Formal Languages

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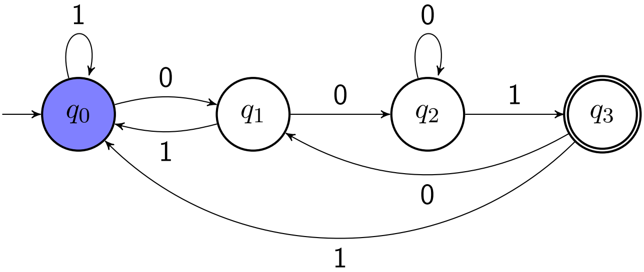
Outline

2. Finite state automata
   - Introduction
   - Deterministic finite state automata
   - Non-deterministic finite state automata
   - Equivalence of NFSA and DFSA
   - Finite state automata with ϵ-transitions
   - Elimination of ϵ-transitions

Introduction



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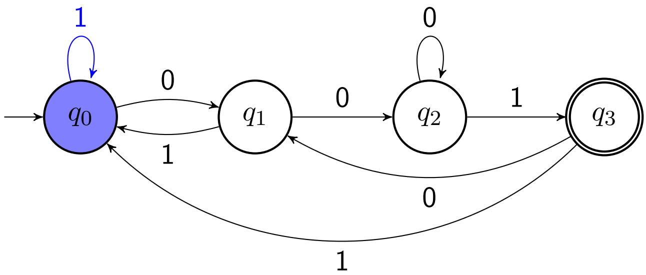

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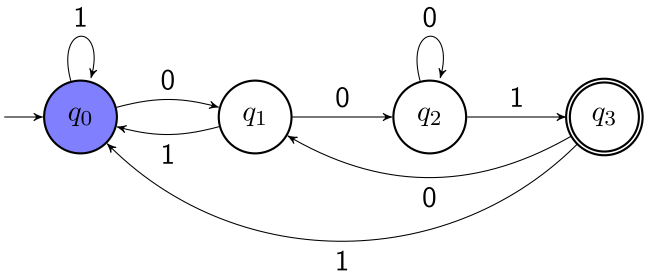

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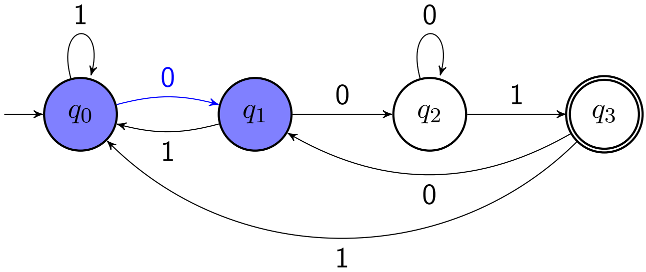

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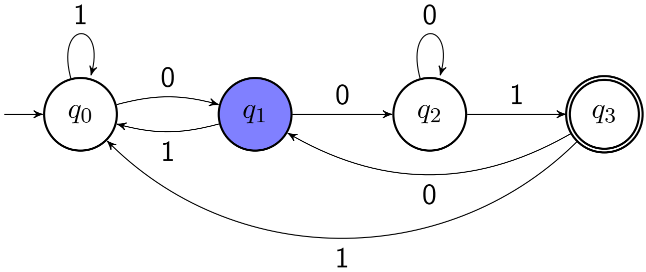

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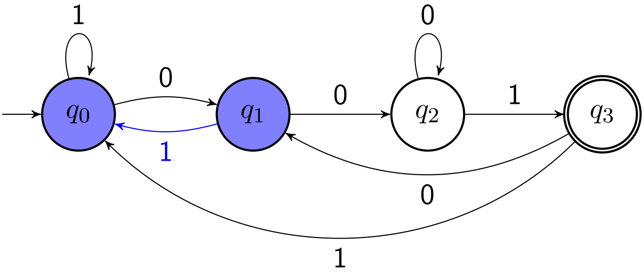

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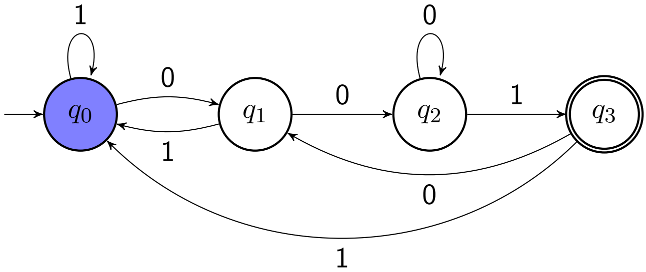

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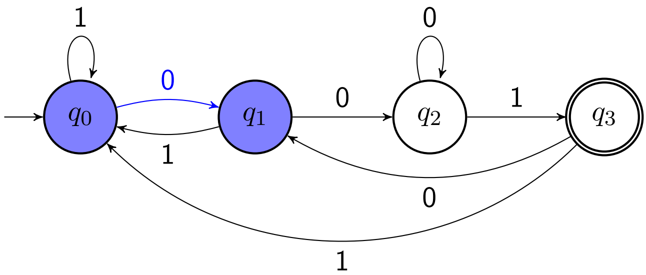

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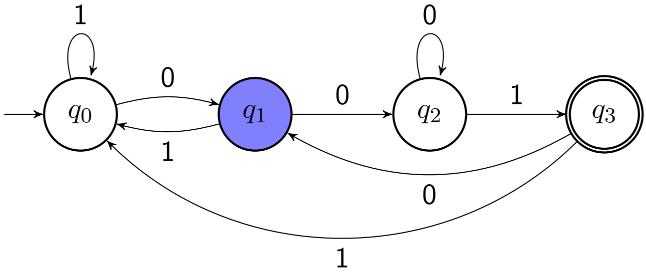

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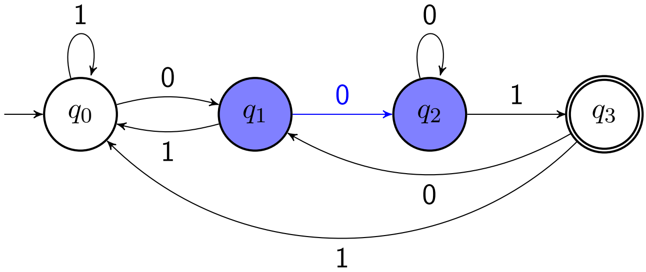

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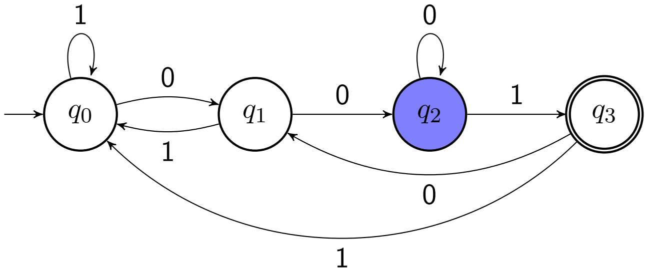

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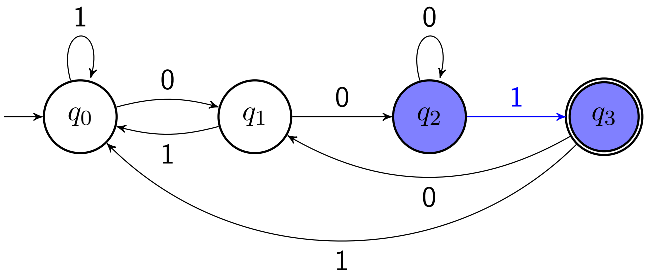

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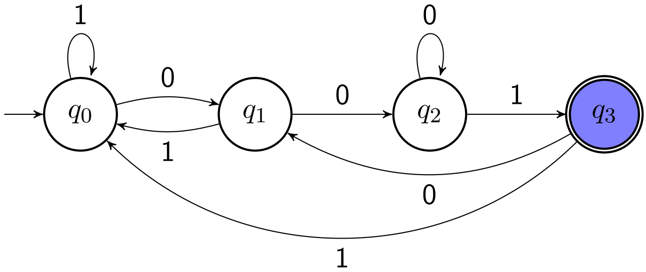

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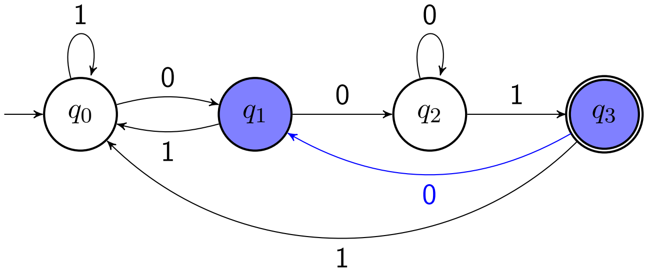

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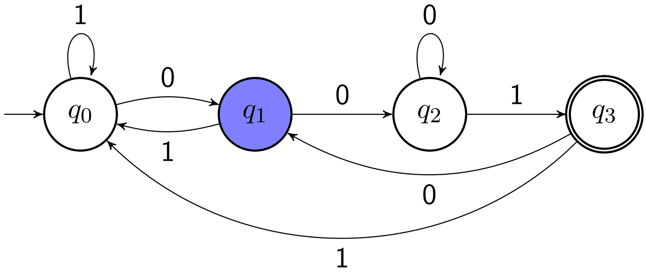

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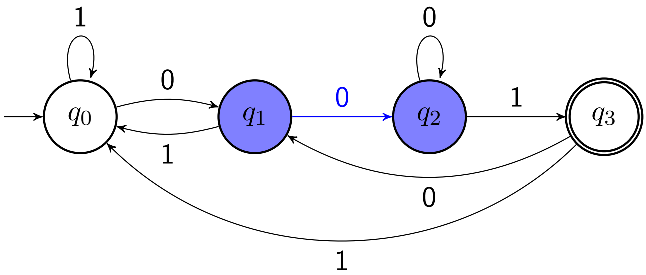

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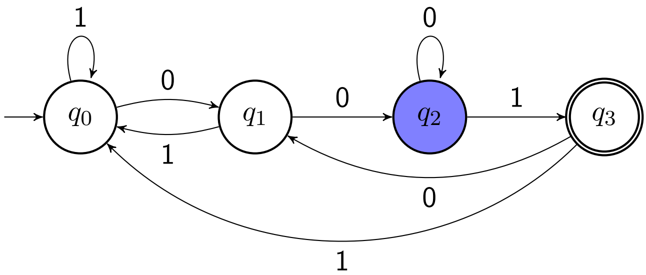

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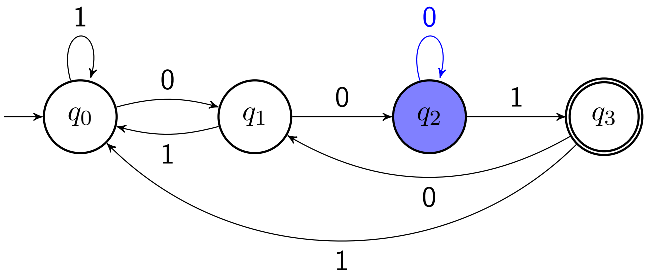

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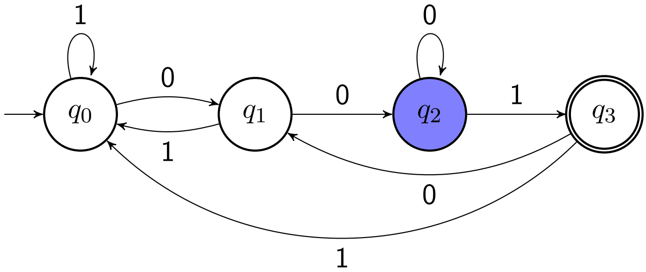

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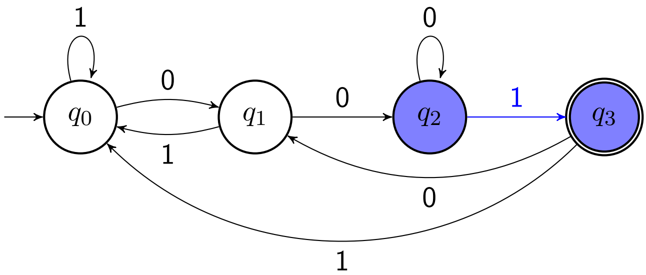

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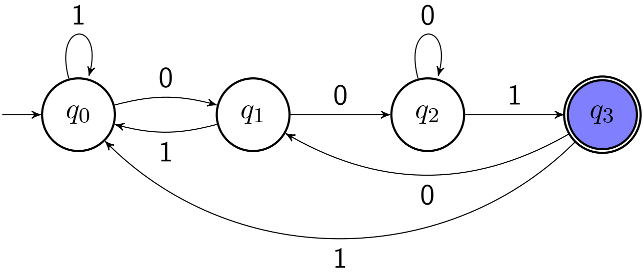

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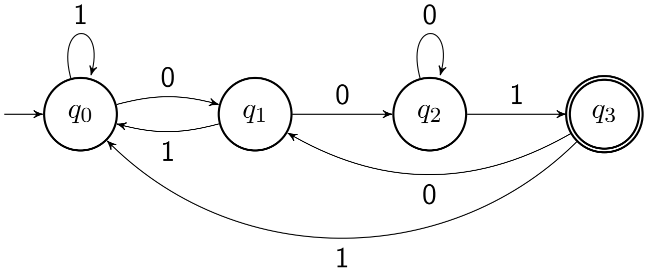
Deterministic finite state automata

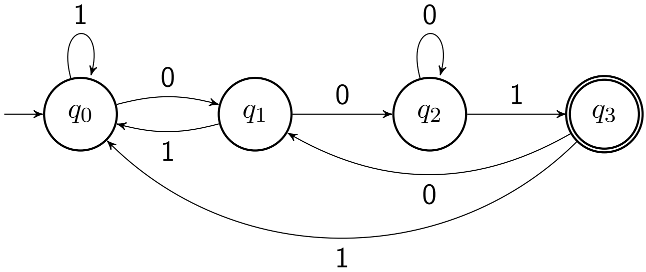
A *deterministic finite state automaton* is a 5-tuple $A = \langle Q, \Sigma, \delta, q_0, F \rangle$, where

- Q is an alphabet, the elements of which are called *states*;
- Σ is an alphabet, the elements of which are called *input symbols*;
- $\delta \in Q^{Q \times \Sigma}$ is called the *transition function*;
- $q_0 \in Q$ is called the *initial state*;
- $F \subset Q$ is a set of *final states*.

Deterministic finite state automata



Deterministic finite state automata



$$Q = \{q_0, q_1, q_2, q_3\}$$
$$\Sigma = \{0, 1\}$$
$$F = \{q_3\}$$

$$\delta(q_0, 0) = q_1 \quad \delta(q_2, 0) = q_2$$
$$\delta(q_0, 1) = q_0 \quad \delta(q_2, 1) = q_3$$
$$\delta(q_1, 0) = q_2 \quad \delta(q_3, 0) = q_1$$
$$\delta(q_1, 1) = q_0 \quad \delta(q_3, 1) = q_0$$

Deterministic finite state automata

Extended transition function:

$$\hat{\delta} \in Q^{Q \times \Sigma^*}$$

$$\begin{aligned} \hat{\delta}(q, \epsilon) &= q \\ \hat{\delta}(q, a\alpha) &= \hat{\delta}(\delta(q, a), \alpha) \end{aligned}$$

Deterministic finite state automata

Extended transition function:

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$$\begin{array}{rcl} \hat{\delta}(q, \epsilon) & = & q \\ \hat{\delta}(q, a\alpha) & = & \hat{\delta}(\delta(q, a), \alpha) \end{array}$$

Language accepted by a DFSA:

$$L(A) = \{\alpha \in \Sigma^* : \hat{\delta}(q_0, \alpha) \in F\}$$

Exercice

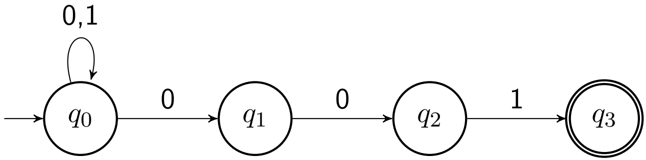
Design a DFSA that recognizes the natural numbers divisible by 3.

Non-deterministic finite state automata

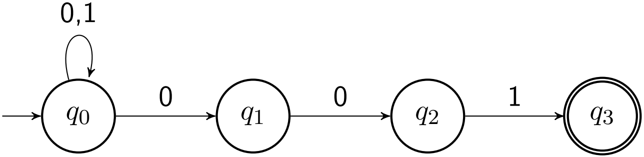
A *non-deterministic finite state automaton* is a 5-tuple $A = \langle Q, \Sigma, \delta, q_0, F \rangle$, where

- Q is an alphabet, the elements of which are called *states*;
- Σ is an alphabet, the elements of which are called *input symbols*;
- $\delta \in (2^Q)^{Q \times \Sigma}$ is a *transition function*;
- $q_0 \in Q$ is called the *initial state*;
- $F \subset Q$ is a set of *final states*.

Non-deterministic finite state automata



Non-deterministic finite state automata



$Q = \{q_0, q_1, q_2, q_3\}$
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$\delta(q_0, 0) = \{q_0, q_1\}$
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$\delta(q_1, 1) = \varnothing$

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Non-deterministic finite state automata

Extended transition function:

$$\hat{\delta} \in (2^Q)^{Q \times \Sigma^*}$$

$$\begin{array}{rcl} \hat{\delta}(q, \epsilon) & = & \{q\} \\ \hat{\delta}(q, a\alpha) & = & \bigcup_{p \in \delta(q,a)} \hat{\delta}(p, \alpha) \end{array}$$

Non-deterministic finite state automata

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Language accepted by an NFSA:

$$L(A) = \{\alpha \in \Sigma^* : \hat{\delta}(q_0, \alpha) \cap F \neq \varnothing\}$$

Non-deterministic finite state automata

Extended transition function:

$$\hat{\delta} \in (2^Q)^{Q \times \Sigma^*}$$

$$\begin{aligned} \hat{\delta}(q, \epsilon) &= \{q\} \\ \hat{\delta}(q, a\alpha) &= \bigcup_{p \in \delta(q,a)} \hat{\delta}(p, \alpha) \end{aligned}$$

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Remark: every DFSA $A = \langle Q, \Sigma, \delta, q_0, F \rangle$ may be seen as an NFSA $A' = \langle Q, \Sigma, \delta', q_0, F \rangle$ by defining $\delta'(q, a) = \{\delta(q, a)\}$.

Equivalence of NFSA and DFSA

Let $A_N = \langle Q_N, \Sigma_N, \delta_N, q_N, F_N \rangle$ be an NFSA. Define a DFSA $A_D = \langle Q_D, \Sigma_D, \delta_D, q_D, F_D \rangle$ as follows:

- $Q_D = 2^{Q_N}$
- $\Sigma_D = \Sigma_N$
- $\delta_D(S, a) = \bigcup_{p \in S} \delta_N(p, a)$
- $q_D = \{q_N\}$
- $F_D = \{S \subset Q : S \cap F_N \neq \varnothing\}$

Equivalence of NFSA and DFSA

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Proposition $L(A_N) = L(A_D)$.

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- $q_D = \{q_N\}$
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Proposition $L(A_N) = L(A_D)$.

Corollary A language is accepted by an NFSA if and only if it is accepted by a DFSA.

Equivalence of NFSA and DFSA

PROOF:

Equivalence of NFSA and DFSA

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We prove by induction on the length of α that $\hat{\delta}_D(S, \alpha) = \bigcup_{q \in S} \hat{\delta}_N(q, \alpha)$

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Basis:

Equivalence of NFSA and DFSA

PROOF:

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Basis:

$$\hat{\delta}_D(S, \epsilon) = S$$

Equivalence of NFSA and DFSA

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Basis:

$$\begin{aligned} \hat{\delta}_D(S, \epsilon) &= S \\ &= \bigcup_{q \in S} \{q\} \end{aligned}$$

Equivalence of NFSA and DFSA

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Induction:

Equivalence of NFSA and DFSA

PROOF:

We prove by induction on the length of α that $\hat{\delta}_D(S, \alpha) = \bigcup_{q \in S} \hat{\delta}_N(q, \alpha)$

Basis:

$$\begin{aligned}\hat{\delta}_D(S, \epsilon) &= S \\ &= \bigcup_{q \in S} \{q\} \\ &= \bigcup_{q \in S} \hat{\delta}_N(q, \epsilon)\end{aligned}$$

Induction:

$$\hat{\delta}_D(S, a\alpha') = \hat{\delta}_D(\delta_D(S, a), \alpha')$$

Equivalence of NFSA and DFSA

PROOF:

We prove by induction on the length of α that $\hat{\delta}_D(S,\alpha) = \bigcup_{q\in S} \hat{\delta}_N(q,\alpha)$

Basis:

$$\begin{aligned}\hat{\delta}_D(S,\epsilon) &= S \\ &= \textstyle\bigcup_{q\in S}\{q\} \\ &= \textstyle\bigcup_{q\in S} \hat{\delta}_N(q,\epsilon)\end{aligned}$$

Induction:

$$\begin{aligned}\hat{\delta}_D(S,a\alpha') &= \hat{\delta}_D(\delta_D(S,a),\alpha') \\ &= \hat{\delta}_D(\textstyle\bigcup_{q\in S} \delta_N(q,a),\alpha') && \text{by definition of } \delta_D\end{aligned}$$

Equivalence of NFSA and DFSA

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We prove by induction on the length of α that $\hat{\delta}_D(S, \alpha) = \bigcup_{q \in S} \hat{\delta}_N(q, \alpha)$

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Induction:

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Equivalence of NFSA and DFSA

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Basis:

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Induction:

$$\begin{aligned}\hat{\delta}_D(S,a\alpha') &= \hat{\delta}_D(\delta_D(S,a),\alpha') \\ &= \hat{\delta}_D(\textstyle\bigcup_{q\in S} \delta_N(q,a),\alpha') && \text{by definition of } \delta_D \\ &= \textstyle\bigcup_{p\in \bigcup_{q\in S} \delta_N(q,a)} \hat{\delta}_N(p,\alpha') && \text{by induction hypothesis} \\ &= \textstyle\bigcup_{q\in S}(\bigcup_{p\in \delta_N(q,a)} \hat{\delta}_N(p,\alpha'))\end{aligned}$$

Equivalence of NFSA and DFSA

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We prove by induction on the length of α that $\hat{\delta}_D(S,\alpha) = \bigcup_{q\in S} \hat{\delta}_N(q,\alpha)$

Basis:

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Induction:

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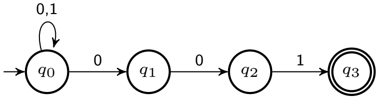
Equivalence of NFSA and DFSA



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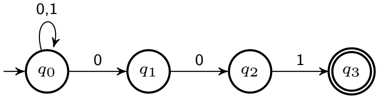

$$\delta_d(\varnothing,0)=\varnothing$$
$$\delta_d(\varnothing,1)=\varnothing$$
$$\delta_d(\{q_0\},0)=\{q_0,q_1\}$$
$$\delta_d(\{q_0\},1)=\{q_0\}$$
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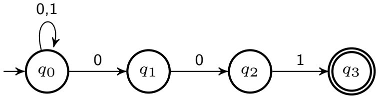
Equivalence of NFSA and DFSA



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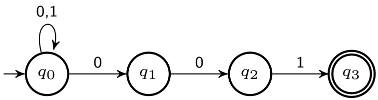

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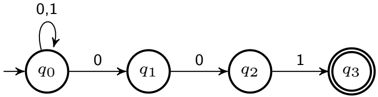

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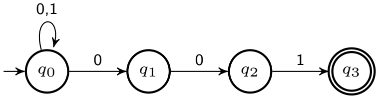
Equivalence of NFSA and DFSA



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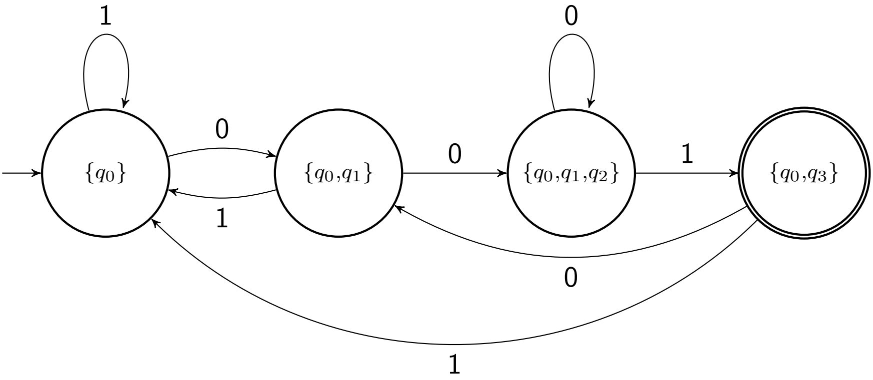
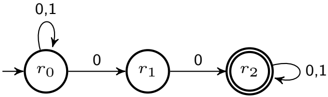


Finite state automata with ϵ-transitions

Finite state automata with ϵ-transitions

An automaton that recognizes binary words with at least two consecutive 0's.



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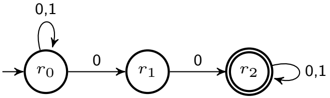
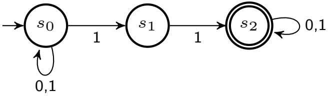


An automaton that recognizes binary words with at least two consecutive 1's.



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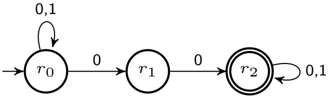

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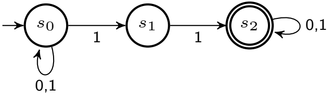
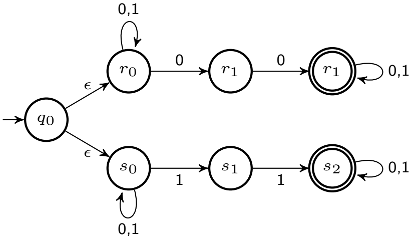


An automaton that recognizes binary words with at least two consecutive identical digits.



Finite state automata with ϵ-transitions

A *finite state automaton with ϵ-transitions* is a 5-tuple $A = \langle Q, \Sigma, \delta, q_0, F \rangle$, where:

- Q is an alphabet, the elements of which are called *states*;
- Σ is an alphabet, the elements of which are called *input symbols*;
- $\delta \in (2^Q)^{Q \times (\Sigma \cup \{\epsilon\})}$ is a *transition function*;
- $q_0 \in Q$ is called the *initial state*;
- $F \subset Q$ is a set of *final states*.

Finite state automata with ϵ-transitions

ϵ-closure of a state q:

- $q \in Cl_\epsilon(q)$;
- if $p \in Cl_\epsilon(q)$ and $s \in \delta(p, \epsilon)$ then $s \in Cl_\epsilon(q)$.

Finite state automata with ϵ-transitions

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Extended transition function:

$$\hat{\delta} \in (2^Q)^{Q \times \Sigma^*}$$

$$\begin{array}{rcl} \hat{\delta}(q, \epsilon) & = & Cl_\epsilon(q) \\ \hat{\delta}(q, a\alpha) & = & \bigcup_{p \in Cl_\epsilon(q)} \bigcup_{s \in \delta(p,a)} \hat{\delta}(s, \alpha) \end{array}$$

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Language accepted by a ϵ-FSA:

$$L(A) = \{\alpha \in \Sigma^* : \hat{\delta}(q_0, \alpha) \cap F \neq \varnothing\}$$

Elimination of ϵ-transitions

Let $A_E = \langle Q_E, \Sigma_E, \delta_E, q_E, F_E \rangle$ be an ϵ-FSA. Define an NFSA $A_N = \langle Q_N, \Sigma_N, \delta_N, q_N, F_N \rangle$ as follows:

- $Q_N = Q_E$
- $\Sigma_N = \Sigma_E$
- $\delta_N(q, a) = \bigcup_{p \in Cl_\epsilon(q)} \delta_E(p, a)$
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Proposition $L(A_E) = L(A_N)$.

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Corollary A language is accepted by an ϵ-FSA if and only if it is accepted by an NFSA.

Elimination of ϵ-transitions

PROOF:

Elimination of ϵ-transitions

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We prove by induction on the length of α that $\hat{\delta}_N(q, \alpha) \cap F_N \neq \varnothing$ if and only if $\hat{\delta}_E(q, \alpha) \cap F_E \neq \varnothing$.

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Basis:

$\hat{\delta}_N(q, \epsilon) \cap F_N \neq \varnothing$ iff $\{q\} \cap F_N \neq \varnothing$

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Elimination of ϵ-transitions

Induction:

Elimination of ϵ-transitions

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$\hat{\delta}_N(q, a\alpha') \cap F_N \neq \varnothing$ iff $(\bigcup_{s \in \delta_N(q,a)} \hat{\delta}_N(s, \alpha')) \cap F_N \neq \varnothing$

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by definition of δ_N

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by induction hypothesis

Elimination of ϵ-transitions

Induction:

$$\hat{\delta}_N(q, a\alpha') \cap F_N \neq \varnothing \text{ iff } (\bigcup_{s\in\delta_N(q,a)} \hat{\delta}_N(s,\alpha')) \cap F_N \neq \varnothing$$

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by induction hypothesis

$$\text{iff } (\bigcup_{p\in Cl_\epsilon(q)} \bigcup_{s\in\delta_E(p,a)} \hat{\delta}_E(s,\alpha')) \cap F_E \neq \varnothing$$

Elimination of ϵ-transitions

Induction:

$$\hat{\delta}_N(q, a\alpha') \cap F_N \neq \varnothing \text{ iff } (\bigcup_{s\in\delta_N(q,a)} \hat{\delta}_N(s,\alpha')) \cap F_N \neq \varnothing$$

$$\text{iff } (\bigcup_{s\in\bigcup_{p\in Cl_\epsilon(q)} \delta_E(p,a)} \hat{\delta}_N(s,\alpha')) \cap F_N \neq \varnothing$$

by definition of δ_N

$$\text{iff } (\bigcup_{p\in Cl_\epsilon(q)} \bigcup_{s\in\delta_E(p,a)} \hat{\delta}_N(s,\alpha')) \cap F_N \neq \varnothing$$

$$\text{iff } \bigcup_{p\in Cl_\epsilon(q)} \bigcup_{s\in\delta_E(p,a)} (\hat{\delta}_N(s,\alpha') \cap F_N) \neq \varnothing$$

$$\text{iff } \bigcup_{p\in Cl_\epsilon(q)} \bigcup_{s\in\delta_E(p,a)} (\hat{\delta}_E(s,\alpha') \cap F_E) \neq \varnothing$$

by induction hypothesis

$$\text{iff } (\bigcup_{p\in Cl_\epsilon(q)} \bigcup_{s\in\delta_E(p,a)} \hat{\delta}_E(s,\alpha')) \cap F_E \neq \varnothing$$

$$\text{iff } \hat{\delta}_E(q, a\alpha') \cap F_E \neq \varnothing$$