubseteq u.\sigma$ then there exist u' and σ' such that:*

1. *u' and σ' are hollow*
2. *$u' \sqsubseteq u$ and $\sigma' \sqsubseteq \sigma$*
3. *$t \sqsubseteq u'.\sigma' \sqsubseteq u.\sigma$*
4. *$|u'| + |\sigma'| \leq |t| + 1$*

Proof. If $\mathrm{dom}(\sigma) = \emptyset$ then $\sigma = Id$ and we just have to take $u' \equiv t$ and $\sigma' = Id$ to get all that is needed. The rest of the proof won't take this trivial case into account, and the condition $\mathrm{dom}(\sigma) \neq \emptyset$ which will allow us to apply the Lemma 11 will be implicitly verified.

We prove this lemma using an induction on the structure of u:

1. In case $u \equiv x$ and $x \in \mathrm{dom}(\sigma)$ we take $u' \equiv x$ and $\sigma' \equiv [x := t]$. Such u' and σ' verify the requiered properties.
2. In case $u \equiv \mathbf{X}$ and $\mathbf{X} \in \mathrm{dom}(\sigma)$ then we also take $u' \equiv \mathbf{X}$ and $\sigma' = [\mathbf{X} := t]$.
3. In case $u \equiv h$ where h is an atomic term which is not in $\mathrm{dom}(\sigma)$, we let $u' \equiv t$ and $\sigma' = Id$.
4. In case $u \equiv \langle u_1, u_2 \rangle$ then $t \equiv \langle t_1, t_2 \rangle$ so that $t_i \sqsubseteq u_i.\sigma$. The induction hypothesis implies the existence of two pairs u'_1, σ_1 and u'_2, σ_2 such that $t_i \sqsubseteq u'_i.\sigma_i \sqsubseteq u_i.\sigma$, $u'_i \sqsubseteq u_i$, $\sigma_i \sqsubseteq \sigma$ and $|u'_i| + |\sigma_i| \leq |t_i| + 1$. As $\sigma_1 \sqsubseteq \sigma$ and $\sigma_2 \sqsubseteq \sigma$, from Lemma 11, there exists σ' such that $\sigma' \sqsubseteq \sigma$, $\sigma_i \sqsubseteq \sigma'$ and $|\sigma'| \leq |\sigma_1| + |\sigma_2| - 1$. By Lemma 5, as $u'_i \sqsubseteq u_i$ and $\sigma_i \sqsubseteq \sigma' \sqsubseteq \sigma$ it comes that $t_i \sqsubseteq u'_i.\sigma_i \sqsubseteq u'_i.\sigma' \sqsubseteq u_i.\sigma$. We let $u' \equiv \langle u'_1, u'_2 \rangle$ and verify that $t \sqsubseteq u'.\sigma' \sqsubseteq u.\sigma$ and :

$$\begin{aligned} |u'| + |\sigma'| &\leq |u'_1| + |u'_2| + |\sigma_1| + |\sigma_2| - 1 \\ &\leq |t_1| + 1 + |t_2| + 1 - 1 \\ &\leq |t| + 1 \end{aligned}$$

5. the case where $u \equiv u_1 u_2$ can be solved in the same way as the previous one.
6. the other cases are straightforward. □

Corollary 2. *If t, t_1 and t_2 are hollow terms, $t = t_1[x := t_2]$ and $t \sqsubseteq t_1[x := t_2]$ then there exists t'_1 and t'_2 such that:*

1. *t'_1 and t'_2 are hollow terms*
2. *$t'_1 \sqsubseteq t_1$ and $t'_2 \sqsubseteq t_2$*
3. *$t = t'_1[x := t'_2]$*
4. *$|t'_1| + |t'_2| \leq |t| + 1$*

Proof. From the previous lemma we know that there exists t'_1 and t'_2 such that $t'_1 \sqsubseteq t_1$, $t'_2 \sqsubseteq t_2$, $t \sqsubseteq t'_1[x := t'_2] \sqsubseteq t_1[x := t_2]$ and $|t'_1| + |t'_2| \leq |t| + 1$. As $t = t_1[x := t_2]$ lemma 9 implies $t = t'_1[x := t'_2]$. □

We now establish the key lemma of this section. We get a bound on the size of the term t' obtained by deleting useless subterms of a term t.

Lemma 13. *If t, u and u' are hollow term, $t \to u$, $u' \sqsubseteq u$ and $u' = u$ then there is some t' such that:*

1. *t' is a hollow term*
2. *$t' \sqsubseteq t$*
3. *$t' = t$*
4. *$|t'| \leq |u'| + 2$*

Proof. We proceed by induction on the structure of t. We just present the cases where t is a redex and u is the result of the contraction of that redex, the other ones are direct consequences of the induction hypothesis:

1. If $t \equiv (\lambda x.t_1)t_2$ and $u \equiv t_1[x := t_2]$, from Corollary 2 there are two hollow terms t'_1 and t'_2 such that $t'_i \sqsubseteq t_i$, $t'_1[x := t'_2] = u' = u = t$ and $|t'_1| + |t'_2| \leq |u'| + 1$. Thus $(\lambda x.t'_1)t'_2$ is a hollow term $(\lambda x.t'_1)t'_2 \sqsubseteq (\lambda x.t_1)t_2$, $(\lambda x.t'_1)t'_2 = t$ and $|(\lambda x.t'_1)t'_2| = |t'_1| + |t'_2| + 1 \leq |u'| + 2$.
2. If $t \equiv \pi_1(\langle t_1, t_2\rangle)$ (*resp.* $t \equiv \pi_2(\langle t_1, t_2\rangle)$) and $u \equiv t_1$ (*resp.* $u \equiv t_2$) then we let $t' \equiv \pi_1(\langle u', \diamondsuit\rangle)$ (*resp.* $t' \equiv \pi_2(\langle \diamondsuit, u'\rangle)$) and we verify that t' fullfills the conditions of the lemma. □

Lemma 14. *If t and u are hollow terms and $t \xrightarrow{n} u$ then there is some t' such that :*

1. *t' is a hollow term*
2. *$t' \sqsubseteq t$*
3. *$t' = t$*
4. *$|t'| \leq |u| + 2n$*

Proof. This result is obtained by iterating the previous lemma. The iteration can be initiated because $u \sqsubseteq u$ and $u = u$. □

Proposition 3. *If t and u are hollow terms, σ is a hollow subsitution and $t.\sigma \xrightarrow{n} u$ then there exists σ' such that:*

1. *σ' is a hollow substitution*
2. *$\sigma' \sqsubseteq \sigma$*
3. *$t.\sigma' = u$*
4. *$|\sigma'| \leq |u| + 2n$*

Proof. From the previous lemma, we know that if $t.\sigma \xrightarrow{n} u$ then there exists a hollow term t' such that $t' \sqsubseteq t.\sigma$, $t' = u$ and $|t'| \leq |u| + 2n$. Lemma 12 leads to the existence of a hollow term t'' and a hollow subsitution σ' such that $t'' \sqsubseteq t$, $\sigma' \sqsubseteq \sigma$, $t' \sqsubseteq t''.\sigma' \sqsubseteq t.\sigma$ and $|t''|+|\sigma'| \leq |t'|+1$. As a consequence $|\sigma'| \leq |u|+2n$.

We now have to verify that $t.\sigma' = u$. Lemma 9 gives $t''.\sigma' = u$ because $t' = u$ and $t.\sigma = u$. As $t'' \sqsubseteq t$ and $\sigma' \sqsubseteq \sigma$ Lemma 5 gives $t''.\sigma' \sqsubseteq t.\sigma' \sqsubseteq t.\sigma$. But $t''.\sigma' = u$ and $t.\sigma = u$, then Lemma 9 leads to what we expected. □

Theorem 1. *Given a matching problem in the linear λ-calculus with pairing (t, u), it is decidable whether it has a solution or not. Furthermore, if u is in normal form, this problem is NP-complete.*

Proof. Let v be the normal form of u. If the matching equation (t, u) admits a solution σ then, from Proposition 2, there exists n such that $t.\sigma \xrightarrow{n} v$ and $n \leq \mu(t.\sigma) \times \#(v)$. If we consider that the terms substituted to unknowns by σ are in normal form then the redices contained in $t.\sigma$ are those contained in t and those created by the substitution. Thus, if $\{\mathbf{X}_1, \ldots, \mathbf{X}_n\}$ is the multiset of unknowns that occure in t, we have:

$$\mu(t.\sigma) \leq \mu(t) + \sum_{i=1}^{n} \rho(\mathbf{X}_i)$$

From proposition 3 we know that there exists σ' such that $\sigma' \sqsubseteq \sigma$, $t.\sigma' = v$ and $|\sigma'| \leq |v|+2n \leq |v|+2\#(v)\mu(t)+\sum_{i=1}^{n} 2\#(v)\rho(\mathbf{X}_i)$. But σ' may substitute to some unknowns terms which contain some $\diamondsuit$. Lemma 4 gives us the existence of a substitution σ'' with the same domain as σ', which substitutes a linear λ-term to each unknown of its domain and such that $\sigma' \sqsubseteq \sigma''$ and:

$$|\sigma''| \leq |\sigma'|^2 \leq (|v| + 2\#(v)\mu(t) + \sum_{i=1}^{n} 2\#(v)\rho(\mathbf{X}_i))^2$$

As $\sigma' \sqsubseteq \sigma''$, Lemma 5 proves that $t.\sigma' \sqsubseteq t.\sigma''$. Finally, since it is a linear λ-term, v does not contain any $\diamondsuit$ and we have (Lemma 8) $t.\sigma'' = t.\sigma' = v$. Hence, if there is a solution to the equation then there is also a solution which is bounded. The problem is then decidable.

Furthermore, if u is in normal form then $|v| = |u|$ and the existence of a solution implies the existence of a polynomially bounded one. And since Proposition 2 entails, in that case, that verifying whether a substitution is a solution or not is polynomial, the problem is in NP if u is in normal form. And as it is an extension of linear λ-calculus which is NP-hard [9], matching in the linear λ-calculus with pairing is NP-complete when u is in normal form. □

We have not found yet the precise complexity of matching in the linear λ-calculus with pairing in the case where the right part of the equation is not in normal form. We managed to prove that this problem was PSPACE-hard, but we did not find a PSPACE-algorithm which solves it. At worst, we still have the EXP-time algorithm which consists in normalizing the right part of the equation and then solving it.

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