Towards a Montagovian Account of Dynamics

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Introduction

An old problem:

A man enters the room. He smiles.

$[\![$A man enters the room$]\!] = \exists x.\mathbf{man}(x) \wedge \mathbf{enters_the_room}(x)$. x is bound.

$[\![$He smiles$]\!] = \mathbf{smiles}(x)$. x is free.

How can we get from these:

$[\![$A man enters the room. He smiles$]\!]$
$= \exists x.\mathbf{man}(x) \wedge \mathbf{enters\_the\_room}(x) \wedge \mathbf{smiles}(x)$.

A well known solution: DRT.

- The reference markers of DRT act as existential quantifiers.
- Nevertheless, from a technical point of view, they must be considered as free variables.

Expressing propositions in context

> "The key idea behind (...) Discourse Representation Theory is that each new sentence of a discourse is interpreted in the context provided by the sentences preceding it."
>
> van Eijck and Kamp.
> Representing Discourse in Context.
> In *Handbook of Logic and Language*.
> Elsevier, 1997.

We go two steps further:

- We will interpret a sentence according to both its left and right contexts.
- These two kinds of contexts will be abstracted over the meaning of the sentences.

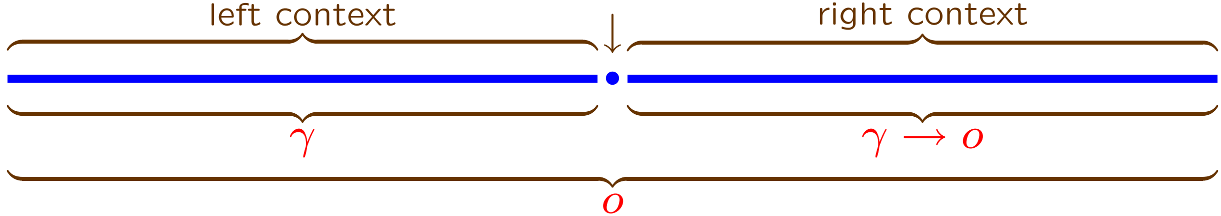
Typing the left and the right contexts

Montague semantics is based on Church's simple type theory, which provides a full hierarchy of functional types built upon two atomic types:

- ι, the type of individuals (a.k.a. entities).
- o, the type of propositions (a.k.a. truth values).

We add a third atomic type, γ, which stands for the type of the left contexts.

What about the type of the right contexts?



Semantic interpretation of the sentences

Let s be the syntactic category of sentences. Remember that we intend to abstract our notions of left and right contexts over the meaning of the sentences.

$$[\![s]\!] = \gamma \rightarrow (\gamma \rightarrow o) \rightarrow o$$

Composition of two sentence interpretations

$$[\![S_1 . S_2]\!] = \lambda e \phi . [\![S_1]\!] \, e \, (\lambda e' . [\![S_2]\!] \, e' \, \phi)$$

Note that this operation is associative!

Back to DRT and DRSs

Consider a DRS:

$$\begin{array}{|c|}\hline x_1 \ldots x_n \\ \hline C_1 \\ \vdots \\ C_m \\ \hline \end{array}$$

To such a structure, corresponds the following λ-term of type $\gamma \rightarrow \gamma \rightarrow o \rightarrow o$:

$$\lambda e\phi.\, \exists x_1 \ldots x_n.\, C_1 \wedge \cdots \wedge C_m \wedge \phi\, e'$$

where e' is a context made of e and of the variables $x_1, \ldots, x_n$.

Updating and accessing the context

John[1] loves Mary[2]. He$_1$ smiles at her$_2$.

$$\begin{array}{rcl} \texttt{nil} & : & \gamma \\ \texttt{push} & : & \mathbb{N} \to \iota \to \gamma \to \gamma \\ \texttt{sel} & : & \mathbb{N} \to \gamma \to \iota \end{array}$$

$$\texttt{sel}\, i\, (\texttt{push}\, j\, a\, l) = \begin{cases} a & \text{if } i = j \\ \texttt{sel}\, i\, l & \text{otherwise} \end{cases}$$

$$[\![\text{John}^1 \text{ loves Mary}^2]\!] = \lambda e\phi.\, \mathbf{love}\, \mathbf{j}\, \mathbf{m} \wedge \phi\, (\texttt{push}\, 2\, \mathbf{m}\, (\texttt{push}\, 1\, \mathbf{j}\, e))$$

$$[\![\text{He}_1 \text{ smiles at her}_2]\!] = \lambda e\phi.\, \mathbf{smile}\, (\texttt{sel}\, 1\, e)\, (\texttt{sel}\, 2\, e) \wedge \phi\, e$$

$$
\begin{aligned}
&\lambda e\phi.\, [\![\textsf{John}^1 \textsf{ loves Mary}^2]\!]\, e\, (\lambda e'.\, [\![\textsf{He}_1 \textsf{ smiles at her}_2]\!]\, e'\, \phi) \\
&\quad = \lambda e\phi.\, (\lambda e\phi.\, \mathbf{love}\, \mathbf{j}\, \mathbf{m} \wedge \phi\, (\mathrm{push}\, 2\, \mathbf{m}\, (\mathrm{push}\, 1\, \mathbf{j}\, e)))\, e\, (\lambda e'.\, [\![\textsf{He}_1 \textsf{ smiles at her}_2]\!]\, e'\, \phi) \\
&\quad \rightarrow_\beta \lambda e\phi.\, (\lambda \phi.\, \mathbf{love}\, \mathbf{j}\, \mathbf{m} \wedge \phi\, (\mathrm{push}\, 2\, \mathbf{m}\, (\mathrm{push}\, 1\, \mathbf{j}\, e)))\, (\lambda e'.\, [\![\textsf{He}_1 \textsf{ smiles at her}_2]\!]\, e'\, \phi) \\
&\quad \rightarrow_\beta \lambda e\phi.\, \mathbf{love}\, \mathbf{j}\, \mathbf{m} \wedge (\lambda e'.\, [\![\textsf{He}_1 \textsf{ smiles at her}_2]\!]\, e'\, \phi)\, (\mathrm{push}\, 2\, \mathbf{m}\, (\mathrm{push}\, 1\, \mathbf{j}\, e)) \\
&\quad \rightarrow_\beta \lambda e\phi.\, \mathbf{love}\, \mathbf{j}\, \mathbf{m} \wedge [\![\textsf{He}_1 \textsf{ smiles at her}_2]\!]\, (\mathrm{push}\, 2\, \mathbf{m}\, (\mathrm{push}\, 1\, \mathbf{j}\, e))\, \phi \\
&\quad = \lambda e\phi.\, \mathbf{love}\, \mathbf{j}\, \mathbf{m} \wedge (\lambda e\phi.\, \mathbf{smile}\, (\mathtt{sel}\, 1\, e)\, (\mathtt{sel}\, 2\, e) \wedge \phi\, e)\, (\mathrm{push}\, 2\, \mathbf{m}\, (\mathrm{push}\, 1\, \mathbf{j}\, e))\, \phi \\
&\quad \rightarrow_\beta \lambda e\phi.\, \mathbf{love}\, \mathbf{j}\, \mathbf{m} \wedge \\
&\qquad\qquad (\lambda \phi.\, \mathbf{smile}\, (\mathtt{sel}\, 1\, (\mathrm{push}\, 2\, \mathbf{m}\, (\mathrm{push}\, 1\, \mathbf{j}\, e)))\, (\mathtt{sel}\, 2\, (\mathrm{push}\, 2\, \mathbf{m}\, (\mathrm{push}\, 1\, \mathbf{j}\, e))) \wedge \\
&\qquad\qquad\quad \phi\, (\mathrm{push}\, 2\, \mathbf{m}\, (\mathrm{push}\, 1\, \mathbf{j}\, e)))\, \phi \\
&\quad \rightarrow_\beta \lambda e\phi.\, \mathbf{love}\, \mathbf{j}\, \mathbf{m} \wedge \\
&\qquad\qquad\quad \mathbf{smile}\, (\mathtt{sel}\, 1\, (\mathrm{push}\, 2\, \mathbf{m}\, (\mathrm{push}\, 1\, \mathbf{j}\, e)))\, (\mathtt{sel}\, 2\, (\mathrm{push}\, 2\, \mathbf{m}\, (\mathrm{push}\, 1\, \mathbf{j}\, e))) \wedge \\
&\qquad\qquad\qquad \phi\, (\mathrm{push}\, 2\, \mathbf{m}\, (\mathrm{push}\, 1\, \mathbf{j}\, e)) \\
&\quad = \lambda e\phi.\, \mathbf{love}\, \mathbf{j}\, \mathbf{m} \wedge \mathbf{smile}\, \mathbf{j}\, (\mathtt{sel}\, 2\, (\mathrm{push}\, 2\, \mathbf{m}\, (\mathrm{push}\, 1\, \mathbf{j}\, e))) \wedge \phi\, (\mathrm{push}\, 2\, \mathbf{m}\, (\mathrm{push}\, 1\, \mathbf{j}\, e)) \\
&\quad = \lambda e\phi.\, \mathbf{love}\, \mathbf{j}\, \mathbf{m} \wedge \mathbf{smile}\, \mathbf{j}\, \mathbf{m} \wedge \phi\, (\mathrm{push}\, 2\, \mathbf{m}\, (\mathrm{push}\, 1\, \mathbf{j}\, e))
\end{aligned}
$$

Assigning a semantics to the lexical entries

$$
\begin{aligned}
[\![s]\!] &= o \\
[\![n]\!] &= \iota \rightarrow o \\
[\![np]\!] &= (\iota \rightarrow o) \rightarrow o
\end{aligned}
$$

$$
\begin{aligned}
[\![s]\!] &= o && (1) \\
[\![n]\!] &= \iota \rightarrow [\![s]\!] && (2) \\
[\![np]\!] &= (\iota \rightarrow [\![s]\!]) \rightarrow [\![s]\!] && (3)
\end{aligned}
$$

Replacing (1) with:

$$[\![s]\!] = \gamma \rightarrow (\gamma \rightarrow o) \rightarrow o$$

we obtain:

$$
\begin{aligned}
[\![n]\!] &= \iota \rightarrow \gamma \rightarrow (\gamma \rightarrow o) \rightarrow o \\
[\![np]\!] &= (\iota \rightarrow \gamma \rightarrow (\gamma \rightarrow o) \rightarrow o) \rightarrow \gamma \rightarrow (\gamma \rightarrow o) \rightarrow o
\end{aligned}
$$

Nouns

$$[\![n]\!] = \iota \to \gamma \to (\gamma \to o) \to o$$

$$[\![\text{man}]\!] = \lambda x e \phi.\, \mathbf{man}\, x \;\wedge\; \phi\, e$$
$$[\![\text{woman}]\!] = \lambda x e \phi.\, \mathbf{woman}\, x \;\wedge\; \phi\, e$$
$$[\![\text{farmer}]\!] = \lambda x e \phi.\, \mathbf{farmer}\, x \;\wedge\; \phi\, e$$
$$[\![\text{donkey}]\!] = \lambda x e \phi.\, \mathbf{donkey}\, x \;\wedge\; \phi\, e$$

Noun phrases

$$[\![np]\!] = (\iota \to \gamma \to (\gamma \to o) \to o) \to \gamma \to (\gamma \to o) \to o$$

$$\begin{aligned}
[\![\text{John}^i]\!] &= \lambda\psi e\phi.\, \psi\, \mathbf{j}\, e\, (\lambda e.\, \phi\, (\mathrm{push}\, i\, \mathbf{j}\, e)) \\
[\![\text{Mary}^i]\!] &= \lambda\psi e\phi.\, \psi\, \mathbf{m}\, e\, (\lambda e.\, \phi\, (\mathrm{push}\, i\, \mathbf{m}\, e)) \\
[\![\text{he}_i]\!] &= \lambda\psi e\phi.\, \psi\, (\mathtt{sel}\, i\, e)\, e\, \phi \\
[\![\text{her}_i]\!] &= \lambda\psi e\phi.\, \psi\, (\mathtt{sel}\, i\, e)\, e\, \phi \\
[\![\text{it}_i]\!] &= \lambda\psi e\phi.\, \psi\, (\mathtt{sel}\, i\, e)\, e\, \phi
\end{aligned}$$

Determiners

$$[\![det]\!] = [\![n]\!] {\rightarrow} [\![np]\!]$$

$$[\![\text{a}^i]\!] = \lambda n \psi e \phi.\, \exists x.\, n\, x\, e\, (\lambda e.\, \psi\, x\, (\texttt{push}\, i\, x\, e)\, \phi)$$

$$[\![\text{every}^i]\!] = \lambda n \psi e \phi.\, (\forall x.\, \neg(n\, x\, e\, (\lambda e.\, \neg(\psi\, x\, (\texttt{push}\, i\, x\, e)\, (\lambda e.\, \top))))) \wedge \phi\, e$$

Transitive verbs

$$[\![tv]\!] = [\![np]\!] \longrightarrow [\![np]\!] \longrightarrow [\![s]\!]$$

$$[\![\text{loves}]\!] = \lambda os.\, s\,(\lambda x.\, o\,(\lambda ye\phi.\, \mathbf{love}\, x\, y \wedge \phi\, e))$$
$$[\![\text{owns}]\!] = \lambda os.\, s\,(\lambda x.\, o\,(\lambda ye\phi.\, \mathbf{own}\, x\, y \wedge \phi\, e))$$
$$[\![\text{beats}]\!] = \lambda os.\, s\,(\lambda x.\, o\,(\lambda ye\phi.\, \mathbf{beat}\, x\, y \wedge \phi\, e))$$

Relative pronouns

$$[\![rel]\!] = ([\![np]\!] \rightarrow [\![s]\!]) \rightarrow [\![n]\!] \rightarrow [\![n]\!]$$

$$[\![\text{who}]\!] = \lambda rnxe\phi.\, n\, x\, e\, (\lambda e.\, r\, (\lambda \psi.\, \psi\, x)\, e\, \phi)$$

$[\![\text{beats}]\!]\,[\![\text{it}_2]\!]\,([\![\text{every}^1]\!]\,([\![\text{who}]\!]\,([\![\text{owns}]\!]\,([\![\text{a}^2]\!]\,[\![\text{donkey}]\!]))\,[\![\text{farmer}]\!]))$

$[\![\text{a}^2]\!]\,[\![\text{donkey}]\!]$

$$
\begin{aligned}
&= (\lambda n\psi e\phi.\,\exists y.\,n\,y\,e\,(\lambda e.\,\psi\,y\,(\mathrm{push}\,2\,y\,e)\,\phi))\,[\![\text{donkey}]\!]\\
&\twoheadrightarrow_\beta \lambda\psi e\phi.\,\exists y.\,[\![\text{donkey}]\!]\,y\,e\,(\lambda e.\,\psi\,y\,(\mathrm{push}\,2\,y\,e)\,\phi)\\
&= \lambda\psi e\phi.\,\exists y.\,(\lambda x e\phi.\,\mathbf{donkey}\,x \wedge \phi\,e)\,y\,e\,(\lambda e.\,\psi\,y\,(\mathrm{push}\,2\,y\,e)\,\phi)\\
&\twoheadrightarrow_\beta \lambda\psi e\phi.\,\exists y.\,\mathbf{donkey}\,y \wedge (\lambda e.\,\psi\,y\,(\mathrm{push}\,2\,y\,e)\,\phi)\,e\\
&\twoheadrightarrow_\beta \lambda\psi e\phi.\,\exists y.\,\mathbf{donkey}\,y \wedge \psi\,y\,(\mathrm{push}\,2\,y\,e)\,\phi
\end{aligned}
$$

$[\![\text{owns}]\!]\,([\![\text{a}^2]\!]\,[\![\text{donkey}]\!])$

$$
\begin{aligned}
&= [\![\text{owns}]\!]\,(\lambda\psi e\phi.\,\exists y.\,\mathbf{donkey}\,y \wedge \psi\,y\,(\mathrm{push}\,2\,y\,e)\,\phi)\\
&= (\lambda o s.\,s\,(\lambda x.\,o\,(\lambda y e\phi.\,\mathbf{own}\,x\,y \wedge \phi\,e)))\,(\lambda\psi e\phi.\,\exists y.\,\mathbf{donkey}\,y \wedge \psi\,y\,(\mathrm{push}\,2\,y\,e)\,\phi)\\
&\twoheadrightarrow_\beta \lambda s.\,s\,(\lambda x.\,(\lambda\psi e\phi.\,\exists y.\,\mathbf{donkey}\,y \wedge \psi\,y\,(\mathrm{push}\,2\,y\,e)\,\phi)\,(\lambda y e\phi.\,\mathbf{own}\,x\,y \wedge \phi\,e))\\
&\twoheadrightarrow_\beta \lambda s.\,s\,(\lambda x e\phi.\,\exists y.\,\mathbf{donkey}\,y \wedge (\lambda y e\phi.\,\mathbf{own}\,x\,y \wedge \phi\,e)\,y\,(\mathrm{push}\,2\,y\,e)\,\phi)\\
&\twoheadrightarrow_\beta \lambda s.\,s\,(\lambda x e\phi.\,\exists y.\,\mathbf{donkey}\,y \wedge \mathbf{own}\,x\,y \wedge \phi\,(\mathrm{push}\,2\,y\,e))
\end{aligned}
$$

$$
\begin{aligned}
&[\![\text{who}]\!]\;([\![\text{owns}]\!]\;([\![\text{a}^2]\!]\;[\![\text{donkey}]\!]))\\
&\quad = \;[\![\text{who}]\!]\;(\lambda s.\, s\,(\lambda x e \phi.\, \exists y.\, \mathbf{donkey}\, y \;\wedge\; \mathbf{own}\, x\, y \;\wedge\; \phi\,(\mathrm{push}\, 2\, y\, e)))\\
&\quad = \;(\lambda r n x e \phi.\, n\, x\, e\,(\lambda e.\, r\,(\lambda \psi.\, \psi\, x)\, e\, \phi))\\
&\qquad\qquad (\lambda s.\, s\,(\lambda x e \phi.\, \exists y.\, \mathbf{donkey}\, y \;\wedge\; \mathbf{own}\, x\, y \;\wedge\; \phi\,(\mathrm{push}\, 2\, y\, e)))\\
&\quad \twoheadrightarrow_\beta \lambda n x e \phi.\, n\, x\, e\,(\lambda e.\\
&\qquad\quad (\lambda s.\, s\,(\lambda x e \phi.\, \exists y.\, \mathbf{donkey}\, y \;\wedge\; \mathbf{own}\, x\, y \;\wedge\; \phi\,(\mathrm{push}\, 2\, y\, e)))\,(\lambda \psi.\, \psi\, x)\, e\, \phi)\\
&\quad \twoheadrightarrow_\beta \lambda n x e \phi.\, n\, x\, e\,(\lambda e.\, (\lambda \psi.\, \psi\, x)\,(\lambda x e \phi.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(\mathrm{push}\, 2\, y\, e))\, e\, \phi)\\
&\quad \twoheadrightarrow_\beta \lambda n x e \phi.\, n\, x\, e\,(\lambda e.\, (\lambda x e \phi.\, \exists y.\, \mathbf{donkey}\, y \;\wedge\; \mathbf{own}\, x\, y \;\wedge\; \phi\,(\mathrm{push}\, 2\, y\, e))\, x\, e\, \phi)\\
&\quad \twoheadrightarrow_\beta \lambda n x e \phi.\, n\, x\, e\,(\lambda e.\, \exists y.\, \mathbf{donkey}\, y \;\wedge\; \mathbf{own}\, x\, y \;\wedge\; \phi\,(\mathrm{push}\, 2\, y\, e))
\end{aligned}
$$

$$
\begin{aligned}
&[\![\text{who}]\!]\;([\![\text{owns}]\!]\;([\![\text{a}^2]\!]\;[\![\text{donkey}]\!]))\;[\![\text{farmer}]\!]\\
&\quad = \;(\lambda n x e \phi.\, n\, x\, e\,(\lambda e.\, \exists y.\, \mathbf{donkey}\, y \;\wedge\; \mathbf{own}\, x\, y \;\wedge\; \phi\,(\mathrm{push}\, 2\, y\, e)))\;[\![\text{farmer}]\!]\\
&\quad \twoheadrightarrow_\beta \lambda x e \phi.\, [\![\text{farmer}]\!]\, x\, e\,(\lambda e.\, \exists y.\, \mathbf{donkey}\, y \;\wedge\; \mathbf{own}\, x\, y \;\wedge\; \phi\,(\mathrm{push}\, 2\, y\, e))\\
&\quad = \;\lambda x e \phi.\, (\lambda x e \phi.\, \mathbf{farmer}\, x \wedge \phi\, e)\, x\, e\,(\lambda e.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(\mathrm{push}\, 2\, y\, e))\\
&\quad \twoheadrightarrow_\beta \lambda x e \phi.\, \mathbf{farmer}\, x \;\wedge\; (\lambda e.\, \exists y.\, \mathbf{donkey}\, y \;\wedge\; \mathbf{own}\, x\, y \;\wedge\; \phi\,(\mathrm{push}\, 2\, y\, e))\, e\\
&\quad \twoheadrightarrow_\beta \lambda x e \phi.\, \mathbf{farmer}\, x \;\wedge\; (\exists y.\, \mathbf{donkey}\, y \;\wedge\; \mathbf{own}\, x\, y \;\wedge\; \phi\,(\mathrm{push}\, 2\, y\, e))
\end{aligned}
$$

$$
\begin{array}{l}
[\![\text{every}^1]\!]\,([\![\text{who}]\!]\,([\![\text{owns}]\!]\,([\![\text{a}^2]\!]\,[\![\text{donkey}]\!]))\,[\![\text{farmer}]\!])\\
\quad = \;\; [\![\text{every}^1]\!]\,(\lambda xe\phi.\,\mathbf{farmer}\,x \wedge (\exists y.\,\mathbf{donkey}\,y \wedge \mathbf{own}\,x\,y \wedge \phi\,(\mathrm{push}\,2\,y\,e)))\\
\quad = \;\; (\lambda n\psi e\phi.\,(\forall x.\,\neg(n\,x\,e\,(\lambda e.\,\neg(\psi\,x\,(\mathrm{push}\,1\,x\,e)\,(\lambda e.\,\top))))) \wedge \phi\,e)\\
\qquad\qquad (\lambda xe\phi.\,\mathbf{farmer}\,x \wedge (\exists y.\,\mathbf{donkey}\,y \wedge \mathbf{own}\,x\,y \wedge \phi\,(\mathrm{push}\,2\,y\,e)))\\
\quad \twoheadrightarrow_\beta \lambda\psi e\phi.\,(\forall x.\,\neg(\\
\qquad\qquad (\lambda xe\phi.\,\mathbf{farmer}\,x \wedge (\exists y.\,\mathbf{donkey}\,y \wedge \mathbf{own}\,x\,y \wedge \phi\,(\mathrm{push}\,2\,y\,e)))\\
\qquad\qquad\quad x\,e\,(\lambda e.\,\neg(\psi\,x\,(\mathrm{push}\,1\,x\,e)\,(\lambda e.\,\top))))) \wedge \phi\,e\\
\quad \twoheadrightarrow_\beta \lambda\psi e\phi.\,(\forall x.\,\neg(\mathbf{farmer}\,x \wedge (\exists y.\,\mathbf{donkey}\,y \wedge \mathbf{own}\,x\,y \wedge\\
\qquad\qquad (\lambda e.\,\neg(\psi\,x\,(\mathrm{push}\,1\,x\,e)\,(\lambda e.\,\top)))\,(\mathrm{push}\,2\,y\,e)))) \wedge \phi\,e\\
\quad \twoheadrightarrow_\beta \lambda\psi e\phi.\,(\forall x.\,\neg(\mathbf{farmer}\,x \wedge (\exists y.\,\mathbf{donkey}\,y \wedge \mathbf{own}\,x\,y \wedge\\
\qquad\qquad \neg(\psi\,x\,(\mathrm{push}\,1\,x\,(\mathrm{push}\,2\,y\,e))\,(\lambda e.\,\top))))) \wedge \phi\,e
\end{array}
$$

$$
\begin{array}{l}
[\![\text{beats}]\!]\,[\![\text{it}_2]\!]\\
\quad = \;\; (\lambda os.\,s\,(\lambda x.\,o\,(\lambda ye\phi.\,\mathbf{beat}\,x\,y \wedge \phi\,e)))\,[\![\text{it}_2]\!]\\
\quad \twoheadrightarrow_\beta \lambda s.\,s\,(\lambda x.\,[\![\text{it}_2]\!]\,(\lambda ye\phi.\,\mathbf{beat}\,x\,y \wedge \phi\,e))\\
\quad = \;\; \lambda s.\,s\,(\lambda x.\,(\lambda\psi e\phi.\,\psi\,(\mathtt{sel}\,2\,e)\,e\,\phi)\,(\lambda ye\phi.\,\mathbf{beat}\,x\,y \wedge \phi\,e))\\
\quad \twoheadrightarrow_\beta \lambda s.\,s\,(\lambda xe\phi.\,(\lambda ye\phi.\,\mathbf{beat}\,x\,y \wedge \phi\,e)\,(\mathtt{sel}\,2\,e)\,e\,\phi)\\
\quad \twoheadrightarrow_\beta \lambda s.\,s\,(\lambda xe\phi.\,\mathbf{beat}\,x\,(\mathtt{sel}\,2\,e) \wedge \phi\,e)
\end{array}
$$

$$
\begin{aligned}
&[\![\mathsf{beats}]\!]\ [\![\mathsf{it}_2]\!]\ ([\![\mathsf{every}^1]\!]\ ([\![\mathsf{who}]\!]\ ([\![\mathsf{owns}]\!]\ ([\![\mathsf{a}^2]\!]\ [\![\mathsf{donkey}]\!]))\ [\![\mathsf{farmer}]\!]))\\
&\quad = \ (\lambda s.\, s\, (\lambda x e \phi.\, \mathbf{beat}\, x\, (\mathtt{sel}\, 2\, e)\ \wedge\ \phi\, e))\\
&\qquad\qquad ([\![\mathsf{every}^1]\!]\ ([\![\mathsf{who}]\!]\ ([\![\mathsf{owns}]\!]\ ([\![\mathsf{a}^2]\!]\ [\![\mathsf{donkey}]\!]))\ [\![\mathsf{farmer}]\!]))\\
&\quad \twoheadrightarrow_\beta [\![\mathsf{every}^1]\!]\ ([\![\mathsf{who}]\!]\ ([\![\mathsf{owns}]\!]\ ([\![\mathsf{a}^2]\!]\ [\![\mathsf{donkey}]\!]))\ [\![\mathsf{farmer}]\!])\\
&\qquad\qquad (\lambda x e \phi.\, \mathbf{beat}\, x\, (\mathtt{sel}\, 2\, e)\ \wedge\ \phi\, e)\\
&\quad = \ (\lambda \psi e \phi.\, (\forall x.\, \neg(\mathbf{farmer}\, x\ \wedge\ (\exists y.\, \mathbf{donkey}\, y\ \wedge\ \mathbf{own}\, x\, y\ \wedge\\
&\qquad\qquad\quad \neg(\psi\, x\, (\mathrm{push}\, 1\, x\, (\mathrm{push}\, 2\, y\, e))\, (\lambda e.\, \top)))))\ \wedge\ \phi\, e)\\
&\qquad\qquad (\lambda x e \phi.\, \mathbf{beat}\, x\, (\mathtt{sel}\, 2\, e)\ \wedge\ \phi\, e)\\
&\quad \twoheadrightarrow_\beta \lambda e \phi.\, (\forall x.\, \neg(\mathbf{farmer}\, x\ \wedge\ (\exists y.\, \mathbf{donkey}\, y\ \wedge\ \mathbf{own}\, x\, y\ \wedge\\
&\qquad\qquad \neg((\lambda x e \phi.\, \mathbf{beat}\, x\, (\mathtt{sel}\, 2\, e) \wedge \phi\, e)\, x\, (\mathrm{push}\, 1\, x\, (\mathrm{push}\, 2\, y\, e))\, (\lambda e.\, \top))))) \wedge \phi\, e\\
&\quad \twoheadrightarrow_\beta \lambda e \phi.\, (\forall x.\, \neg(\mathbf{farmer}\, x\ \wedge\ (\exists y.\, \mathbf{donkey}\, y\ \wedge\ \mathbf{own}\, x\, y\ \wedge\\
&\qquad\qquad \neg(\mathbf{beat}\, x\, (\mathtt{sel}\, 2\, (\mathrm{push}\, 1\, x\, (\mathrm{push}\, 2\, y\, e))) \wedge (\lambda e.\, \top)\, (\mathrm{push}\, 1\, x\, (\mathrm{push}\, 2\, y\, e))))))\\
&\qquad\quad \wedge\ \phi\, e\\
&\quad \twoheadrightarrow_\beta \lambda e \phi.\, (\forall x.\, \neg(\mathbf{farmer}\, x\ \wedge\ (\exists y.\, \mathbf{donkey}\, y\ \wedge\ \mathbf{own}\, x\, y\ \wedge\\
&\qquad\qquad \neg(\mathbf{beat}\, x\, (\mathtt{sel}\, 2\, (\mathrm{push}\, 1\, x\, (\mathrm{push}\, 2\, y\, e)))\ \wedge\ \top))))\ \wedge\ \phi\, e\\
&\quad = \ \lambda e \phi.\, (\forall x.\, \neg(\mathbf{farmer}\, x \wedge (\exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \neg(\mathbf{beat}\, x\, y\ \wedge\ \top))))\ \wedge\ \phi\, e\\
&\quad \equiv \ \lambda e \phi.\, (\forall x.\, \mathbf{farmer}\, x\ \supset\ (\forall y.\, (\mathbf{donkey}\, y\ \wedge\ \mathbf{own}\, x\, y)\ \supset\ \mathbf{beat}\, x\, y))\ \wedge\ \phi\, e
\end{aligned}
$$