Partially commutative linear logic: sequent calculus and phase semantics

Philippe de Groote
Projet Calligramme
INRIA-Lorraine & CRIN – CNRS
615 rue du Jardin Botanique - B.P. 101
F 54602 Villers-lès-Nancy Cedex – FRANCE
e-mail: Philippe.de.Groote@loria.fr

We define a variant of intuitionistic multiplicative linear logic that features the non-commutative connectives of the Lambek calculus together with the usual commutative connectives of linear logic. We give a sequent calculus, define a notion of phase space, and prove the completeness of the sequent calculus with respect to the phase semantics.

1 Introduction

In this short paper we introduce a variant of intuitionistic multiplicative linear logic that features both commutative and non-commutative connectives. This new logic extends conservatively both Girard's multiplicative linear logic [2] and Lambek's syntactic calculus [3]. Ultimately, our goal will be to fulfil the programme achieved by Girard in [2], i.e., to provide our system with: (1) a sequent calculus, (2) a proof-net syntax together with a correctness criterion and a sequentialisation theorem, (3) semantics of provability together with a completeness theorem, (4) denotational semantics, i.e., semantics of proofs. Nevertheless, in this first report, we only concentrate on (1) and (3).

Related works on mixing commutativity and non-commutativity in linear logic include [1] and [5].

2 Formal system

2.1 Formulas and sequents

The formulas of our system are built from a set $\mathcal{V}$ of propositional variables (or atomic formulas), according to the definition that follows.

Definition 1 (Fromula) *The set $\mathcal{F}$ of formulas obeys the following grammar:*

$$\mathcal{F} \ ::= \ \mathcal{V} \mid (\mathcal{F} \otimes \mathcal{F}) \mid (\mathcal{F} \multimap \mathcal{F}) \mid (\mathcal{F} \bullet \mathcal{F}) \mid (\mathcal{F} \mathbin{-\!\bullet} \mathcal{F}) \mid (\mathcal{F} \mathbin{\bullet\!-} \mathcal{F}) \qquad \blacksquare$$

As it will be clear from the sequent calculus, the connectives "$\otimes$" (*tensor*) and "$\multimap$" (*implication*) are commutative. They correspond exactly to Girard's multiplicative conjunction and implication [2], which justifies the notation. The other connectives, namely, "$\bullet$" (*product*), "$-\!\!\bullet$" (*direct implication*), and "$\bullet\!\!-$" (*retro implication*) are non-commutative. They correspond to Lambek's conjunction and implications [3]. More precisely, formulas of the form $(A \bullet B)$, $(A -\!\!\bullet B)$, and $(A \bullet\!\!- B)$ correspond respectively to $(A \bullet B)$, $(A\backslash B)$, and (A/B) in Lambek's notation.

The fact that we are dealing with two sorts of conjunctions (a commutative one and a non-commutative one) is reflected at the level of the sequents that are built by means of two meta-connectives: "," (*parallel composition*) and ";" (*serial composition*). The precise definition of sequent is as follows.

Definition 2 (Sequent) *The set $\mathcal{S}$ of sequents is inductively defined as follows:*

$$\begin{array}{rcl} \mathcal{S} & ::= & \mathcal{H} \vdash \mathcal{F} \\ \mathcal{H} & ::= & () \mid \mathcal{H}_0 \\ \mathcal{H}_0 & ::= & \mathcal{F} \mid (\mathcal{H}_0, \mathcal{H}_0) \mid (\mathcal{H}_0; \mathcal{H}_0) \end{array}$$

$\mathcal{H}$ and $\mathcal{H}_0$ are respectively the set of antecedents (or sequences of hypotheses) and the set of non-empty antecedents (or non-empty sequences of hypotheses). "$()$" is the empty antecedent. ■

We use the following notational conventions. Roman uppercases ($A, B, \ldots$) denote formulas while Greek uppercases ($\Gamma, \Delta, \ldots$), possibly with subscripts, denote (possibly empty) antecedents. If "Γ" denotes the empty antecedent then "(Γ, Δ)", "(Δ, Γ)", "$(\Gamma; \Delta)$", and "$(\Delta; \Gamma)$" denote the the antecedent denoted by "Δ". In other words, the empty antecedent acts as the identity element of the parallel and serial compositions. Usually, we do not write the empty antecedent: we write "$\vdash A$" instead of "$() \vdash A$". We also use a square bracket notation to denote *contexts*, i.e., antecedents with one "hole". Then, if "$\Gamma[\,]$" denotes such a context and "Δ" denotes a non-empty antecedent, "$\Gamma[\Delta]$" denotes the antecedent obtained by "filling the hole" with "Δ". We omit the formal definition that would rely on the notion of binary tree.

2.2 Sequent calculus

Following Girard, we organize our sequent calculus into three groups of rules: the structural rules, the identity rules, and the logical rules.

The first four structural rules (pa1, pa2, sa1, sa2) express that both the parallel and serial compositions are associative. The next one (pc) says that the parallel composition is commutative. Consequently, the *tensor* ($\otimes$) is associative and commutative while the product ($\bullet$) is only associative. Finally, there is an *entropy* rule (ent) that specifies that the parallel composition is weaker than the serial composition. As a consequence, the formula

$$(A \otimes B) \multimap (A \bullet B)$$

is provable.

The logical and identity rules are as usual.

Structural rules

$$\frac{\Gamma[((\Delta_1, \Delta_2), \Delta_3)] \vdash A}{\Gamma[(\Delta_1, (\Delta_2, \Delta_3))] \vdash A} \ (\mathrm{pa1}) \qquad \frac{\Gamma[(\Delta_1, (\Delta_2, \Delta_3))] \vdash A}{\Gamma[((\Delta_1, \Delta_2), \Delta_3)] \vdash A} \ (\mathrm{pa2})$$

$$\frac{\Gamma[((\Delta_1;\Delta_2);\Delta_3)] \vdash A}{\Gamma[(\Delta_1;(\Delta_2;\Delta_3))] \vdash A}\ (\mathrm{sa1}) \qquad \frac{\Gamma[(\Delta_1;(\Delta_2;\Delta_3))] \vdash A}{\Gamma[((\Delta_1;\Delta_2);\Delta_3)] \vdash A}\ (\mathrm{sa2})$$

$$\frac{\Gamma[(\Delta_1,\Delta_2)] \vdash A}{\Gamma[(\Delta_2,\Delta_1)] \vdash A}\ (\mathrm{pc}) \qquad \frac{\Gamma[(\Delta_1;\Delta_2)] \vdash A}{\Gamma[(\Delta_1,\Delta_2)] \vdash A}\ (\mathrm{ent})$$

Identity rules

$$A \vdash A\ \ (\mathrm{id}) \qquad \frac{\Gamma \vdash A \quad \Delta[A] \vdash B}{\Delta[\Gamma] \vdash B}\ (\mathrm{cut})$$

Logical rules

$$\frac{\Gamma[(A,B)] \vdash C}{\Gamma[(A \otimes B)] \vdash C}\ (\mathrm{tl}) \qquad \frac{\Gamma \vdash A \quad \Delta \vdash B}{(\Gamma,\Delta) \vdash A \otimes B}\ (\mathrm{tr})$$

$$\frac{\Gamma \vdash A \quad \Delta[B] \vdash C}{\Delta[(\Gamma, A \multimap B)] \vdash C}\ (\mathrm{il}) \qquad \frac{(A,\Gamma) \vdash B}{\Gamma \vdash A \multimap B}\ (\mathrm{ir})$$

$$\frac{\Gamma[(A;B)] \vdash C}{\Gamma[(A \bullet B)] \vdash C}\ (\mathrm{pl}) \qquad \frac{\Gamma \vdash A \quad \Delta \vdash B}{(\Gamma;\Delta) \vdash A \bullet B}\ (\mathrm{pr})$$

$$\frac{\Gamma \vdash A \quad \Delta[B] \vdash C}{\Delta[(\Gamma; A \mathbin{-\!\bullet} B)] \vdash C}\ (\mathrm{dl}) \qquad \frac{(A;\Gamma) \vdash B}{\Gamma \vdash A \mathbin{-\!\bullet} B}\ (\mathrm{dr})$$

$$\frac{\Gamma \vdash A \quad \Delta[B] \vdash C}{\Delta[(B \mathbin{\bullet\!-} A;\Gamma)] \vdash C}\ (\mathrm{rl}) \qquad \frac{(\Gamma;A) \vdash B}{\Gamma \vdash B \mathbin{\bullet\!-} A}\ (\mathrm{rr})$$

We write "$\vdash (\Gamma \vdash A)$" when the sequent "$\Gamma \vdash A$" is provable with respect to the above rules. When it is provable without using Rule (cut), i.e., provable in the cut-free calculus, we write "$\vdash_{\mathrm{cf}} (\Gamma \vdash A)$".

It is almost immediate to establish that the above calculus satisfies the cut elimination property.[1] Then, one may easily establish, as a corollary of the subformula property, that our calculus is a conservative extension of both Lambek's and Girard's intuitionistic multiplicative calculi.

We end this section by establishing two technical lemmas that will be needed in the sequel.

Lemma 3 *The following properties hold:*

1. $\vdash_{\mathrm{cf}} (\Gamma[(A,B)] \vdash C)$ *if and only if* $\vdash_{\mathrm{cf}} (\Gamma[(A \otimes B)] \vdash C)$.
2. $\vdash_{\mathrm{cf}} (\Gamma[(A;B)] \vdash C)$ *if and only if* $\vdash_{\mathrm{cf}} (\Gamma[(A \bullet B)] \vdash C)$.
3. $\vdash_{\mathrm{cf}} ((A,\Gamma) \vdash B)$ *if and only if* $\vdash_{\mathrm{cf}} (\Gamma \vdash (A \multimap B))$.
4. $\vdash_{\mathrm{cf}} ((A;\Gamma) \vdash B)$ *if and only if* $\vdash_{\mathrm{cf}} (\Gamma \vdash (A \mathbin{-\!\bullet} B))$.

[1] In fact, cut elimination is a consequence of the completeness theorem of Section 3.3 because we establish the soudness of the cut rule, on the one hand, and the completeness of the cut-free calculus, on the other hand. But this is quite a *détour* with respect to the direct syntactic proof.

5. $\vdash_{\mathrm{cf}} ((\Gamma; A) \vdash B)$ *if and only if* $\vdash_{\mathrm{cf}} (\Gamma \vdash (B \bullet\!- A))$.

Proof. The *only if* parts of these five equivalences are obvious since they correspond to Rules (tl), (pl), (ir), (dr) and (rr). The *if* parts are established by routine inductions. □

Lemma 4 *Let Γ be an antecedent and γ be the formula obtained by replacing in Γ each occurrence of ";" by "$\bullet$" and each occurrence of "," by "$\otimes$".*

1. $\vdash_{\mathrm{cf}} ((\Gamma, \Delta) \vdash A)$ *if and only if* $\vdash_{\mathrm{cf}} (\Delta \vdash (\gamma \multimap A))$.
2. $\vdash_{\mathrm{cf}} ((\Gamma; \Delta) \vdash A)$ *if and only if* $\vdash_{\mathrm{cf}} (\Delta \vdash (\gamma -\!\bullet A))$.
3. $\vdash_{\mathrm{cf}} ((\Delta; \Gamma) \vdash A)$ *if and only if* $\vdash_{\mathrm{cf}} (\Delta \vdash (A \bullet\!- \gamma))$.

Proof. By iterating Lemma 3. □

3 Phase semantics

3.1 Definition

Phase spaces for the multiplicative intutionistic fragment of Girard's linear logic are made of commutative monoids while phase spaces for Lambek's syntactic calculus are made of non-commutative ones. Here, we consider sets with two monoidal products: a commutative one, and a non-commutative one. These sets, in addition, are partially ordered. This allows the consequence relation existing between the commutative and the non-commutative connectives to be taken into account. The exact definition of our notion of phase space is the following.

Definition 5 (Phase space) *A phase space $\mathbf{P} = \langle M, \leq, \cdot, \star, \mathbf{1}, \mathcal{F} \rangle$ is a structure such that:*

1. $\langle M, \leq \rangle$ *is a poset;*
2. $\langle M, \cdot, \mathbf{1} \rangle$ *is a monoid;*
3. $\langle M, \star, \mathbf{1} \rangle$ *is a commutative monoid;*
4. $\forall a, b, c, d \in M$*:*
   (a) *if* $a \leq b, c \leq d$ *then* $a \cdot c \leq b \cdot d$*,*
   (b) *if* $a \leq b, c \leq d$ *then* $a \star c \leq b \star d$*,*
   (c) $a \star b \leq a \cdot b$*;*
5. $\mathcal{F}$*, which is called the set of facts, is a set of subsets of* M*;*
6. *each fact is an order ideal, i.e.,* $\forall F \in \mathcal{F}, \forall a \in F, \forall b \in M$*, if* $b \leq a$ *then* $b \in F$*;*
7. *the set of facts* $\mathcal{F}$ *is closed by arbitrary intersection;*
8. $M \in \mathcal{F}$*;*
9. $\forall a \in M, \forall F \in \mathcal{F}$*:*

(a) $\exists G \in \mathcal{F}, \forall b \in M, a \cdot b \in F$ *iff* $b \in G$,

(b) $\exists G \in \mathcal{F}, \forall b \in M, b \cdot a \in F$ *iff* $b \in G$,

(c) $\exists G \in \mathcal{F}, \forall b \in M, a \star b \in F$ *iff* $b \in G$. ■

As usual, the formulas will be interpreted as facts, i.e., as subsets of M. To this end, we introduce operations on subsets that will allow the different connective to be interpreted.

Definition 6 (Operations on subsets of M) *Let* $\mathbf{P} = \langle M, \leq, \cdot, \star, \mathbf{1}, \mathcal{F} \rangle$ *be a phase space. Let* $A, B \subset M$. *We define the following operations:*

1. $A \cdot B \quad := \quad \{x \in M : \exists a \in A, \exists b \in B, x = a \cdot b\}$;
2. $A \star B \quad := \quad \{x \in M : \exists a \in A, \exists b \in B, x = a \star b\}$;
3. $A \multimap\!\!\bullet\, B \quad := \quad \{x \in M : \forall a \in A, a \cdot x \in B\}$;
4. $B \bullet\!\!- A \quad := \quad \{x \in M : \forall a \in A, x \cdot a \in B\}$;
5. $A \multimap B \quad := \quad \{x \in M : \forall a \in A, a \star x \in B\}$;
6. $\mathrm{C}_{\mathcal{F}} A \quad := \quad \bigcap \{F \in \mathcal{F} : A \subset F\}$. ■

The following lemma state some fundamental properties of the above operations.

Lemma 7 *Let* $\mathbf{P} = \langle M, \leq, \cdot, \star, \mathbf{1}, \mathcal{F} \rangle$ *be a phase space and* $A, B, C, D \subset M$.

1. $A \cdot B \subset C$ *if and only if* $B \subset (A \multimap\!\!\bullet\, C)$.
2. $A \cdot B \subset C$ *if and only if* $A \subset (C \bullet\!\!- B)$.
3. $A \star B \subset C$ *if and only if* $B \subset (A \multimap C)$.
4. *if* $A \subset C$ *and* $B \subset D$ *then* $(A \cdot B) \subset (C \cdot D)$.
5. *if* $A \subset C$ *and* $B \subset D$ *then* $(A \star B) \subset (C \star D)$.
6. $(A \star B) \subset \mathrm{C}_{\mathcal{F}}(A \cdot B)$.

Proof. Properties 1 to 5 are direct consequences of Definition 6. Property 6 follows from Conditions 4.c and 6 in Definition 5. □

Condition 9 in Definition 5 ensures that whenever F is a fact so are $(A \multimap\!\!\bullet\, F)$, $(F \bullet\!\!- A)$, and $(A \multimap F)$. This property is established by the next lemma.

Lemma 8 *Let* $\mathbf{P} = \langle M, \leq, \cdot, \star, \mathbf{1}, \mathcal{F} \rangle$ *be a phase space. If* $A \subset M$*, and* $F \in \mathcal{F}$ *then:*

1. $A \multimap\!\!\bullet\, F \in \mathcal{F}$;
2. $F \bullet\!\!- A \in \mathcal{F}$;
3. $A \multimap F \in \mathcal{F}$.

Proof. We prove Property 1, the proofs of properties 2 and 3 being similar.

Since the set of facts is closed by arbitrary intersection, we have that for any $X \subset M$, $\mathrm{C}_{\mathcal{F}}X$ is the smallest fact containing X. Consequently, any $X \subset M$ is a fact if and only if $\mathrm{C}_{\mathcal{F}}X \subset X$. Therefore, we have to show that $\mathrm{C}_{\mathcal{F}}(A \multimap F) \subset (A \multimap F)$.

Suppose that $x \in \mathrm{C}_{\mathcal{F}}(A \multimap F)$ or, equivalently, suppose that:

$$\forall H \in \mathcal{F}, \text{ if } (A \multimap F) \subset H \text{ then } x \in H \qquad \text{(a)}$$

Suppose also that $a \in A$, we have to show that $a \cdot x \in F$.

Now, by Condition 9.a in Definition 5, there exists $G \in \mathcal{F}$ such that:

$$\forall y \in M, a \cdot y \in F \text{ iff } y \in G \qquad \text{(b)}$$

By instantiating H with G in (a), we obtain:

$$\text{if } (A \multimap F) \subset G \text{ then } x \in G$$

Then, using (b), we obtain:

$$\text{if } (A \multimap F) \subset G \text{ then } a \cdot x \in F,$$

and it remains to show that $(A \multimap F) \subset G$. To this end, suppose that $z \in (A \multimap F)$, i.e., $\forall a' \in A, a' \cdot z \in F$. Since $a \in A$, we have that $a \cdot z \in F$. Then, by (b), we have that $z \in G$. □

As we stressed in the preceeding proof, $\mathrm{C}_{\mathcal{F}}$ is the operator that yields the smallest fact containing a given subset. In fact, Conditions 7 and 8 in Definition 5 stipulates that $\mathcal{F}$ is a topped $\bigcap$-structure, and $\mathrm{C}_{\mathcal{F}}$ is the closure operator to which that structure gives rise. This is stated by the following lemma.

Lemma 9 *Let* $\mathbf{P} = \langle M, \leq, \cdot, \star, \mathbf{1}, \mathcal{F} \rangle$ *be a phase space. If* $A, B \subset M$ *then:*

1. $A \subset \mathrm{C}_{\mathcal{F}}A$;
2. *if* $A \subset B$ *then* $\mathrm{C}_{\mathcal{F}}A \subset \mathrm{C}_{\mathcal{F}}B$;
3. $\mathrm{C}_{\mathcal{F}}\mathrm{C}_{\mathcal{F}}A \subset \mathrm{C}_{\mathcal{F}}A$;
4. $(\mathrm{C}_{\mathcal{F}}A \cdot \mathrm{C}_{\mathcal{F}}B) \subset \mathrm{C}_{\mathcal{F}}(A \cdot B)$;
5. $(\mathrm{C}_{\mathcal{F}}A \star \mathrm{C}_{\mathcal{F}}B) \subset \mathrm{C}_{\mathcal{F}}(A \star B)$.

Proof. Properties 1 to 3, which are the definitional properties of a closure operator, are obvious. We only prove Property 4, the proof of Property 5 being similar.

By Property 1, we have that $(A \cdot B) \subset \mathrm{C}_{\mathcal{F}}(A \cdot B)$. Then, by Lemma 7, we have that $B \subset (A \multimap \mathrm{C}_{\mathcal{F}}(A \cdot B))$ and, using Property 2, we obtain that $\mathrm{C}_{\mathcal{F}}B \subset \mathrm{C}_{\mathcal{F}}(A \multimap \mathrm{C}_{\mathcal{F}}(A \cdot B))$. Now, by Lemma 8, $(A \multimap \mathrm{C}_{\mathcal{F}}(A \cdot B))$ is a fact and, consequently, $\mathrm{C}_{\mathcal{F}}(A \multimap \mathrm{C}_{\mathcal{F}}(A \cdot B)) = (A \multimap \mathrm{C}_{\mathcal{F}}(A \cdot B))$. This implies that $\mathrm{C}_{\mathcal{F}}B \subset (A \multimap \mathrm{C}_{\mathcal{F}}(A \cdot B))$ and, using Lemma 7, that $(A \cdot \mathrm{C}_{\mathcal{F}}B) \subset \mathrm{C}_{\mathcal{F}}(A \cdot B)$. From this, by a similar reasoning, we obtain that $\mathrm{C}_{\mathcal{F}}A \subset (\mathrm{C}_{\mathcal{F}}(A \cdot B) \multimapinv \mathrm{C}_{\mathcal{F}}B)$. Finally, using again Lemma 7, we conclude that $(\mathrm{C}_{\mathcal{F}}A \cdot \mathrm{C}_{\mathcal{F}}B) \subset \mathrm{C}_{\mathcal{F}}(A \cdot B)$. □

We are now in the position of defining the interpretation of the formulas.

Definition 10 (Interpretation) *Let $\mathbf{P} = \langle M, \leq, \cdot, \star, \mathbf{1}, \mathcal{F} \rangle$ be a phase space. Let $\eta : \mathcal{V} \to \mathcal{F}$ be a valuation that assigns a fact to each atomic formula. We inductively define the interpereation $[\![A]\!]\eta$ of a formula A as follows:*

1. $[\![a]\!]\eta \quad := \quad \eta(a)$;
2. $[\![A \bullet B]\!]\eta \quad := \quad \mathrm{C}_{\mathcal{F}}([\![A]\!]\eta \cdot [\![B]\!]\eta)$;
3. $[\![A \otimes B]\!]\eta \quad := \quad \mathrm{C}_{\mathcal{F}}([\![A]\!]\eta \star [\![B]\!]\eta)$;
4. $[\![A \multimap\!\!\bullet\, B]\!]\eta := [\![A]\!]\eta \multimap\!\!\bullet\, [\![B]\!]\eta$;
5. $[\![A \bullet\!\!\!- B]\!]\eta := [\![A]\!]\eta \bullet\!\!\!- [\![B]\!]\eta$;
6. $[\![A \multimap B]\!]\eta := [\![A]\!]\eta \multimap [\![B]\!]\eta$.

We define the interpereation $[\![\Gamma]\!]\eta$ of an antecedent Γ similarly:

1. $[\![(\Gamma, \Delta)]\!]\eta \quad := \quad \mathrm{C}_{\mathcal{F}}([\![\Gamma]\!]\eta \star [\![\Delta]\!]\eta)$;
2. $[\![(\Gamma; \Delta)]\!]\eta \quad := \quad \mathrm{C}_{\mathcal{F}}([\![\Gamma]\!]\eta \cdot [\![\Delta]\!]\eta)$.

Finally, we define the interpretation of a sequent as follows:

$$[\![\Gamma \vdash A]\!]\eta \quad := \quad [\![\Gamma]\!]\eta \multimap [\![A]\!]\eta,$$

where $[\![\Gamma]\!]\eta = \{\mathbf{1}\}$ when Γ is empty. ■

Remark that Lemma 8 implies that the interpretation of any formula, context, or sequent is a fact.

The notion of validity is defined as usual.

Definition 11 (Validity) *Let $\mathbf{P} = \langle M, \leq, \cdot, \star, \mathbf{1}, \mathcal{F} \rangle$ be a phase space, and let $\eta : \mathrm{A} \to \mathcal{F}$ be a valuation.*

1. $\mathbf{P}, \eta \models (\Gamma \vdash A)$ *if and only if* $\mathbf{1} \in [\![\Gamma \vdash A]\!]\eta$. *($\eta$ satifies the sequent $\Gamma \vdash A$.)*
2. $\mathbf{P} \models (\Gamma \vdash A)$ *if and only* $\mathbf{P}, \rho \models (\Gamma \vdash A)$ *for any valuation ρ. (The sequent $\Gamma \vdash A$ is valid in* $\mathbf{P}$.*)*
3. $\models (\Gamma \vdash A)$ *if and only* $\mathbf{Q} \models (\Gamma \vdash A)$ *for any phase space* $\mathbf{Q}$. *(The sequent $\Gamma \vdash A$ is valid.)*

■

3.2 Soundness

Formulas are interpreted by facts that are partially ordered by set inclusion. This inclusion order, which corresponds to a consequence relation, allows another characterization of validity to be given.

Lemma 12 *Let $\mathbf{P} = \langle M, \leq, \cdot, \star, \mathbf{1}, \mathcal{F} \rangle$ be a phase space and η be a valuation. $\mathbf{P}, \eta \models (\Gamma \vdash A)$ if and only if $[\![\Gamma]\!]\eta \subset [\![A]\!]\eta$.*

Proof. $\mathbf{1} \in [\![\Gamma \vdash A]\!]\eta$ iff $\forall \gamma \in [\![\Gamma]\!]\eta, \mathbf{1} \star \gamma \in [\![A]\!]\eta$ iff $\forall \gamma \in [\![\Gamma]\!]\eta, \gamma \in [\![A]\!]\eta$. □

We now prove that the interpretation given in Definition 11 is sound. To this end, we state several lemmas. Their proofs, which are easy to establish using Lemmas 7 and 9, are left to the reader.

Lemma 13 *Let* $\mathbf{P} = \langle M, \leq, \cdot, \star, \mathbf{1}, \mathcal{F} \rangle$ *be a phase space and let* η *be a valuation. The following properties hold:*

1. $[\![((\Delta_1; \Delta_2); \Delta_3)]\!]\eta = [\![(\Delta_1; (\Delta_2; \Delta_3))]\!]\eta$*;*
2. $[\![((\Delta_1, \Delta_2), \Delta_3)]\!]\eta = [\![(\Delta_1, (\Delta_2, \Delta_3))]\!]\eta$*;*
3. $[\![(\Delta_1, \Delta_2)]\!]\eta = [\![(\Delta_2, \Delta_1)]\!]\eta$*;*
4. $[\![(\Delta_1, \Delta_2)]\!]\eta \subset [\![(\Delta_1; \Delta_2)]\!]\eta$*;*
5. $[\![\Delta_1]\!]\eta \subset [\![\Delta_2]\!]\eta$ *implies that* $[\![\Gamma[\Delta_1]]\!]\eta \subset [\![\Gamma[\Delta_2]]\!]\eta$*.* □

Lemma 14 *Let* $\mathbf{P} = \langle M, \leq, \cdot, \star, \mathbf{1}, \mathcal{F} \rangle$ *be a phase space and let* η *be a valuation. The following properties hold:*

1. $[\![(A; A \multimap\!\!\bullet\, B)]\!]\eta \subset [\![B]\!]\eta$*;*
2. $[\![(A \bullet\!\!- B; B)]\!]\eta \subset [\![A]\!]\eta$*;*
3. $[\![(A, A \multimap B)]\!]\eta \subset [\![B]\!]\eta$*.* □

Theorem 15 (Soundness) $\vdash (\Gamma \;\vdash\; A)$ *implies that* $\models (\Gamma \;\vdash\; A)$.

Proof. A straightforward induction on the derivation of $\Gamma \;\vdash\; A$, using Lemmas 12, 13, and 14. □

3.3 Completeness

In this section, we prove the converse of Theorem 15: any valid sequent is derivable. Our proof, which is adapted from [4], undertakes the usual construction of a syntactic phase $\mathbf{S} = \langle M_S, \leq_S, \cdot_S, \star_S, \mathbf{1}_S, \mathcal{F}_S \rangle$ in which the notions of provability and validity coincide:

1. The set M_S is the set of contexts modulo the associativity law of serial composition, and the associativity and commutativity laws of parallel composition. The two monoidal products $\cdot_S$ and $\star_S$ correspond respectively to serial composition ";" and parallel composition ",". The identity element is the empty context.
2. The partial order $\leq_S$ is defined to be the least order such that:
   (a) if $\Delta_1 \leq_S \Gamma_1$ and $\Delta_2 \leq_S \Gamma_2$ then $(\Delta_1; \Delta_2) \leq_S (\Gamma_1; \Gamma_2)$,
   (b) if $\Delta_1 \leq_S \Gamma_1$ and $\Delta_2 \leq_S \Gamma_2$ then $(\Delta_1, \Delta_2) \leq_S (\Gamma_1, \Gamma_2)$,
   (c) $(\Delta, \Gamma_1) \leq_S (\Delta; \Gamma_1)$;
3. For each formula A, we define $[A] := \{\Gamma : \;\vdash_{\mathrm{cf}} (\Gamma \;\vdash\; A)\}$. The set of facts $\mathcal{F}_S$ is defined to be the least set such that:
   (a) $[A] \in \mathcal{F}_S$, for any formula A,

(b) $M_S \in \mathcal{F}_S$,

(c) $\mathcal{F}_S$ is closed by arbitrary intersection.

We must now check that the above contruction gives rise to a phase space.

Lemma 16 *The structure* $\mathbf{S} = \langle M_S, \leq_S, \cdot_S, \star_S, \mathbf{1}_S, \mathcal{F}_S \rangle$ *defined as above is a phase space.*

Proof. Conditions 1 to 5, Condition 7, and Condition 8 of Definition 5 are satisfied by construction. Condition 6, which stipulates that each fact must be an order ideal is also satisfied because of Rule (ent). Therefore, it remains to show that Condition 9 is satisfied. We only prove 9.a, the proofs of 9.b and 9.c being similar.

Let $\Gamma \in M_S$ and let $F \in \mathcal{F}_S$. If $F = M_S$ then we take $G = M_S$. Otherwise, there exists a family $\mathcal{A}$ of formulas such that $F = \bigcap_{A \in \mathcal{A}} [A]$. Let γ be the formula obtained by replacing in Γ each occurrence of ";" by "$\bullet$" and each occurrence of "," by "$\otimes$". Then, consider $G = \bigcap_{A \in \mathcal{A}} [\gamma \multimap\!\!\bullet\, A]$. By Lemma 4, we have that for any $A \in \mathcal{A}$, $\vdash_{\text{cf}} (\Gamma; \Delta \vdash A)$ if and only if $\vdash_{\text{cf}} (\Delta \vdash \gamma \multimap\!\!\bullet\, A)$. From this, we conclude that for all $\Delta \in M_S$,

$$\forall A \in \mathcal{A},\ \vdash_{\text{cf}} (\Gamma; \Delta \vdash A) \quad \text{if and only if} \quad \forall A \in \mathcal{A},\ \vdash_{\text{cf}} (\Delta \vdash \gamma \multimap\!\!\bullet\, A).$$

$\square$

We are now in the position of establishing that validity in the phase space $\mathbf{S}$ implies provability.

Lemma 17 *Let η be the valuation that assigns the fact $[a]$ to each atomic formula a and write simply $[\![A]\!]$ for $[\![A]\!]\eta$. For any formula A, the following holds:*

1. $[\![A]\!] \subset [A]$;
2. $A \in [\![A]\!]$.

Proof. Preliminary remark: Property 1 implies that the interpretation of each formula is contained in at least one fact of the form $[D]$. Therefore, for each formula A, there exists a family of formula $\mathcal{A}$ such that $[\![A]\!] = \bigcap_{C \in \mathcal{A}} [D]$, since the interpretation of any formula is a fact. We will use this property when establishing Property 2.

The proof is by induction on the structure of A. For the base case, Property 1 holds by definition of the valuation, and Property 2 holds because of the axiom (id). We only give the inductive cases corresponding to the connectives $\bullet$ and $\multimap\!\!\bullet$, the others being similar.

$A \equiv B \bullet C$. Suppose $\Gamma \in [\![B \bullet C]\!]$. By definition, this means that for any fact F such that $[\![B]\!] \star_S [\![C]\!] \subset F$, we have $\Gamma \in F$. In particular, consider the fact $[B \bullet C]$. We obtain that if $[\![B]\!] \star_S [\![C]\!] \subset [B \bullet C]$ then $\Gamma \in [B \bullet C]$. Therefore, it remains to show that $[\![B]\!] \star_S [\![C]\!] \subset [B \bullet C]$. To this end, suppose $\Gamma_1 \in [\![B]\!]$ and $\Gamma_2 \in [\![C]\!]$. By induction hypothesis, we have $\Gamma_1 \in [B]$ and $\Gamma_2 \in [C]$, which implies $\vdash_{\text{cf}} (\Gamma_1 \vdash B)$ and $\vdash_{\text{cf}} (\Gamma_2 \vdash C)$. Hence, by Rule (pr), we obtain $\vdash_{\text{cf}} (\Gamma_1; \Gamma_2 \vdash B \bullet C)$, which implies $(\Gamma_1; \Gamma_2) \in [B \bullet C]$. This establishes Property 1.

To establish Property 2, we must show that $B \bullet C \in \mathcal{F}_S([\![B]\!] \star_S [\![C]\!])$. According to the prelimnary remark, it is sufficient to show that for any formula D such that $[\![B]\!] \star_S [\![C]\!] \subset [D]$, we have $B \bullet C \in [D]$. Now, by induction hypothesis, $B \in [\![B]\!]$ and $C \in [\![C]\!]$. Therefore, for any formula D such that $[\![B]\!] \star_S [\![C]\!] \subset [D]$, we have $B; C \in [D]$. This means that $\vdash_{\text{cf}} (B; C \vdash D)$. Hence, by Rule (pl), $\vdash_{\text{cf}} (B \bullet C \vdash D)$, which implies $B \bullet C \in [D]$.

$A \equiv B \multimap C$. Suppose $\Gamma \in [\![B \multimap C]\!]$. By definition, this means that for all $\Delta \in [\![B]\!]$, $(\Delta; \Gamma) \in [\![C]\!]$. By induction hypothesis, $B \in [\![B]\!]$. Hence, $(B; \Gamma) \in [\![C]\!]$. By induction hypothesis again, $[\![C]\!] \subset [C]$. This implies $B; \Gamma \in [C]$, which means $\vdash_{\text{cf}} (B; \Gamma \vdash C)$. Then, using Rule (dr), we obtain $\vdash_{\text{cf}} (\Gamma \vdash B \multimap C)$, which establishes that $\Gamma \in [B \multimap C]$.

To establish Property 2, we must show that $\forall \Gamma \in [\![B]\!], (\Gamma; B \multimap C) \in [\![C]\!]$. Suppose $\Gamma \in [\![B]\!]$. By induction hypothesis, $[\![B]\!] \subset [B]$, which implies:

$$\vdash_{\text{cf}} (\Gamma \vdash B) \qquad \text{(a)}$$

Now, according to the preliminary remark, in order to show that $(\Gamma; B \multimap C) \in [\![C]\!]$, we may suppose $[\![C]\!] \subset [D]$ and show that $(\Gamma; B \multimap C) \in [D]$. By induction hypothesis, $C \in [\![C]\!]$. This implies $C \in [D]$, which means that:

$$\vdash_{\text{cf}} (C \vdash D) \qquad \text{(b)}$$

From (a), and (b), using Rule (dl), we obtain $\vdash_{\text{cf}} (\Gamma; B \multimap C \vdash D)$, which establishes that $(\Gamma; B \multimap C) \in [D]$. □

Theorem 18 (Completeness) *The following statements are equivalent:*

1. $\models (\Gamma \vdash A)$
2. $\vdash_{\text{cf}} (\Gamma \vdash A)$
3. $\vdash (\Gamma \vdash A)$

Proof. We have that 2 implies 3 trivially and that 3 implies 1 by Theorem 15. To show that 1 implies 2, suppose that $\models (\Gamma \vdash A)$. This means, by Lemma 12, that $[\![\Gamma]\!]\eta \subset [\![A]\!]\eta$ for any phase space **P** and any valuation η. In particular, this inclusion holds for the syntactic phase space **S** and the valuation of Lemma 17. Then, by Lemma 17, Property 1, we have $[\![A]\!] \subset [A]$, which consequently implies $[\![\Gamma]\!] \subset [A]$. On the other hand, as a consequence of Lemma 17, Property 2, we have that $\Gamma \in [\![\Gamma]\!]$. Therefore, we have $\Gamma \in [A]$, which implies $\vdash_{\text{cf}} (\Gamma \vdash A)$. □

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