From PDA to CFG

Let $P = \langle Q, \Sigma, \Gamma, \delta, q_0, Z_0, F \rangle$ be a PDA. Define a context-free grammar $G = \langle N, \Sigma, P, S \rangle$ as follows:

- $N = \{S\} \cup \{[pXq] : p, q \in Q \wedge X \in \Gamma\}$;
- P contains the following rules:
  - for every $p \in Q$:
    $$S \to [q_0 Z_0 p]$$
  - for every $(r, \epsilon) \in \delta(q, a, Y)$:
    $$[qYr] \to a$$
    where $r_0, r_1, \ldots r_{n-2}, r_{n-1} \in Q$
  - for every $(r, Y_1 Y_2 \ldots Y_n) \in \delta(q, a, Y)$:
    $$[qYr_n] \to a[rY_1r_1][r_1Y_2r_2] \ldots [r_{n-1}Y_nr_n]$$
    where $r_0, r_1, \ldots r_{n-2}, r_{n-1} \in Q$

PDA example

$P = \langle Q, \Sigma, \Gamma, \delta, q_0, Z_0, F \rangle$ where

- $Q = \{q, p\}$
- $\Sigma = \{a, b\}$
- $\Gamma = \{Z, A\}$
- $\delta(q, a, Z) = \{(q, AZ)\}$
  $\delta(q, a, A) = \{(q, AA)\}$
  $\delta(q, \epsilon, A) = \{(p, A)\}$
  $\delta(p, b, A) = \{(p, \epsilon)\}$
  $\delta(p, \epsilon, Z) = \{(p, \epsilon)\}$
- $q_0 = q$
- $Z_0 = Z$
- $F = \{p\}$

Corresponding CFG

$$
\begin{aligned}
S &\rightarrow [qZq] \\
S &\rightarrow [qZp]
\end{aligned}
$$

$$
\begin{aligned}
[qZq] &\rightarrow a\,[qAq]\,[qZq] \\
[qZq] &\rightarrow a\,[qAp]\,[pZq] \\
[qZp] &\rightarrow a\,[qAq]\,[qZp] \\
[qZp] &\rightarrow a\,[qAp]\,[pZp]
\end{aligned}
$$

$$
\begin{aligned}
[qAq] &\rightarrow a\,[qAq]\,[qAq] \\
[qAq] &\rightarrow a\,[qAp]\,[pAq] \\
[qAp] &\rightarrow a\,[qAq]\,[qAp] \\
[qAp] &\rightarrow a\,[qAp]\,[pAp]
\end{aligned}
$$

$$
\begin{aligned}
[qAq] &\rightarrow [pAq] \\
[qAp] &\rightarrow [pAp]
\end{aligned}
$$

$$[pAp] \rightarrow b$$

$$[pZp] \rightarrow \epsilon$$

Reduced CFG (elimination of non-productive rules)

$$S \rightarrow [qZp]$$

$$[qZp] \rightarrow a\,[qAp]\,[pZp]$$

$$[qAp] \rightarrow a\,[qAp]\,[pAp]$$

$$[qAp] \rightarrow [pAp]$$

$$[pAp] \rightarrow b$$

$$[pZp] \rightarrow \epsilon$$

Paralleling transitions and productions

Automaton	*Grammar*
$(q, aabb, Z) \vdash (q, abb, AZ)$	$S \Rightarrow [qZp] \Rightarrow a\,[qAp]\,[pZp]$
$\vdash (q, bb, AAZ)$	$\Rightarrow aa\,[qAp]\,[pAp]\,[pZp]$
$\vdash (p, bb, AAZ)$	$\Rightarrow aa\,[pAp]\,[pAp]\,[pZp]$
$\vdash (p, b, AZ)$	$\Rightarrow aab\,[pAp]\,[pZp]$
$\vdash (p, \epsilon, Z)$	$\Rightarrow aabb\,[pZp]$
$\vdash (p, \epsilon, \epsilon)$	$\Rightarrow aabb$