$\mathbf{Ty_2}$ revisited (abstract)

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Gallin, in [6], provides a faithfull translation of Montague intensional logic [8], **IL**, into $\mathbf{Ty_2}$ (a two-sorted simple theory types) :

$$
\begin{aligned}
(x^{\alpha})^* &= x^{\alpha} & \qquad (\lambda x^{\alpha}.\, t^{\beta})^* &= \lambda x^{\alpha}.\, (t^{\beta})^* \\
(c^{\alpha})^* &= c^{\mathbf{s}\to\alpha}\, \mathbf{x}^{\mathbf{s}} & \qquad ({}^{\wedge}t^{\alpha})^* &= \lambda \mathbf{x}^{\mathbf{s}}.\, (t^{\alpha})^* \\
(t^{\alpha\to\beta}\, u^{\alpha})^* &= (t^{\alpha\to\beta})^*\, (u^{\alpha})^* & \qquad ({}^{\vee}t^{\mathbf{s}\to\alpha})^* &= (t^{\mathbf{s}\to\alpha})^*\, \mathbf{x}^{\mathbf{s}}
\end{aligned}
$$

In [5], another translation of **IL** into $\mathbf{Ty_2}$is proposed. This translation, which has the property of translating a closed term by a closed term, interprets extensions as terms of type $\mathbf{s} \to \alpha$, and intensions as terms of type $\mathbf{s} \to \mathbf{s} \to \alpha$. Montague's intension and extension operators correspond then to Curry's elementary cancelator (K) and duplicator (W), respectively [4]:

$$
\begin{aligned}
(x^{\alpha})^{\dagger} &= \lambda i^{\mathbf{s}}.\, x^{\alpha} & \qquad (\lambda x^{\alpha}.\, t^{\beta})^{\dagger} &= \lambda i^{\mathbf{s}} x^{\alpha}.\, (t^{\beta})^{\dagger}\, i^{\mathbf{s}} \\
(c^{\alpha})^{\dagger} &= \lambda i^{\mathbf{s}}.\, c^{\mathbf{s}\to\alpha}\, i^{\mathbf{s}} & \qquad ({}^{\wedge}t^{\alpha})^{\dagger} &= \lambda i^{\mathbf{s}}.\, (t^{\alpha})^{\dagger} \\
(t^{\alpha\to\beta}\, u^{\alpha})^{\dagger} &= \lambda i^{\mathbf{s}}.\, (t^{\alpha\to\beta})^{\dagger}\, i^{\mathbf{s}}\, ((u^{\alpha})^{\dagger}\, i^{\mathbf{s}}) & \qquad ({}^{\vee}t^{\mathbf{s}\to\alpha})^{\dagger} &= \lambda i^{\mathbf{s}}.\, (t^{\mathbf{s}\to\alpha})^{\dagger}\, i^{\mathbf{s}}\, i^{\mathbf{s}}
\end{aligned}
$$

In this paper, we use the same kind of interpretation in order to translate hybrid logic [2] into $\mathbf{Ty_2}$:

$$
\begin{aligned}
(p)^{\ddagger} &= \lambda i^{\mathbf{s}}.\, p^{\mathbf{s}\to\mathbf{t}}\, i^{\mathbf{s}} & \qquad (\varphi \wedge \psi)^{\ddagger} &= \lambda i^{\mathbf{s}}.\, ((\varphi)^{\ddagger}\, i^{\mathbf{s}}) \wedge ((\psi)^{\ddagger}\, i^{\mathbf{s}}) \\
(j)^{\ddagger} &= \lambda i^{\mathbf{s}}.\, j^{\mathbf{s}} = i^{\mathbf{s}} & \qquad (@_j\, \varphi)^{\ddagger} &= \lambda i^{\mathbf{s}}.\, (\varphi)^{\ddagger}\, j^{\mathbf{s}} \\
(\neg\varphi)^{\ddagger} &= \lambda i^{\mathbf{s}}.\, \neg((\varphi)^{\ddagger}\, i^{\mathbf{s}}) & \qquad (\downarrow_j.\, \varphi)^{\ddagger} &= \lambda j^{\mathbf{s}}.\, (\varphi)^{\ddagger}\, j^{\mathbf{s}}
\end{aligned}
$$

where $i^{\mathbf{s}}$ is a fresh variable.

Translation $(\cdot)^{\ddagger}$, which is such that $\mathfrak{M}, g, w \vDash \varphi$ if and only if $[\![(\varphi)^{\ddagger}\, i]\!]^{\mathfrak{M}}_{g[i:=w]} = 1$, offers several advantages:

1. It allows to establish results on hybrid formulas simply by using β-reduction. For instance, among the axioms given in [2]: Axioms *Q1*, $K_@$, and *Scope* appear to be mere instances of the propositional tautology $A \to A$; Axioms *Q2*, *Q3*, *Introduction*, *Label*, *Nom*, and *Swap* derive from elementary equality rules; Axioms *Self Dual*$_\downarrow$, and *Self Dual*$_@$ are instances of the double negation rule.
2. It may be easily extended to cover the higher-order case, resulting in a rebuilding of the Hybrid Type Theory of Areces et al. [1]. A completeness theorem may then be derived as a corollary of Henkin's classical result [7].
3. It may be mixed with translation $(\cdot)^{\dagger}$, resulting in translations of (higher-order) Intensional Hybrid Logics [3, CHAP. 7].
4. From a more practical point of view, it provides a way of easily incorporating hybrid logic constructs in a Montague grammar.

References

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