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Classical Non-Associative Lambek Calculus

Abstract. We introduce non-associative linear logic, which may be seen as the classical version of the non-associative Lambek calculus. We define its sequent calculus, its theory of proof-nets, for which we give a correctness criterion and a sequentialization theorem, and we show proof search in it is polynomial.

Keywords: non-associative Lambek calculus, linear logic, proof-net

1. Introduction

During the last ten years or so, the study of substructural logics like the Lambek calculus has enjoyed a remarkable revival, there being two reasons for this. One is the resurgence of interest in the study of categorial grammars because of their applications in natural language processing. Since its introduction [15, 16] the Lambek calculus has been the core logic of the great majority of categorial grammatical formalisms. The other reason is the discovery of linear logic [12].

Linear logic has brought many new insight into the study of proof theory and logic formalisms in general, and among these, the one that matters most for us, is that an intuitionistic logical system whose structural rules are sufficiently restricted[1] is absolutely compatible with an involutive negation. In purely formal terms, the addition of such a negation, that turns the intuitionistic system into a "classical" one, is a conservative extension of the old system. So there are more logics around than we ever thought. But the real interest of what could be only a formal game, is that, as usual in mathematics and physics, the discovery of symmetry in the world is a tremendous help in understanding it. The extension of an intuitionistic substructural system to a classical one gives us new ways of looking at the old one. The theory of proof-nets is only an example of this: not only do we have a very compact and intrinsic way of presenting proofs, but we can use it to solve problems that were intractable before.

The relationship between the original, associative Lambek calculus **L** [15] and linear logic has already been explored extensively in the litterature [5, 10, 14, 21, 22, 23]. One interesting thing about it is that it can be

[1] So far this has meant: without weakening or contraction, but it is probable that this can be relaxed.

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extended in two ways into a system with negation. In one case, which is often called the Girard-Yetter calculus [28], negation has the usual property of involutiveness. But one can also have a conservative extension of **L** that has *two* negations, that are not involutive, but that cancel each other, this being sufficient for defining a classical one-sided sequent calculus. This has been explored in detail by Abrusci [2]. This latter case is more natural from the point of view of category theory, since it corresponds to the most elementary notion (the one with the fewest assumptions) of a dualizing object in a monoidal biclosed category. Both approaches give rise to a theory of proof-nets, with the usual correctness criterion. One interesting feature of non-commutative linear logic is that the geometric aspect of proof-nets is even more pronounced than in the commutative case: the concepts associated with *planarity* of graphs become essential. This geometric side of things has made itself useful in problems like proof search [9, 20].

On the other hand, very little has been done in the way of proof-nets for the non-associative Lambek calculus, **NL** [16]. This system, whose reason for being is the description/generation of trees instead of strings, has not received the same amount of attention as its associative ancestor, although its linguistic interes, the common denominator of all linguistic calculi, has been stressed by Moortgat [17]. It has been studied from the point of view of generative power by Buszkowski [8] and Kandulski [11], and benefits from a completeness theorem due to Szczerba [26]. From the point of view of non-associative proof-nets, there is the paper by Moortgat and Oehrle [19], which is inside an intuitionistic framework (but the use of polarities is very indicative that the classical world is not very far away), and lacks a correctness criterion in the tradition of linear logic; it is fair to say that it is closer to the theory of categorical combinators for the lambda calculus. During the long gestation time between the first draft of this paper and its present form we also learned about the work of Puite and Moot [18], also for the intutionistic calculus, but in this case there is a correctness criterion, expressed via rewriting.

So the aim of this paper is to present a classical version of **NL**, which we will call **CNL**, along with its associated theory of proof-nets, that includes a correctness criterion. **CNL** has one involutive negation, which surely is the simplest way of "going classical"; we are aware that there might be other ways of doing so, and obtaining finer systems. **NL** can easily be embedded into **CNL**, via a the usual definition of its implications. One thing that tells us we are doing something right is that the technique of polarities allows us to consider **NL** as a *subsystem* of **CNL** the latter being a conservative extension of the former. **CNL** is also formally simpler than

NL, and indeed makes explicit some symmetries that were latent in **NL** but not directly visible. Indeed, the discovery of **CNL** is due to the observation that the algebra of contexts which is necessary to define the sequent calculus of **NL** can be turned on its head, so to speak, by forgetting the difference between the input of a context marker (the places where types/formulas are plugged) and its output (the resulting context). One result of this is that the distinction between several introduction rules of **NL** disappear, as we will see. In addition, as soon as this step is taken, everything acquires a very geometric flavor, very much in the spirit of linear logic. This includes a theory of proof-nets, where the usual "splitting lemma" is much easier to prove than usual, given the more constrained character of the logic.

We should mention that this paper is mainly concerned with the cut free fragment of **CNL**. This does not mean that **CNL** does not satisfy cut-elimination; it does. Nevertheless, the correctness criterion we give is defined on the cut-free proof-structure only. The cut-link is very well known to be problematic when non-commutativity is involved, and the only work we know about that has a full correctness criterion, compatible with every cut-elimination step, is by Abrusci and Maringelli [3]. As is standard in this context, our work, however, allows a weak form of the cut-elimination property to be proved: whenever two cut-free proof-nets are connected by a cut, this cut may be eliminated and the resulting cut-free proof-structure is correct.

Another interesting property of our calculus is that proof search in it is polynomial. It was already known that the implicative fragment of **NL** was polynomial [1], and here again we have an example of simplification by the use of a generalization that shows more symmetries. We should mention that the complexity of the original **L** is still an open problem, and that the study of calculi that are related to it (both "from above", and "from below") is worthwile for the insights it may bring in one of the few gray areas left in the complexity theory of pure calculi.

We would like to thank Richard Moot and Quintijn Puite for the very close and helpful reading they gave to an early version of this paper.

This paper is dedicated to Jim Lambek for the occasion of his 75th birthday. The intellectual debt this work owes him should already be blatant, and in addition we should say how much we have learned about mathematical elegance, taste and clarity through the study of his work.

2. Frames, sequent trees and pre-nets

The syntax we present has both a traditional syntactical side (formulas, sequents), and a geometric one where sequents are given a tree-like presentation. We will call these specialized trees *sequent trees.* The fact that trees are a very natural way of presenting sequents is due to the presence of stuctured contexts in the sequent calculus. The geometric presentation has the advantage of being more intrinsic, everything being in normal form from the start.

We intend to present the system from that geometric point of view first. First let us point out that the word "tree" has different meanings, according to the community that uses it. The most common definition in graph theory[2] is as follows: a tree is a non-directed graph (and in this paper everything will be finite) which is connected and acyclic.

In other word there is a finite set of *nodes* (often called vertices), and *edges*, each edge connecting two *different* vertices, and the lack of direction is the same as saying that given nodes a, b we do not make the difference between edge ab and edge ba. The defining properties of "treeness", connectedness and acyclicity, can be rephrased as:

Given any two nodes a and b, there is a unique path connecting a to b

The notion of path should be obvious, but anyway we will be more precise very soon.

The trees as defined above are the ones that contain the least structure. They can be enriched in very many ways; for instance it is very often the case that a tree has a notion of "up" and "down". For example, in the syntactic tree of a formula/type in logic, the variables or constants are thought of as being "above" (they are the leaves) and the formula itself (or its outermost connector) is thought of being "at the bottom" (it is the root).

Given a non-directed tree it is easy to turn it into one that has a root: choose any node a, and decree it is the root. This will give an ordering to the rest of the nodes, such that for any b the set of nodes below it will always be a linear order with a at the bottom ... a tree ordering. Also note that given a non-directed tree, the choice of any edge ab gives rise to *two* such rooted trees, one with root a and one with root b: they are the connected components obtained by removing the edge.

In this paper we want to consider trees that are hybrids of the rooted and non-directed varieties. The vertices of our trees will always be labeled by *node types*, the equivalent of symbols in syntax, and the same way symbols

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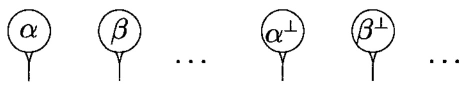
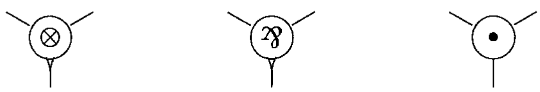
[2] The graph-theoretic notions and terminology we use in this paper follows [6].

have arities in ordinary syntax, a node type has a *valence* associated to it once and for all: in a tree a node decorated by a symbol of valence n will have exactly n edges connected to it. In a sequent tree an ordinary syntactical symbol of arity n corresponds to a node of valence $n+1$, the increment being due to the need to take the "business end" of the symbol in consideration. In particular the constants and variables of logic become nodes of valence 1. We will call these "atomic nodes" *terminal nodes.* The edges connecting nodes will be called *wires*; a wire is connected to a node via a *port*, in other words a node has exactly as many ports as its valence.

As is traditional in linear logic the language has a set of type variables and negavariables $\mathcal{V} = \{\alpha, \beta, \ldots, \alpha^\perp, \beta^\perp, \ldots\}$, and from now on they will *be the only node types with valence one:*



The minimal system for non-associative linear logic, i.e. the par-tensor fragment, need three other node types, all of valence three:



that are respectively called *tensor*, *par* and *context.*

We will denote this set of three node types by $\mathcal{N}_M$, but the theory of trees that follows is valid for for any set $\mathcal{N}$ of node types, provided they all have valences ≥ 2.

Given a node type of $\mathcal{N}_M$ which is a connector, i.e. a Tensor or a Par, it has a distinguished port, the *principal port,*[3] marked by a "Y" which is the root of the subformula it represents, i.e., its "business end" (the same goes for variables whose only port is principal). The other two ports (which we call the *auxiliary ports*) are connected to the two daughter subformulas. They also have a notion of "left daughter" and "right daughter". This left-right distinction is a bit too strict in general because a node does not necessarily have a principal port. Therefore, we replace this notion of "bottom + left & right" by a cyclic order on the ports, i.e., a cyclic permutation,[4] whose intuitive meaning is "next node when going counterclockwise" (see Figure 1).

[3] The terminology of ports and principal ports is taken from Lafont's interaction nets [13], the present work being very much in that spirit.

[4] A cyclic permutation π on a finite set A is a bijection $\pi\colon A \to A$ such that for any $a, b \in A$, $\pi^n(a) = b$ for some natural number n.

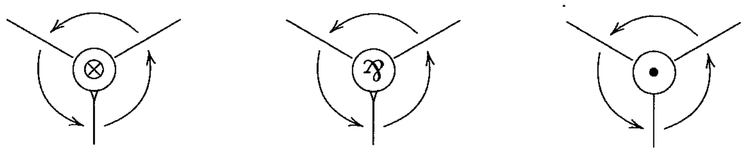


Figure 1.

Then a node type that has a principal port can use this cyclic order to know which of the auxiliaries is to the left (the one that immediately precedes the principal port for the cyclic order), and which is to the right (the one that immediately follows). The nodes that do not have a principal port (here there is only Context)[5] also have that cyclic order, and so, while being without a notion of top/bottom, they nevertheless have an orientation.

This distinction will hold if $\mathcal{N}_M$ is replaced by a larger $\mathcal{N} \supseteq \mathcal{N}_M$, as we have mentioned above: so in that more general case we assume that every node type $\mathbf{n} \in \mathcal{N}$ is given, in addition to its valence $\mathrm{val}(\mathbf{n}) \geq 2$, a cyclic order on its set of ports, and some nodes (which are called *connector nodes*) are given a principal port, while those without a principal port are called *context nodes*. The notion of auxiliary port of a connector node generalizes accordingly, and it is easy to see that connector nodes of valence $n+1$ can be seen as type constructors of arity n, the cyclic order along with the choice of the principal port allowing a precise definition of i-th daughter, for $1 \leq i \leq n$.

In what follows we work with a given set $\mathcal{N}$, of node types, with the necessary choice of valences and principal ports, but the reader can imagine that $\mathcal{N}$ is just $\mathcal{N}_M$.

DEFINITION 2.1. Given a set $\mathcal{N}$ of nodes types as above, an *$\mathcal{N}$-frame* is defined to be a connected acyclic simple graph G such that:

- each vertex of G is assigned a node type belonging to $\mathcal{N} \cup \mathcal{V}$;
- the number of edges incident to a vertex v is equal to the valence of the node type assigned to v;
- each edge incident to a vertex v is assigned one of the ports of the node type assigned to v, and this port assignment is such that any two different edges are assigned different ports.

[5] The idea of using a tree which is only "partially directed" for describing contexts in a classical calculus was presented by the second author in a Roma Workshop in February 1998. This idea turns out to be fruitful in more than just the non-associative case, as has been shown by Ruet for non-commutative linear logic [25]

We now may be more precise about the notions of node and wire. Given a $\mathcal{N}$-frame T, a node of T is a vertex v of the graph underlying T together with the node type assigned to this vertex v. Similarly, a wire corresponds to an edge of the underlying graph together with the ports that are assigned to this edge. This notion of wire may be formalised as an unordered pair of triples $\{(v_1, t_1, p_1), (v_2, t_2, p_2)\}$, where v_1 and v_2 are two different vertices, t_1 and t_2 are the node types that are respectively assigned to v_1 and v_2, and p_1 and p_2 are ports of t_1 and t_2 respectively.

Sometimes, we will want to describe frames that have one or two wires that are not connected to a terminal. This is done by introducing a special new terminal symbol $\circledast$ that means "ignore me, the wire I'm connected to is actually free".

It should be clear that a frame is already a kind of tree, but it does not yet have all the necessary properties to be considered a well-formed syntactical object. Now if in general our trees are not directed we intend to make the difference between directed paths and non-directed ones, which we will call *segments.*

DEFINITION 2.2. Let T be an $\mathcal{N}$-frame, and $a \neq b$ *distinct* nodes in it. The *path* from a to b, denoted $[a, b]$, is the uniquely defined sequence of nodes $a = a_0, a_1, \ldots, a_n = b$ such that, for all $1 \leq i \leq n$, a_{i-1} is connected to a_i by a wire.

DEFINITION 2.3. With the same notation as above, the *segment* $\langle a, b\rangle$ is the set of nodes in path $[a, b]$.

In other words a segment is obtained by forgetting the order of traversal of a path. We get that $\langle a, b\rangle = \langle b, a\rangle$ always, but $[a, b] = [b, a]$ never, since by definition $a \neq b$. Notice also that a segment can be turned into a path in exactly two ways, depending on which end is chosen as the starting point. Clearly, there is a one-one-correspondence between the wires of a frame and its segments that have only two nodes. Consequently, if a and b are two nodes connected by a wire, we will write $\langle a, b\rangle$ (or, equivalently, $\langle b, a\rangle$) for this wire. Then a two-node path will be called a *directed wire.* Also, if in a segment $\langle a, b\rangle$ the node a has a principal port such that the wire attached to this principal port connects a to a node that belongs to $\langle a, b\rangle$, we say that a *points towards* b.

Given a frame, the cyclic orders on the ports induced a cyclic order on the set of terminals of the frame on Figure 2.

Notice how, given a terminal node a the next terminal is obtained: walk from node a with your left hand to the frame, following the cyclic order on

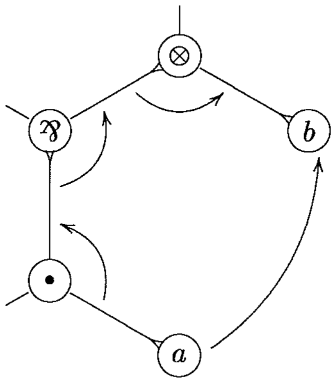


Figure 2.

the ports, until you reach a terminal (this assumes that the frame is drawn as a planar graph in such a way that the cyclic orders on the ports of the nodes follow the counterclockwise orientation).

We can extend that cyclic order to the set of *all directed wires*. Given a two-node path $[a, b]$, where b is not a terminal, its immediate successor in that big cyclic order is the unique two-node path $[b, c]$ such that the wire $\langle b, c\rangle$ immediately follows $\langle a, b\rangle$ on the cyclic order of ports on b. If b is a terminal, then the immediate successor of $[a, b]$ is $[b, a]$. This amounts to walking around the whole frame, always keeping your left hand on it, doing counterclockwise turns around terminals. Every wire will be touched twice, once for each of its directions.

There are also notions of subframes for which the cyclic order restricts naturally. Let T be an $\mathcal{N}$-frame and $a, b, c \in T$ three distinct nodes. Looking at the intersection $I = \langle a, b\rangle \cap \langle a, c\rangle$, three things may happen:

- $I = \{a\}$, meaning that $\langle a, b\rangle, \langle a, c\rangle$ go through different ports of a. We define $C(a, b, c) = a$.
- $I = \langle a, b\rangle$ meaning that $\langle a, b\rangle \subseteq \langle a, c\rangle$. We define $C(a, b, c) = b$.
- $I = \langle a, c\rangle$, meaning that $\langle a, b\rangle \supseteq \langle a, c\rangle$. We define $C(a, b, c) = c$.
- $I = \langle a, d\rangle$, for $d \neq b, c$. We define $C(a, b, c) = d$.

So we have defined a ternary partial operation $C(-, -, -)$ on T. What is remarkable (and easy to show) is that the value of $C(a, b, c)$ is totally independent of the order of the three parameters a, b, c. It is the "center node" in the tree structure of the triangle determined by the three distinct a, b, c. In particular, if they form a line, $C(a, b, c)$ will pick out the node in $\{a, b, c\}$ which is between the other two.

DEFINITION 2.4. Let T be an $\mathcal{N}$-frame. A *subframe* of T is given by a subset S of the set of nodes of T such that,

- given any $a \in S$ and any port of a there is $b \in S$ and a segment $\langle a, b \rangle$ of T that goes through that port.
- given three (distinct) $a, b, c \in S$, then $C(a, b, c) \in S$.

A subframe S has a frame structure on its own which is obtained by forgetting some of the nodes of T, thus turning some segments of T into wires of S, but in such a way that there are enough remaining nodes for their valence rules to be respected. Thus a subframe inherits both a tree structure and cyclic ordering from T.

PROPOSITION 2.5. *Let the node types $\mathcal{N}$ all be of valence ≥ 3 and let T be an $\mathcal{N}$-frame, A its set of terminals, and S a (nonempty) subframe of T. Then S is entirely determined by $S \cap A$, in the sense that it is the smallest subframe containing all its terminals.*

PROOF. Clearly, $S \cap A$ has to have more than one element, since the valence rules could not be respected otherwise. If it has only two elements, the only way the valence rules can be respected is if $S = S \cap A$ is made of two terminals, and then the result holds trivially. So let us assume that $S \cap A$ contains at least three elements. Then we know that for all $a, b, c \in S \cap A$ we have that $C(a, b, c) \in S$. Conversely, Let $d \in S$ not be a terminal, and choose a port p of it. We claim there is $a \in S \cap A$ such that $\langle d, a \rangle$ goes through that port. By definition we can find a segment $\langle d, d_1 \rangle$ of T whose other extremity d_1 is in S. If $d_1 \in S \cap A$ we stop. If not we choose *another* port of d_1 than the one that links it to d and we repeat the procedure, getting d_2 such that $\langle d_1, d_2 \rangle$ goes through that new port. This way construct a sequence $d_1, d_2, \ldots d_i$, with $\langle d, d_i \rangle \subseteq \langle d, d_{i+1} \rangle$, and this cannot go infinitely (or loop) so we end up with a $d_n \in S \cap A$, such that $\langle d, d_n \rangle$ goes through the chosen port p of d. By assumption d has at least three ports p_1, p_2, p_3, so we can find $a_1, a_2, a_3 \in S \cap A$ such that $\langle d, a_i \rangle$ goes through port p_i, and then we necessarily have $d = C(a_1, a_2, a_3)$. We have shown that every $d \in S$ can be captured from the information contained in $S \cap A$, and this concludes the proof. ∎

Remark 2.6. If all the nodes of $\mathcal{N}$ have valence exactly three, more is true. Indeed, one may prove that any subset of the terminals with at least three elements uniquely determines a subframe. More precisely, let T be an $\mathcal{N}$-frame, A its set of terminals, and $B \subset A$ be any subset of terminals containing at least three elements. Then there exists a unique subframe S such that $S \cap A = B$. This stronger property is not valid when valences greater than three are allowed.

On the other hand, Proposition 2.5 does not hold when there are nodes whose valence is two. This is due to the fact that, in this case, there may be subframes with more than two nodes but with only two terminals.

Here is a natural way of constructing subframes:

DEFINITION 2.7. Let T be a frame and $a \neq b \in T$ nodes. We define the following subsets:

$$a\rangle_b = \{\, x \in T \mid \langle x, a\rangle \cap \langle a, b\rangle = \{a\} \,\} \cup \{a\}$$
$$a \,\rangle\langle\, b = a\rangle_b \cup b\rangle_a$$

Thus $a \,\rangle\langle\, b$ is the set of all nodes "that are away from $\langle a, b\rangle$", with $a\rangle_b$ being those that "hide away from b behind a," and $b\rangle_a$ those that "hide away from a behind b." It is not hard to see $a \,\rangle\langle\, b$ is a subframe, because a segment $\langle x, y\rangle$ with $x, y \in a \,\rangle\langle\, b$ which is not entirely contained in $a \,\rangle\langle\, b$ necessarily has to contain the whole of $\langle a, b\rangle$. One particular case is when a, b are neighbours: then $a \,\rangle\langle\, b$ is the whole of T, which has been split in two.

We can now state the missing well-formedness condition:

DEFINITION 2.8. An $\mathcal{N}$-frame is said to be an *$\mathcal{N}$-sequent tree* if, given any connector node in it, its auxiliary ports are connected to the principal port of a connector or terminal node. It is said to be a *$\mathcal{N}$-quasi-sequent tree* if one of the wires is actually free, and the corresponding $\circledast$ is connected to a context node or a principal port. A $\mathcal{N}$-quasi-sequent is said to be a *$\mathcal{N}$-formula tree* if that free wire is connected to a principal port.

Thus, in a well-formed tree, to any connector node one can associate a formula (in ordinary syntax, the subformula for which the connector is outermost), given by the nodes (necessarily connectors or variables) that are "upstream" from it, going up principal ports. This also works for a terminal node, the formula that it defines being itself.

DEFINITION 2.9. A *context wire* is a wire which is connected to a context node, or to two principal ports.

PROPOSITION 2.10. *A $\mathcal{N}$-sequent tree without context nodes has exactly one context wire.*

PROOF. Take any node a_0, and look at the neighbour a_1 towards which it points. Then either

- a_1 points towards a_0 and we have found a context wire, or
- a_1 does not point towards a_0, so it points toward another neighbouring node, call it a_2.

Then we can start again, replacing a_0 above by a_1, constructing a path $[a_0, a_1, \ldots, a_n \ldots]$, stopping only when a_{n+1} points towards a_n. Since the tree is finite this has to stop, otherwise we'll have a loop (if one reaches a terminal, the search is obviously over).

For uniqueness, let $\langle a_1, a_2 \rangle$ and $\langle b_1, b_2 \rangle$ be two context wires in a frame without context node. There is a (unique) path/segment connecting one a_i with one b_i, in such a way that the other two nodes (call them a_j, b_j) are not in it. Then the principal port of the a_i, b_i are pointing *outside* the segment, and thus somewhere in there there have to be two atoms and a wire between them such that no principal port is involved, contradicting the well-formedness condition. ∎

In the same way, one can show that given two context nodes, the path between them has to contain only context nodes. This is because a connector node in that path has to point away from one of the two context nodes, contradicting well-formednesss. Thus the set of all context nodes has to be either connected or empty: it is a subtree of the sequent tree.

Remark 2.11. Let a, b be two terminal nodes. We know there is a unique path between them, and that the extremities a and b point towards the interior of that path. And well-formedness ensures that this path can only have two possible shapes:

- there is a middle made of context wires and context nodes, and all the other nodes point towards this middle.
- there are no context wires in the path, but still the path can be divided in two segments that both point inwards, with exactly one connector node x at the middle which is being pointed to by its neighbours via x's auxiliary ports.

PROPOSITION 2.12. *Let T be an $\mathcal{N}$-sequent tree, and $S \subseteq T$ a subframe. Then S is a sequent tree.*

PROOF. Given a connector node $x \in S$, an auxiliary port of it, and the node y connected to it in S, we get by induction that because T is a sequent tree and $x = x_0$ a connector node, the path $[x, y] = \{x_0, x_1, \ldots, x_n\}$ is such that x_i is always connected via an auxiliary port to a principal port of x_{i+1}. Thus it is also the case for $[x_0, x_n]_S = \{x, y\}$. ∎

We will be rather pedantic about the presentation of pairings and bijective pairings, since this notion is going to have more than the usual list of incarnations. The readers who are a little familiar with linear logic are expecting to meet the usual pairings given by axiom links, and they will not be disappointed.

DEFINITION 2.13. Let S be a set. A *pairing* on it is given by a binary relation ρ such that

Uniqueness	$x \rho y$ and $x \rho y'$ implies $y = y'$.
Antireflexivity	$x \rho y$ implies $x \neq y$,
Symmetry	$x \rho y$ implies $y \rho x$.

A pairing is said to be *total*, or *bijective* if in addition every x has a y such that $x \rho y$.

Thus, given a total pairing ρ, defining $\rho(x)$ as the unique y such that $x \rho y$ allows ρ to be seen as a bijection on S which is involutive ($\rho(\rho(x)) = x$) but has no fixed point. In general a pairing is only a partial bijection with these properties.

DEFINITION 2.14. A pre-net is a pair (T, τ), where T is a $\mathcal{N}_M$-sequent tree, and τ a bijective pairing on its set of terminals, sending a variable to its negation. A *sub-pre-net* of T is a subframe $S \subseteq T$ whose terminals are closed under the action of τ. Given a a terminal of T, the pair $(a, \tau(a))$ is called an *axiom link*.

The most standard term for a pre-nets is proof structure, but some people prefer our terminology for structures that are not necessarily associated with a proof.

In [14, 22] the atomic formulas are thought of lying on a circle, as they do here, but the axiom links are drawn *inside* while the syntactic trees of formulas ar drawn *outside* of the circle. Given the presence of context nodes, the reverse has to be done here. From a purely formal point of view there is absolutely no difference between these two approaches, since when a circle divides the sphere (obtained by adding a point at infinity to the plane) the distinction between inside and outside is arbitrary. But from the practical point of view of drawing figures, things are very different ....

3. The sequent calculus

In this section we present a more traditional syntax in terms of a sequent calculus that we call **CNL**. The formulas of this calculus obey the following grammar, where $\mathcal{V}$ is the aforementioned alphabet of atomic propositions:

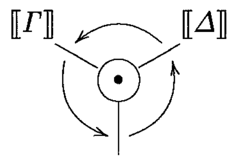
$$\mathcal{F} \ ::= \ \mathcal{V} \mid (\mathcal{F} \otimes \mathcal{F}) \mid (\mathcal{F} \parr \mathcal{F})$$

It is clear that there is a one-one correspondence between these formulas and the formula tree of Definition 2.8. Let A, B, C, ... range over formulas, we write $[\![A]\!]$ to denote the uniquely determined formula tree corresponding to A.

We then define a notion of *quasi-sequent*:

$$\mathcal{QS} \ ::= \ \mathcal{F} \mid (\mathcal{QS}, \mathcal{QS})$$

and let Γ, Δ, ... range over quasi-sequents. Remember that a formula tree has a unique port that is not connected to any other node, namely the principal port of the root of the tree. The correspondence $[\![-]\!]$ naturally extends to the quasi-sequents as follows:

$[\![(\Gamma, \Delta)]\!] =$ 

where $[\![\Gamma]\!]$ and $[\![\Delta]\!]$ are connected to a new context node through their respective unconnected ports, the unconnected port of this new context node becoming the unique unconnected port of $[\![(\Gamma, \Delta)]\!]$. This allows us to construct quasi-sequent trees in the sense of Definition 2.8.

Finally, the notion of *sequent* is defined as follows:

$$\mathcal{S} \ ::= \ \mathcal{QS}, \mathcal{QS}$$

Notice that there is not such a thing as an empty quasi-sequent. Consequently, any sequent contains at least two formulas.[6] Although the notations are quite close, there is a fundamental difference between a quasi-sequent (Γ, Δ) and the corresponding sequent Γ, Δ. Indeed, the correspondence

[6] This is the classical counterpart of the intutionistic sequents of the original Lambek calculus whose succedents consist of exactly one formula, and whose antecedents are not allowed to be empty. Interestingly, here this condition is imposed by the geometrical nature of the sequent trees

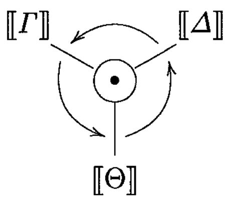
$[\![-]\!]$ is extended to the sequents by defining $[\![\Gamma, \Delta]\!]$ to be the sequent tree obtained by connecting the unconnected port of $[\![\Gamma]\!]$ to the unconnected port of $[\![\Delta]\!]$. Then, contrarily to the quasi-sequent trees, such sequent trees do not have any unconnected port. Because of this difference, the correspondence $[\![-]\!]$, when extended to sequents, is no longer one-one.

PROPOSITION 3.1. *Let Γ, Δ and Θ be quasi-sequents. The following identities hold:*

1. $[\![\Gamma, \Delta]\!] = [\![\Delta, \Gamma]\!]$
2. $[\![(\Gamma, \Delta), \Theta]\!] = [\![\Gamma, (\Delta, \Theta)]\!]$

PROOF.

$$[\![\Gamma, \Delta]\!] = [\![\Gamma]\!] \quad [\![\Delta]\!] = [\![\Delta, \Gamma]\!]$$

$[\![(\Gamma, \Delta), \Theta]\!] =$  $= [\![\Gamma, (\Delta, \Theta)]\!]$

■

In fact the identities of the above proposition characterize exactly what is quotiented out on the set of sequents by $[\![-]\!]$.

PROPOSITION 3.2. *Let "$\sim$" be the least equivalence relation between sequents such that:*

1. $\Gamma, \Delta \sim \Delta, \Gamma$
2. $(\Gamma, \Delta), \Theta \sim \Gamma, (\Delta, \Theta)$

If S_1 and S_2 are two sequents such that $[\![S_1]\!] = [\![S_2]\!]$ then $S_1 \sim S_2$.

PROOF. Let $S_1 = \Gamma_1, \Delta_1$, and let $[a, b]$ be the directed context wire that connects $[\![\Gamma_1]\!]$ to $[\![\Delta_1]\!]$ in $[\![S_1]\!]$. Similarly, let $S_2 = \Gamma_2, \Delta_2$, and let $[c, d]$ be the directed context wire that connects $[\![\Gamma_2]\!]$ to $[\![\Delta_2]\!]$. By connectedness and acyclicity, there is a unique minimal segment s that contains both $\langle a, b\rangle$ and $\langle c, d\rangle$. Let $|s|$ be the number of nodes in s. We proceed by induction on $|s|$.

If $|s| = 2$, there are two cases. Either $[a, b] = [c, d]$ and we are done, or $[a, b] = [d, c]$. In this second case, we have $\Gamma_1 = \Delta_2$ and $\Delta_1 = \Gamma_2$. Consequently, $S_1 \sim S_2$ by Identity 1.

If $|s| > 2$, there are also two cases. First, assume that $\langle c, d \rangle$ is in $[\![\Gamma_1]\!]$. Then, we must have $\Gamma_1 = (\Gamma_{11}, \Gamma_{12})$. There are two subcases:

- $\langle c, d \rangle$ is in $[\![\Gamma_{11}]\!]$ (including the case where $\langle c, d \rangle$ is the free wire of $[\![\Gamma_{11}]\!]$). Then, we have:

$$\begin{aligned} S_1 &= (\Gamma_{11}, \Gamma_{12}), \Delta_1 \\ &\sim \Gamma_{11}, (\Gamma_{12}, \Delta_1) \\ &\sim S_2 \quad \text{by induction hypothesis.} \end{aligned}$$

- $\langle c, d \rangle$ is in $[\![\Gamma_{12}]\!]$ (including the case where $\langle c, d \rangle$ is the free wire of $[\![\Gamma_{12}]\!]$). Then, we have:

$$\begin{aligned} S_1 &= (\Gamma_{11}, \Gamma_{12}), \Delta_1 \\ &\sim \Delta_1, (\Gamma_{11}, \Gamma_{12}) \\ &\sim (\Delta_1, \Gamma_{11}), \Gamma_{12} \\ &\sim S_2 \quad \text{by induction hypothesis.} \end{aligned}$$

The second case, where $\langle c, d \rangle$ is in $[\![\Delta_1]\!]$, is symmetric. ■

We are now in a position of giving the rules of the sequent calculus.

Axiom

$$\vdash \alpha, \alpha^{\perp} \quad \text{(Id)}$$

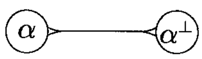
Logical rules

$$\frac{\vdash A, \Gamma \quad \vdash B, \Delta}{\vdash (A \otimes B), (\Delta, \Gamma)} \ (\otimes\text{-Intro}) \qquad \frac{\vdash (A, B), \Gamma}{\vdash (A \parr B), \Gamma} \ (\parr\text{-Intro})$$

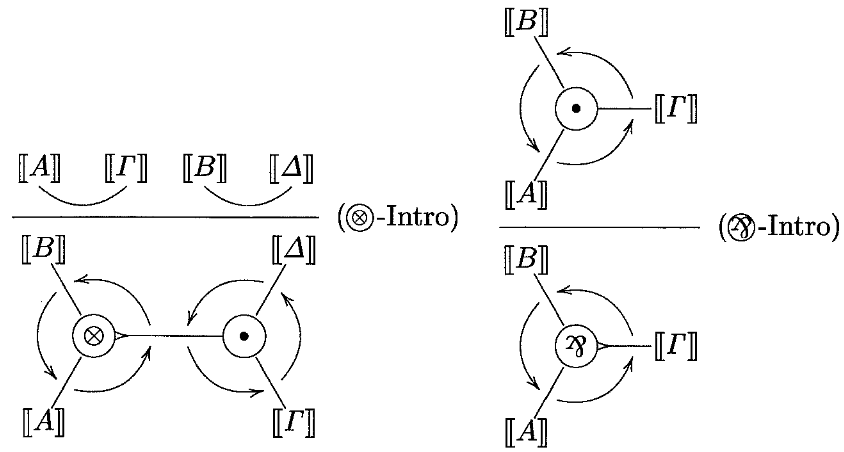
Structural rules

$$\frac{\vdash \Gamma, \Delta}{\vdash \Delta, \Gamma} \ \text{(Perm)} \qquad \frac{\vdash \Gamma, (\Delta, \Theta)}{\vdash (\Gamma, \Delta), \Theta} \ \text{(R-Shift)} \qquad \frac{\vdash (\Gamma, \Delta), \Theta}{\vdash \Gamma, (\Delta, \Theta)} \ \text{(L-Shift)}$$

The above inference rules may be intrepreted as sequent tree construction rules by using the correspondence $[\![-]\!]$:



(Id)



In this system, the structural rules do not play any part because of proposition 3.1. Now, if we interpreted the axioms of **CNL** as defining a pairing on the terminals, any **CNL**-derivation may be interpreted as a pre-net. Consider, for instance, the following derivation:

$$\dfrac{\dfrac{\dfrac{\vdash \alpha,\alpha^\perp}{\vdash \alpha^\perp,\alpha}\,(\text{Perm}) \qquad \vdash \gamma,\gamma^\perp}{\vdash (\alpha^\perp\otimes\gamma),(\gamma^\perp,\alpha)}\,(\otimes\text{-Intro}) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdash \beta,\beta^\perp \quad \dfrac{\vdash \alpha,\alpha^\perp}{\vdash \alpha^\perp,\alpha}\,(\text{Perm})}{\vdash (\beta\otimes\alpha^\perp),(\alpha,\beta^\perp)}\,(\otimes\text{-Intro})}{\vdash ((\beta\otimes\alpha^\perp),\alpha),\beta^\perp}\,(\text{R-Shift})}{\vdash ((\beta\otimes\alpha^\perp)\parr\alpha),\beta^\perp}\,(\parr\text{-Intro}) \qquad \vdash \alpha,\alpha^\perp}{\vdash (((\beta\otimes\alpha^\perp)\parr\alpha)\otimes\alpha),(\alpha^\perp,\beta^\perp)}\,(\otimes\text{-Intro})}{\vdash (\alpha^\perp,\beta^\perp),(((\beta\otimes\alpha^\perp)\parr\alpha)\otimes\alpha)}\,(\text{Perm})}{\vdash (\alpha^\perp\parr\beta^\perp),(((\beta\otimes\alpha^\perp)\parr\alpha)\otimes\alpha)}\,(\parr\text{-Intro})}{\vdash ((\alpha^\perp\otimes\gamma)\otimes(\alpha^\perp\parr\beta^\perp)),((((\beta\otimes\alpha^\perp)\parr\alpha)\otimes\alpha),(\gamma^\perp,\alpha))}\,(\otimes\text{-Intro})$$

It corresponds to the pre-net on Figure 3, where the pairing on the terminals is pictured as dotted lines.

Though any **CNL**-derivation may be interpreted as a pre-net, it is not the case that to any pre-net corresponds a **CNL**-derivation. It is therefore necessary, in order to define an adequate notion of proof-net, to give some criterion that allows the *correct pre-nets* (i.e., the ones that correspond to **CNL**-derivations) to be discriminated from the other ones. This is the goal of the next section.

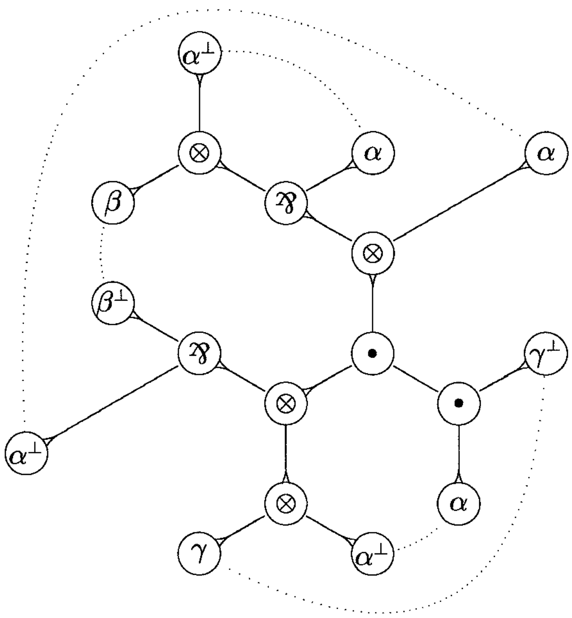


Figure 3.

4. Stating correctness

Well-bracketed pairings on terminals play a central role in the theory of non-commutative proof-nets [14, 22]. In this theory of non-associative proof-nets, well-bracketed pairings defined on all the nodes of the proof-nets will be used.

DEFINITION 4.1. A *well-bracketing* on an $\mathcal{N}$-frame T is a bijective pairing ρ on its set of nodes such that given any node a, the segment $\langle a, \rho(a)\rangle$ is such that for any node $b \in \langle a, \rho(a)\rangle$ we also have $\rho(b) \in \langle a, \rho(a)\rangle$.

PROPOSITION 4.2. *If ρ is a well-bracketing on $\mathcal{N}$-frame T, then for any nodes $a \in T$ the path $[a, \rho(a)]$ is well-bracketed in the usual sense by the restriction of ρ on it.*

PROOF. The proof is very easy, if not trivial. ■

We can also define well-bracketings in the universe of cyclic orders as opposed to trees, as in [14, 22]. Let A be a set equipped with a cyclic order, ρ a total pairing on it and $a \in A$. The pair $a, \rho(a)$ does not uniquely define a "segment" on A, but almost so, cutting the circle in two halves, which we can denote

$$\lfloor a, \rho(a) \rfloor = \text{the } x \in A \text{ between } a \text{ and } \rho(a) \text{ when going counterclockwise}$$
$$\lfloor \rho(a), a \rfloor = \text{the } x \in A \text{ between } \rho(a) \text{ and } a \text{ when going counterclockwise.}$$

DEFINITION 4.3. Given A, ρ as above, we say ρ is a *well-bracketing* of the cyclic order A when, for any $a, b \in A$, we always have $b, \rho(b)$ both in $\lfloor a, \rho(a) \rfloor$ or both in $\lfloor \rho(a), a \rfloor$.

In other words, $b, \rho(b)$ are always on the same side of $a, \rho(a)$.
There is yet another kind of pairing we will consider.

DEFINITION 4.4. A pairing ρ on a $\mathcal{N}_M$-frame is said to be *dualizing* if whenever a is a $\otimes$ then $\rho(a)$ is either a $\invamp$ or a $\odot$, and whenever a is a terminal node, thus typed by a variable or negavariable, then $\rho(a)$ is the negation of the same variable.

Thus a dualizing pairing is an extension of the notion of axiom linkage for a pre-net, linking opposite nodes as well as opposite variables.

DEFINITION 4.5. Let (T, τ) be a pre-net. On the set of nodes of T define a relation ρ_T

$$x \;\rho_T\; y \quad \text{iff} \quad \begin{array}{l} \text{there are axiom links } (a, a') \text{ and } (b, b') \text{ such that} \\ x, y \text{ are the extremities of the intersection segment} \\ \langle a, a' \rangle \cap \langle b, b' \rangle. \end{array}$$

This relation ρ_T is certainly symmetric, and it is easy to see that it extends the pairing given by the axiom links, since one can take a, a' and b, b' above to be the same axiom link.

Despite its trivial proof the following is an important observation.

LEMMA 4.6 (Gluing Lemma). *Let (T, τ) be a pre-net, and $S_1, S_2 \subseteq T$ two sub-pre-nets such that*

1. *If A is the set of terminals of T then $(S_1 \cup S_2) \cap A = A$.*
2. *given $i \neq j \in \{1, 2\}$, $(a, a') \in \tau_i$, $(b, b') \in \tau_j$ such that $\langle x, y \rangle = \langle a, a' \rangle \cap \langle b, b' \rangle$ is nonempty and $x, y \in S_1 \cup S_2$, there is $(c, c') \in \tau_i$ such that $\langle x, y \rangle = \langle a, a' \rangle \cap \langle c, c' \rangle$.*

Then the relation ρ_T restricted to $S_1 \cup S_2$ is the union of ρ_{T_1}, ρ_{T_2}.

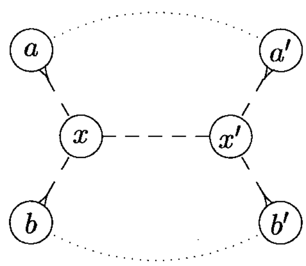
PROOF. We want to show that, given $x, y \in S_1 \cup S_2$, we have $x \;\rho_T\; y$ iff $x \;\rho_{T_1}\; y$ or $x \;\rho_{T_2}\; y$ and this is trivial to show using the conditions. ■

DEFINITION 4.7. A pre-net (T, τ) is said to be *correct*, or a *proof-net*, if the relation ρ_T defined as above from its axioms is a dualizing well-bracketing[7] of the sequent tree, such that in addition

PTR any $\otimes$-node x points *towards* its matching $⅋$- or $\odot$-node $\rho(x)$.

WB$_4$ given any pair $(a, a'), (b, b')$ of axiom links such that the intersection $\langle a, a' \rangle \cap \langle b, b' \rangle$ is nonempty, the axiom links form a well-bracketing on the four-element cyclic order on $\{a, a', b, b'\}$ obtained by restriction from the one on terminals.

This is a very strong definition, that makes sense only in the non-associative world. We should mention that if condition WB$_4$ were removed we would get a theory of proof-nets for *commutative, non-associative* linear logic, whose intuitionistic version has been studied under the name **NLP**[27]. This condition can be pictured by saying that, given any pair of axiom links $(a, a'), (b, b')$ as above, if $\langle x, x' \rangle$ is their intersection segment, then (a, a') lies on one side of $\langle x, x' \rangle$ while (b, b') lies on the other side.



Note that WB$_4$ is weaker than saying that the whole pairing ρ is a well-bracketing on the cyclic ordering of the full set of terminals, thus easier to verify, but it turns out as we will soon see that in correct nets we always do get a well-bracketing on the terminals.

The Gluing lemma can be interpreted as a simple compatibility condition that eases the assembly of two subnets that are correct individually, by giving a minimal procedure for checking that the net obtained by taking the frame completion of the union will be correct. Naturally part of that procedure, which is not stated explicitly in the lemma, is checking that the pairing extends to the nodes that are in the sum pre-net but not in the subframes.

[7] Since for the time being all our node types have valence 3, the axiom of Antireflexivity will always automatically hold here, but this is not necessarily the case for extended systems, i.e., when modalities are added.

THEOREM 4.8 (Sequentialization). *A pre-net which is constructed from the sequent calculus is correct and the axiom links form a well-bracketing of the cyclic order on terminals; moreover the converse holds: any proof-net can be obtained via the sequent calculus.*

5. Proving sequentialization

DEFINITION 5.1. Let T be a sequent tree and x a connector node. If x's principal port is connected to a context wire we say x is a *conclusive connector.*

PROPOSITION 5.2. *If (T, τ) is a proof-net and x a conclusive connector of it, and $\langle x, y\rangle$ its context wire, there exists an axiom link $(a, a') \in \tau$ such that $\langle x, y\rangle \in \langle a, a'\rangle$.*

PROOF. Look at $\rho_\tau(x)$. There has to be a pair of axiom links (a_i, a_i'), $i = 1$, 2, that justifies this pairing relation. It is then easy to see that x, y, has to belong to at least one of the $\langle a_i, a_i'\rangle$. ■

5.1. Necessity

It should be obvious that the pre-net obtained from an axiom is a proof-net. Also, a $⅋$-introduction does not change the structure of the well-bracketing and cyclic order at all, and thus preserves correctness. So the only case that needs a real proof is tensor-introduction.

Let (T, τ) be the pre-net obtained by doing a tensor introduction on the correct pre-nets (S_1, τ_1), (S_2, τ_2), thus adding a $\otimes$-node x and a $\odot$-node y. By induction we assume that the τ_i are well-bracketings for their cyclic orders. In each of the S_i there is a context wire $\langle a_i, b_i\rangle$ such that a_1, a_2 are connector nodes, and such that x, y have been inserted in between, with x on the side of the a_i, thus producing in T the paths $[a_1, x, y, b_1]$ and $[a_2,, x, y, b_2]$, and x pointing towards y. It is easy to see that both inclusion maps $S_i \to T$ turn the S_i into subframes of T, and thus sub-pre-nets because the axiom links do not change. It is also easy to see that the conditions of the Gluing Lemma apply, the second one being vacuous because a nonempty intersection of an axiom link in S_1 and one in S_2 can only be $\langle x, y\rangle$ and these are not in $S_1 \cup S_2$. Therefore the pairing relation ρ_τ restricted to $S_1 \cup S_2$ is the union of τ_1, τ_2.

Because of the previous remark we can show $x \ \rho \ y$, by choosing axiom links $(c_i, d_i) \in S_i$ for $i = 1, 2$, whose intersection in nonempty. Such links exist because of Proposition 5.2 and the fact that a_1, a_2 are conclusive connectors in S_1, S_2. Thus we have shown that ρ_τ is a dualizing pairing.

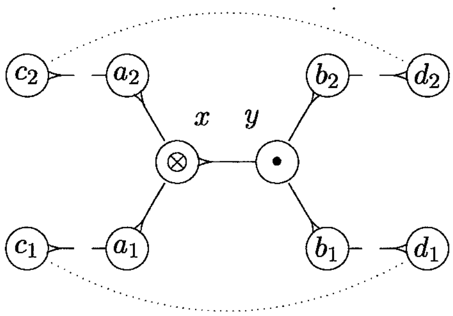


Figure 4.

Condition PTR still holds on the subnets S_1, S_2, since the addition of x, y inside segments cannot affect it, and the introduction rule has ensured that PTR holds for x, y.

As for cyclic order the shape of the introduction rule obviously preserves it, S_1 and S_2 splitting the circle in two "disjoint but connected intervals", and we also get the weaker WB$_4$ at the same time.

5.2. Sufficiency

Let (T, τ) be a proof-net that has a conclusive ⅋. Replacing it by a context node obviously produces a net which is still correct, since well-formedness is respected and nothing changes at the level of well-bracketings. So this can be done until there is no conclusive ⅋ left. One possibility then is that there is no connector node left at all. Then there can be no context node either since $\rho_\mathcal{T}$ maps context nodes to tensors. So in this case the frame can only be two variables connected by a context wire, and the presence of a dualizing τ guarantees they are each other's negation: we have an axiom link, and we are done.

The other possibility is that there are connector nodes left, but the only ones that point to a context wire are tensors, and from now on we assume this.

So assume x is a $\otimes$-node that not only points to an immediate neighbour y which is a $\odot$-node, but in addition such that $y = \rho_\mathcal{T}(x)$. Let a_1 be the left daughter node of x (the meaning should be obvious), and let a_2 be its right daughter. By the definition of the pairing $\rho_\mathcal{T}$ there are axiom links (c_1, d_1) and (c_2, d_2) such that $\langle x, y\rangle = \langle c_1, d_1\rangle \cap \langle c_2, d_2\rangle$. We can choose them in such a way that $\langle a_i, x\rangle \subseteq \langle c_i, x\rangle$, for $i = 1, 2$. There are also neighbours b_1, b_2 of y, and condition WB$_4$ tells us that $[a_i, x, y, b_i]$ has to be a subpath of $[c_i, d_i], i = 1, 2$ (see previous picture), instead of $[c_1, d_2]$ or $[c_2, d_1]$.

Let now, for $i = 1, 2$ $S_i = a_i\rangle, \langle b_i$, as in 2.7. As we have said there these subsets are subframes. We claim that the terminals of both S_1, S_2 are closed under the pairing τ, turning them into sub-pre-nets. Let $a \in S_1$ (say) and assume for a contradiction that $\tau(a) \in S_2$. Obviously $\langle a, \tau(a)\rangle$ has to contain at least one of x, y, and in particular we have to have that $\langle a, \tau(a)\rangle \cap \langle c_1, d_1\rangle$ is nonempty. Let $\langle u, v\rangle$ be that intersection. It cannot be that $\{u, v\} = \{x, y\}$, because then both $a, \tau(a)$ would be in S_2. It cannot be that $\{u, v\} \cap \{x, y\}$ is empty because then we would have $\{u, v\} \supset \{x, y\}$ and this would show $a, \tau(a)$ are both in S_1. So $\{u, v\} \cap \{x, y\}$ contains only one or either x, y, which we will show is also impossible. If for example $u = x, y \neq v$, by the definition of ρ_τ we have $\rho_\tau(x) = v$, which contradicts the assumption $\rho_\tau(x) = y$. The other cases are treated exactly the same way.

The fact that S_1, S_2 are sub-pre-nets tells us T can be obtained by applying tensor-introduction to them, and the Gluing Lemma tells us they are correct subnets. Thus, it is only fair to call a conclusive $\otimes$ which is paired by ρ_τ to its neighbour a *splitting tensor.* All that is left to do is prove the traditional Splitting Lemma; given the constraints on the non-associative calculus, the proof is comparatively trivial.

LEMMA 5.3 (Splitting Lemma). *Every proof-net all whose conclusive connectors are tensors has a splitting tensor.*

PROOF. Let x be a tensor that points to a context wire, and look at $\rho_\tau(x)$. Either $\rho_\tau(x)$ is x's neighbour and we are done, or not. If not, look at $\langle \rho_\tau(x), x\rangle$. There are more than one context nodes in that segment, given that the node z in there which is x's neighbour is a context node. Let $\langle z', z\rangle$ be the largest sub segment all whose elements are context nodes, and let $w \in \langle \rho_\tau(x), z'\rangle$ be the immediate neighbour of z'. It is necessarily a tensor node, by well-formedness it is pointing towards z', and we claim $\rho_\tau(w) = z'$, showing that w is a splitting tensor. If not, the inside of the segment $\{\, t \in \langle w, \rho_\tau(w)\rangle \mid t \neq w, \rho_\tau(w) \,\}$ contains only context nodes, but is also closed under ρ_τ, a contradiction. ■

This terminates the proof, given that any correct net can be decomposed using the rules of the calculus backwards into smaller nets.

6. Polynomiality

In this section, we prove that the decidability problem for non-associative mutltiplicative linear logic is polynomial. The fact that the provability of a given formula is decidable is almost obvious. Nevertheless a naive bottom-up proof-search strategy based on the sequent calculus **CNL** would waste a lot of time because of the structural rules. Moreover, because of these structural rules, the backward application of the inference rules of **CNL** is not deterministic. For instance, the following sequent:

$$((\beta^\perp \parr \alpha) \otimes \alpha^\perp), ((\alpha^\perp \otimes (\alpha \parr (\alpha^\perp \otimes \beta))), \alpha)$$

has two essentially different derivations:

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdash \alpha, \alpha^\perp}{\vdash \alpha^\perp, \alpha}\,(\text{Perm}) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdash \alpha, \alpha^\perp}{\vdash \alpha^\perp, \alpha}\,(\text{Perm}) \quad \vdash \beta, \beta^\perp}{\vdash (\alpha^\perp \otimes \beta), (\beta^\perp, \alpha)}\,(\otimes\text{-Intro})}{\vdash ((\alpha^\perp \otimes \beta), \beta^\perp), \alpha}\,(\text{R-Shift})}{\vdash \alpha, ((\alpha^\perp \otimes \beta), \beta^\perp)}\,(\text{Perm})}{\vdash (\alpha, (\alpha^\perp \otimes \beta)), \beta^\perp}\,(\text{R-Shift})}{\vdash (\alpha \parr (\alpha^\perp \otimes \beta)), \beta^\perp}\,(\parr\text{-Intro})}{\vdash (\alpha^\perp \otimes (\alpha \parr (\alpha^\perp \otimes \beta))), (\beta^\perp, \alpha)}\,(\otimes\text{-Intro})}{\vdash (\beta^\perp, \alpha), (\alpha^\perp \otimes (\alpha \parr (\alpha^\perp \otimes \beta)))}\,(\text{Perm})}{\vdash (\beta^\perp \parr \alpha), (\alpha^\perp \otimes (\alpha \parr (\alpha^\perp \otimes \beta)))}\,(\parr\text{-Intro}) \qquad \dfrac{\vdash \alpha, \alpha^\perp}{\vdash \alpha^\perp, \alpha}\,(\text{Perm})}{\vdash ((\beta^\perp \parr \alpha) \otimes \alpha^\perp), (\alpha, (\alpha^\perp \otimes (\alpha \parr (\alpha^\perp \otimes \beta))))}\,(\otimes\text{-Intro})$$

and

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdash \alpha, \alpha^\perp}{\vdash \alpha^\perp, \alpha}\,(\text{Perm}) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdash \alpha, \alpha^\perp}{\vdash \alpha^\perp, \alpha}\,(\text{Perm}) \quad \vdash \beta, \beta^\perp}{\vdash (\alpha^\perp \otimes \beta), (\beta^\perp, \alpha)}\,(\otimes\text{-Intro})}{\vdash (\beta^\perp, \alpha), (\alpha^\perp \otimes \beta)}\,(\text{Perm})}{\vdash (\beta^\perp \parr \alpha), (\alpha^\perp \otimes \beta)}\,(\parr\text{-Intro}) \qquad \dfrac{\vdash \alpha, \alpha^\perp}{\vdash \alpha^\perp, \alpha}\,(\text{Perm})}{\vdash ((\beta^\perp \parr \alpha) \otimes \alpha^\perp), (\alpha, (\alpha^\perp \otimes \beta))}\,(\otimes\text{-Intro})}{\vdash (\alpha, (\alpha^\perp \otimes \beta)), ((\beta^\perp \parr \alpha) \otimes \alpha^\perp)}\,(\text{Perm})}{\vdash (\alpha \parr (\alpha^\perp \otimes \beta)), ((\beta^\perp \parr \alpha) \otimes \alpha^\perp)}\,(\parr\text{-Intro})}{\vdash (\alpha^\perp \otimes (\alpha \parr (\alpha^\perp \otimes \beta))), (((\beta^\perp \parr \alpha) \otimes \alpha^\perp), \alpha)}\,(\otimes\text{-Intro})}{\vdash (((\beta^\perp \parr \alpha) \otimes \alpha^\perp), \alpha), (\alpha^\perp \otimes (\alpha \parr (\alpha^\perp \otimes \beta)))}\,(\text{Perm})}{\vdash ((\beta^\perp \parr \alpha) \otimes \alpha^\perp), (\alpha, (\alpha^\perp \otimes (\alpha \parr (\alpha^\perp \otimes \beta))))}\,(\text{L-Shift})$$

It is then easy to construct, from the above example, sequents with an exponential number of possible proofs for which any brute force algorithm based on **CNL** would answer in exponential time. For this reason, we introduce an alternative sequent calculus, called **C-CNL**, that does not need any structural rules. This sequent calculus manipulates contexts that are formulas with a "hole":

$$\mathcal{C} \ ::= \ [\,] \mid (\mathcal{C} \otimes \mathcal{F}) \mid (\mathcal{F} \otimes \mathcal{C}) \mid (\mathcal{C} \parr \mathcal{F}) \mid (\mathcal{F} \parr \mathcal{C})$$

where $[\,]$ stands for the empty context, i.e. the context consisting only of a hole. We let $\mathscr{C}[\,]$, $\mathscr{D}[\,]$, ... range over contexts, and we write $\mathscr{C}[A]$ for the formula obtained by filling the hole of a context $\mathscr{C}[\,]$ with a formula A.

Identity rules

$$\vdash a, a^{\perp} \quad (\mathrm{Id}_1) \qquad\qquad \vdash a^{\perp}, a \quad (\mathrm{Id}_2)$$

Logical rules

$$\frac{\vdash A, B \quad \vdash C, D}{\vdash (A \otimes C), (D \parr B)}\ (\otimes\parr\text{-Intro}_1) \qquad \frac{\vdash A, B \quad \vdash C, D}{\vdash (D \parr B), (A \otimes C)}\ (\otimes\parr\text{-Intro}_2)$$

$$\frac{\vdash A, B \quad \vdash \mathscr{C}[\,]}{\vdash \mathscr{C}[A], B}\ (\mathrm{Cont}_1) \qquad \frac{\vdash A, B \quad \vdash \mathscr{C}[\,]}{\vdash B, \mathscr{C}[A]}\ (\mathrm{Cont}_2)$$

Context derivation rules

$$\vdash [\,] \quad (\text{Empty-Cont})$$

$$\frac{\vdash A, B \quad \vdash \mathscr{C}[\,] \quad \vdash \mathscr{D}[\,]}{\vdash (\mathscr{C}[\mathscr{D}[\,] \otimes A] \parr B)}\ (\otimes\parr\text{-Cont}_1) \qquad \frac{\vdash A, B \quad \vdash \mathscr{C}[\,] \quad \vdash \mathscr{D}[\,]}{\vdash (B \parr \mathscr{C}[A \otimes \mathscr{D}[\,]])}\ (\otimes\parr\text{-Cont}_2)$$

Remark that the above calculus manipulates sequents made of exactly two formulas. We know, from the sequentialization theorem, that a proof-net with a conclusive $\bigcirc\!\!\!\!\parr$-node is correct if and only if the proof-net obtained by replacing this $\bigcirc\!\!\!\!\parr$-node with a $\odot$-node is correct. In the realm of the sequent calculus, this means that Rule ($\parr$-Intro) is invertible, which allows to reduce the provability of any sequent to the provability of an equivalent two-formula sequent.

It remains to prove that **CNL** and **C-CNL** are theorem equivalent. Let us write "$\vdash_{\mathrm{CNL}}$" and "$\vdash_{\text{C-CNL}}$" respectively for **CNL**- and **C-CNL**-derivability. We first prove the soundness of **C-CNL** with respect of **CNL**.

PROPOSITION 6.1. *Let A, B, and C be formulas, and let Γ be a quasi-sequent.*

1. *If* $\vdash_{\text{C-CNL}} A, B$ *then* $\vdash_{\text{CNL}} A, B$.
2. *If* $\vdash_{\text{CNL}} C, \Gamma$ *then* $\vdash_{\text{CNL}} \mathscr{E}[C], \Gamma$, *for any context* $\mathscr{E}[\,]$ *such that* $\vdash_{\text{C-CNL}} \mathscr{E}[\,]$.

PROOF. Property 1 is proved by induction on the structure of the **C-CNL**-derivation of A, B, using Property 2 to handle the cases of Rules (Cont_1) and (Cont_2).

Property 2 is proved by induction on the **C-CNL**-derivation of $\mathscr{E}[\,]$, using induction hypothesis 1 when needed. The case of the empty context is straightforward. Let $\mathscr{E}[\,] \equiv (\mathscr{C}[\mathscr{D}[\,] \otimes A] \parr B)$ be obtained as a conclusion of Rule ($\otimes\parr$-Cont_1). We proceed as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdash C, \Gamma}{\vdash \mathscr{D}[C], \Gamma}\text{Ind. Hyp.} \quad \vdash A, B}{\vdash \mathscr{D}[C] \otimes A, (B, \Gamma)}\otimes\text{-Intro}}{\vdash \mathscr{C}[\mathscr{D}[C] \otimes A], (B, \Gamma)}\text{Ind. Hyp.}}{\vdash (\mathscr{C}[\mathscr{D}[C] \otimes A], B), \Gamma}\text{R-Shift}}{\vdash (\mathscr{C}[\mathscr{D}[C] \otimes A] \parr B), \Gamma}\parr\text{-Intro}$$

The case where $\mathscr{E}[\,]$ is obtained as a conclusion of Rule ($\otimes\parr$-Cont_2) is similar. ■

In order to prove the completeness of **C-CNL** with respect to **CNL**, we introduce an intermediate calculus that we called **2-CNL**. This calculus mimics the rules of **CNL** and shares with **C-CNL** the property of manipulating only two-formula sequents.

Identity and logical rules

$$\vdash a, a^{\perp} \quad (\text{Id}) \qquad\qquad \dfrac{\vdash A, B \quad \vdash C, D}{\vdash (A \otimes C), (D \parr B)} \quad (\otimes\parr\text{-Intro})$$

Structural rules

$$\dfrac{\vdash A, B}{\vdash B, A}\;(\text{Perm}') \qquad \dfrac{\vdash A, (B \parr C)}{\vdash (A \parr B), C}\;(\text{R-Shift}') \qquad \dfrac{\vdash (A \parr B), C}{\vdash A, (B \parr C)}\;(\text{L-Shift}')$$

The similarity between **CNL** and **2-CNL** yields immediately the following property.

LEMMA 6.2. *Let A and B be two formulas. If* $\vdash_{\text{CNL}} A, B$ *then* $\vdash_{\text{2-CNL}} A, B$.

PROOF. Let $\overline{\Gamma}$ denote the formula obtained by replacing, in the tree of formulas Γ, each comma by $⅋$. It is straightforward that any **CNL**-derivation of a sequent $\vdash \Gamma, \Delta$ may be transformed into a **2-CNL**-derivation of the sequent $\vdash \overline{\Gamma}, \overline{\Delta}$. ■

The next step is to prove that any **2-CNL**-derivation may be turned into a **C-CNL**-derivation. To this end, we first establish the following technical lemma.

LEMMA 6.3. *Let $\mathscr{E}[\,]$ and $\mathscr{F}[\,]$ be two contexts such that* $\vdash_{\text{C-CNL}} \mathscr{E}[\,]$ *and* $\vdash_{\text{C-CNL}} \mathscr{F}[\,]$. *Then,* $\vdash_{\text{C-CNL}} \mathscr{E}[\mathscr{F}[\,]]$.

PROOF. The proof is performed by induction on the **C-CNL**-derivation of $\mathscr{E}[\,]$. The case of Axiom (Empty-cont) is obvious. If $\mathscr{E}[\,] \equiv (\mathscr{C}[\mathscr{D}[\,] \otimes A] ⅋ B)$ is obtained as a conclusion of Rule ($\otimes⅋$-Cont$_1$), we have that $\vdash \mathscr{D}[[\mathscr{F}[\,]]$ is **C-CNL**-derivable, by induction hypothesis. Hence:

$$\frac{\vdash A, B \quad \vdash \mathscr{C}[\,] \quad \vdash \mathscr{D}[\mathscr{F}[\,]]}{\vdash (\mathscr{C}[\mathscr{D}[\mathscr{F}[\,]] \otimes A] ⅋ B)}\ (\otimes⅋\text{-Cont}_1)$$

The case of Rule ($\otimes⅋$-Cont$_2$) is similar. ■

Now, we say that a **C-CNL**-derivation is *normal* if the two following conditions hold:

1. it is not the case that the right premise of any occurrence of Rule (Cont$_1$) or (Cont$_2$) is obtained by Axiom (Empty-Cont);
2. it is not the case that the left premise of any occurrence of Rule (Cont$_1$) or Rule (Cont$_2$) is obtained as the conclusion of an occurrence of Rule (Cont$_1$).

It is immediate, from Lemma 6.3 that any **C-CNL**-derivation may be turned into a normal **C-CNL**-derivation. This property is used to prove the next lemma, which shows how to transform a **2-CNL**-derivation into a **C-CNL**-derivation.

LEMMA 6.4. *Let A and B be formulas. If* $\vdash_{\text{2-CNL}} A, B$ *then* $\vdash_{\text{C-CNL}} A, B$.

PROOF. We show that any **2-CNL** rule is **C-CNL**-admissible. This is obvious for Rule ($\otimes⅋$-Intro), which identical to Rule ($\otimes⅋$-Intro$_1$). The case of Rule (Perm') is also straightforward because of the symmetry of **C-CNL**.

Consequently, it remains to show that both Rules (R-Shift') and (L-Shift') are **C-CNL**-admissible. We establish the admissibility of Rule (R-Shift'), the case of Rule (L-Shift') being similar.

We have to show that $\vdash (A \parr B), C$ is **C-CNL**-derivable whenever $\vdash A, (B \parr C)$ is. We proceed by induction on the structure of the normal **C-CNL**-derivations. There are three cases.

1. The sequent $\vdash A, (B \parr C)$ *is obtained as a conclusion of Rule* ($\otimes\parr$*-Intro*$_1$)*:*

$$\dfrac{\begin{array}{c}\Pi_1\\ \vdots\\ \vdash A_1, C\end{array} \qquad \begin{array}{c}\Pi_2\\ \vdots\\ \vdash A_2, B\end{array}}{\vdash (A_1 \otimes A_2), (B \parr C)}\ (\otimes\parr\text{-Intro}_1)$$

The derivation may be transformed as follows:

$$\dfrac{\begin{array}{c}\Pi_1\\ \vdots\\ \vdash A_1, C\end{array} \qquad \dfrac{\begin{array}{c}\Pi_2\\ \vdots\\ \vdash A_2, B\end{array} \quad \vdash [\,] \quad \vdash [\,]}{\vdash (([\,] \otimes A_2) \parr B)}\ (\otimes\parr\text{-Cont}_1)}{\vdash ((A_1 \otimes A_2) \parr B), C}\ (\text{Cont}_1)$$

2. The sequent $\vdash A, (B \parr C)$ *is obtained as a conclusion of Rule (Cont*$_1$*).* We distinguish between two subcases.

2.1. The left premise of Rule (Cont$_1$*) is obtained as a conclusion of Rule* ($\otimes\parr$*-Intro*$_1$)*:*

$$\dfrac{\dfrac{\begin{array}{c}\Pi_1\\ \vdots\\ \vdash A_1, C\end{array} \qquad \begin{array}{c}\Pi_2\\ \vdots\\ \vdash A_2, B\end{array}}{\vdash (A_1 \otimes A_2), (B \parr C)}\ (\otimes\parr\text{-Intro}_1) \qquad \begin{array}{c}\Pi_3\\ \vdots\\ \vdash \mathscr{C}[\,]\end{array}}{\vdash \mathscr{C}[(A_1 \otimes A_2)], (B \parr C)}\ (\text{Cont}_1)$$

The derivation may be transformed as follows:

$$\dfrac{\begin{array}{c}\Pi_1\\ \vdots\\ \vdash A_1, C\end{array} \qquad \dfrac{\begin{array}{c}\Pi_2\\ \vdots\\ \vdash A_2, B\end{array} \quad \begin{array}{c}\Pi_3\\ \vdots\\ \vdash \mathscr{C}[\,]\end{array} \quad \vdash [\,]}{\vdash (\mathscr{C}[([\,] \otimes A_2)] \parr B)}\ (\otimes\parr\text{-Cont}_1)}{\vdash (\mathscr{C}[(A_1 \otimes A_2)] \parr B), C}\ (\text{Cont}_1)$$

2.2. The left premise of Rule (Cont$_1$) is obtained as a conclusion of Rule (Cont$_2$). This case may be reduced to Case 3, below, by permuting the two inference rules.

3. The sequent $\vdash A, (B \parr C)$ is obtained as a conclusion of Rule (Cont$_2$). We distinguish between two subcases.

3.1. The right premise of Rule (Cont$_2$) is obtained as a conclusion of Rule ($\otimes\parr$-Cont$_1$):

$$\dfrac{\begin{array}{c}\Pi_1\\ \vdots\\ \vdash B_1, A\end{array} \qquad \dfrac{\begin{array}{c}\Pi_2\\ \vdots\\ \vdash B_2, C\end{array} \quad \begin{array}{c}\Pi_3\\ \vdots\\ \vdash \mathscr{C}[\,]\end{array} \quad \begin{array}{c}\Pi_4\\ \vdots\\ \vdash \mathscr{D}[\,]\end{array}}{\vdash (\mathscr{C}[\mathscr{D}[\,] \otimes B_2] \parr C)}\,(\otimes\parr\text{-Cont}_1)}{\vdash A, (\mathscr{C}[\mathscr{D}[B_1] \otimes B_2] \parr C)}\,(\text{Cont}_2)$$

The derivation may be transformed as follows:

$$\dfrac{\begin{array}{c}\Pi_2\\ \vdots\\ \vdash B_2, C\end{array} \qquad \dfrac{\dfrac{\begin{array}{c}\Pi_1\\ \vdots\\ \vdash B_1, A\end{array} \quad \begin{array}{c}\Pi_4\\ \vdots\\ \vdash \mathscr{D}[\,]\end{array}}{\vdash \mathscr{D}[B_1], A}\,(\text{Cont}_1) \quad \begin{array}{c}\Pi_3\\ \vdots\\ \vdash \mathscr{C}[\,]\end{array} \quad \vdash [\,]}{\vdash (A \parr \mathscr{C}[\mathscr{D}[B_1] \otimes [\,]])}\,(\otimes\parr\text{-Cont}_2)}{\vdash (A \parr \mathscr{C}[\mathscr{D}[B_1] \otimes B_2]), C}\,(\text{Cont}_1)$$

3.2. The right premise of Rule (Cont$_2$) is obtained as a conclusion of Rule ($\otimes\parr$-Cont$_2$):

$$\dfrac{\begin{array}{c}\Pi_1\\ \vdots\\ \vdash C_2, A\end{array} \qquad \dfrac{\begin{array}{c}\Pi_2\\ \vdots\\ \vdash C_1, B\end{array} \quad \begin{array}{c}\Pi_3\\ \vdots\\ \vdash \mathscr{C}[\,]\end{array} \quad \begin{array}{c}\Pi_4\\ \vdots\\ \vdash \mathscr{D}[\,]\end{array}}{\vdash (B \parr \mathscr{C}[C_1 \otimes \mathscr{D}[\,]])}\,(\otimes\parr\text{-Cont}_2)}{\vdash A, (B \parr \mathscr{C}[C_1 \otimes \mathscr{D}[C_2]])}\,(\text{Cont}_2)$$

The derivation may be transformed as follows:

$$\dfrac{\dfrac{\begin{array}{c}\Pi_2\\ \vdots\\ \vdash C_1, B\end{array} \qquad \dfrac{\begin{array}{c}\Pi_1\\ \vdots\\ \vdash C_2, A\end{array} \quad \begin{array}{c}\Pi_4\\ \vdots\\ \vdash \mathscr{D}[\,]\end{array}}{\vdash \mathscr{D}[C_2], A}\,(\text{Cont}_1)}{\vdash (C_1 \otimes \mathscr{D}[C_2]), (A \parr B)}\,(\otimes\parr\text{-Intro}_1) \qquad \begin{array}{c}\Pi_3\\ \vdots\\ \mathscr{C}[\,]\end{array}}{\vdash (A \parr B), \mathscr{C}[C_1 \otimes \mathscr{D}[C_2]])}\,(\text{Cont}_2)\blacksquare$$

We obtain the completeness of **C-CNL** with respect to **CNL** as an immediate consequence of this lemma.

PROPOSITION 6.5. *Let A and B be two formulas. If $\vdash_{\text{CNL}} A, B$ then $\vdash_{\text{C-CNL}} A, B$.*

PROOF. By Lemmas 6.2 and 6.4. ∎

Finally, we may prove the main result of this section.

THEOREM 6.6. *Non-associative mutltiplicative linear logic is decidable in polynomial time.*

PROOF. By Propositions 6.1 and 6.5, any sequent "Γ, Δ" is provable if and only if the corresponding two-formula sequent "$\overline{\Gamma}, \overline{\Delta}$" is **C-CNL**-derivable. Now, any possible **C-CNL**-derivation of "$\overline{\Gamma}, \overline{\Delta}$" will be made of two sorts of expressions:

1. sequents of the form $\vdash A, B$, where A is a subformula of $\overline{\Gamma}$ and B a subformula of $\overline{\Delta}$ or, conversely, A is a subformula of $\overline{\Delta}$ and B a subformula of $\overline{\Gamma}$.
2. expressions of the form $\vdash \mathscr{C}[\,]$ such that $\mathscr{C}[C]$ is a subformula of either $\overline{\Gamma}$ or $\overline{\Delta}$, for some formula C.

The number of expressions of the form $\vdash A, B$ is bounded by $|\Gamma| \times |\Delta|$, where $|\Gamma|$ and $|\Delta|$ are the length of $\overline{\Gamma}$ and $\overline{\Delta}$ respectively. Similarly, the number of expressions of the form $\vdash \mathscr{C}[\,]$ is bounded by $|\Gamma|^2 + |\Delta|^2$. Consequently, a naive proof-search algorithm based on **C-CNL** will terminate in polynomial time if its search space is organised in such a way that different possible derivations share the sub-derivations they have in common. ∎

7. Relation to the non-associative Lambek calculus NL

The non-associative Lambek calculus **NL** [16] may be seen as the intuitionistic fragment of **CNL**. Its formula obey the following grammar:

$$\mathcal{F} \ ::= \ \mathcal{V}^\circ \mid (\mathcal{F} \backslash \mathcal{F}) \mid (\mathcal{F} / \mathcal{F}) \mid (\mathcal{F} \bullet \mathcal{F})$$

where $\mathcal{V}^\circ = \{\alpha, \beta, \ldots\}$ is the alphabet of variables.

The deduction relation of **NL** may be specified by means of a calculus whose sequents have the form $\Gamma \vdash A$, the antecedent Γ being a quasi-sequent made of **NL** formulas, and the succedent A consisting of a single formula. Let Γ be a **NL** quasi-sequent in which occurs some formula A at a given position. One writes $\Gamma[A]$ to emphasize this occurrence of the formula A in Γ, and one writes $\Gamma[B]$ to denote the quasi-sequent obtained by replacing this given occurrence of A by the occurrence of a formula B.

$$A \vdash A \quad (\mathrm{Id})$$

$$\frac{\Gamma \vdash A \quad \Delta[B] \vdash C}{\Delta[(\Gamma, (A \setminus B))] \vdash C}\ (\backslash\text{-L}) \qquad \frac{(A, \Gamma) \vdash B}{\Gamma \vdash (A \setminus B)}\ (\backslash\text{-R})$$

$$\frac{\Gamma \vdash A \quad \Delta[B] \vdash C}{\Delta[((B/A), \Gamma)] \vdash C}\ (/\text{-L}) \qquad \frac{(\Gamma, A) \vdash B}{\Gamma \vdash (B/A)}\ (/\text{-R})$$

$$\frac{\Gamma[(A,B)] \vdash C}{\Gamma[(A \bullet B)] \vdash C}\ (\bullet\text{-L}) \qquad \frac{\Gamma \vdash A \quad \Delta \vdash B}{(\Gamma, \Delta) \vdash (A \bullet B)}\ (\bullet\text{-R})$$

Any **NL** formula, quasi-sequent or sequent may be translated into a **CNL** formula, quasi-sequent or sequent as follows:

$$[\Gamma \vdash A]^\circ \;=\; \vdash [A]^\circ, [\Gamma]^\bullet$$

where:

$$\begin{aligned}
[(\Gamma, \Delta)]^\bullet &= (\Delta^\bullet, \Gamma^\bullet) & & \\
[\alpha]^\bullet &= \alpha^\perp & [\alpha]^\circ &= \alpha \\
[(A \setminus B)]^\bullet &= (B^\bullet \otimes A^\circ) & [(A \setminus B)]^\circ &= (A^\bullet \parr B^\circ) \\
[(A/B)]^\bullet &= (B^\circ \otimes A^\bullet) & [(A/B)]^\circ &= (A^\circ \parr B^\bullet) \\
[(A \bullet B)]^\bullet &= (B^\bullet \parr A^\bullet) & [(A \bullet B)]^\circ &= (A^\circ \otimes B^\circ)
\end{aligned}$$

The soudness of this translation may be established by a routine induction that we leave to the reader.

PROPOSITION 7.1. *Let Γ be a* **NL** *quasi-sequent and A be a* **NL** *formula such that $\Gamma \vdash A$ is* **NL**-*derivable. Then $[\Gamma \vdash A]^\circ$ is* **CNL**-*derivable.*

The converse of this proposition allows **CNL** to be seen as a conservative extension of **NL**. In order to prove this property, we first establish two technical lemmas. Let A' be a **CNL** formula. We say that A' is *positively polarizable* (respectively, *negatively polarizable*) if there exists a **NL**-formula A such that $A' = A^\circ$ (respectively, $A' = A^\bullet$). A first lemma shows that there is a unique way of polarizing a **CNL** formula, if any.

LEMMA 7.2. *Let A' be a* **CNL**-*formula. Then there exist at most one* **NL**-*formula A such that either $A' = A^\circ$ or $A' = A^\bullet$.*

PROOF. The proof is done by an easy induction on the structure of the formula A'. The base case, when A' is a variable or a negavariable is straightforward. The inductive cases are summarized by the following tables that gives the possibility of polarizing a formula in terms of the possibility of polarizing its direct sub-formulas.

$⅋$	$\circ$	$\bullet$
$\circ$		$\circ$
$\bullet$	$\circ$	$\bullet$

$\otimes$	$\circ$	$\bullet$
$\circ$	$\circ$	$\bullet$
$\bullet$	$\bullet$	

■

We say that a **CNL** quasi-sequent consisting of one formula A is *negatively polarizable* if A, as a formula, is negatively polarizable. On the other hand, if A is positively polarizable, we say that the quasi-sequent consisting of A is *intuitionistically polarizable*. Then, we say that a quasi-sequent (Γ, Δ) is *negatively polarizable* if both Γ and Δ are negatively polarizable, and we say that it is *intuitionistically polarizable* if Γ (respectively, Δ) is intuitionistically polarizable while Δ (respectively, Γ) is negatively polarizable. Similarly, we say that a sequent Γ, Δ is *intuitionistically polarizable* if one of the two quasi-sequent it consists of is intuitionistically polarizable while the other is negatively polarizable.

LEMMA 7.3. *Any instance of any inference rule of* **CNL** *is such that its premise(s) is (are) intuitionistically polarizable whenever its conclusion is.*

PROOF. The proof consists in a straightforward case analysis. ■

PROPOSITION 7.4. *Let Γ be a* **NL** *quasi-sequent and A be a* **NL** *formula such that $[\Gamma \vdash A]^\circ$ is* **CNL**-*derivable. Then $\Gamma \vdash A$ is* **NL**-*derivable.*

PROOF. Let Π be a **CNL**-derivation of $[\Gamma \vdash A]^\circ$. By applying the translation $[\![-]\!]$ to each sequent in Π (which is not the conclusion of a structural rule), we obtain an inductive construction $[\![\Pi]\!]$ of the sequent tree corresponding to $[\Gamma \vdash A]^\circ$.

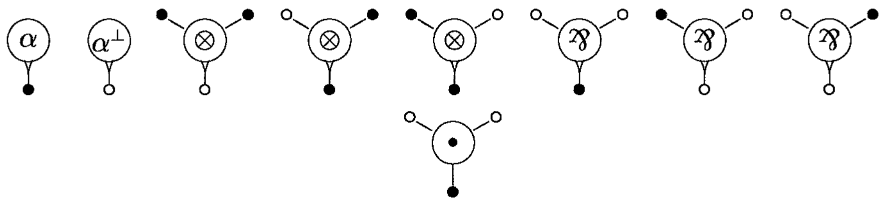
Any sequent tree involved in $[\![\Pi]\!]$, say π, may be transformed into a **NL**-sequent as follows. By Lemmas 7.3, π is intuitionistically polarizable. Therefore, there exists one formula tree in π, say T, that is positively polarizable and that is maximal with respect to subformula ordering. Let a be the root of T and let b be the node to which the principal port of a

is connected. Because of the maximality of T, the wire connecting a to b must be a context wire. Consequently, we have that $a\rangle_b = T$ and that $b\rangle_a$ is a quasi-sequent tree that is negatively polarizable. Hence, by lemma 7.2, there exist a unique **NL** formula B and a unique **NL** quasi-sequent Δ such that $B^\circ = a\rangle_b$ and $\Delta^\bullet = b\rangle_a$. This allows π to be transformed into the **NL** sequent $\Delta \vdash B$.

It is not difficult to see that applying the above transformation to all the sequent trees occurring in $[\![\Pi]\!]$ yields a valid **NL**-derivation of $\Gamma \vdash A$. ■

Propositions 7.1 and 7.4 allows proof-nets to be constructed for **NL** by interpreting the **NL**-derivations as **CNL**-derivations, and by transforming the latter into proof-nets. The proof-nets that are in the range of this construction may be characterized as follows:

1. assign polaraties to the ports of the nodes as follows:



   where $\circ$ is called the *output polarity* and $\bullet$ the *input polarity*;
2. stipulate the typing condition that the wires must connect output to input.

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