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I. Introduction

- Artificial Intelligence
- Machine Learning
- Neural Network and Deep Learning
- Applications

II. Background

III. Fitting a Model

IV. Supervised Learning

V. Unsupervised Learning

VI. Fantastic DNNs: How to choose them, how to train them

VII. Machine Learning in Robot Audition

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- Multivalued Multivariate Functions
- Tensors
- Differential Calculus
- Exercises

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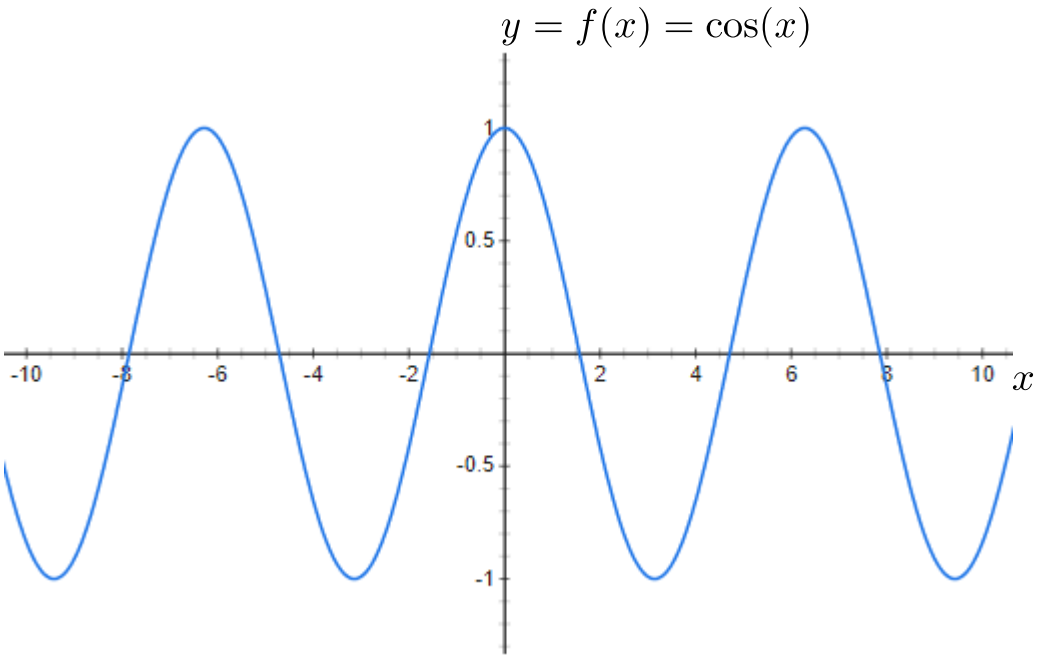

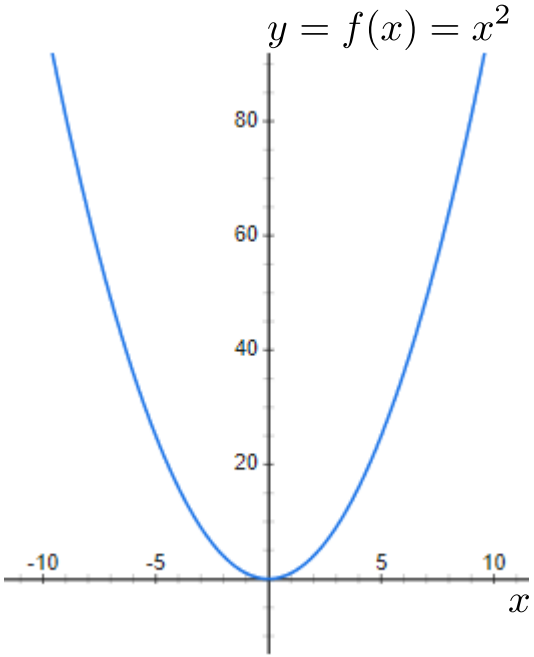

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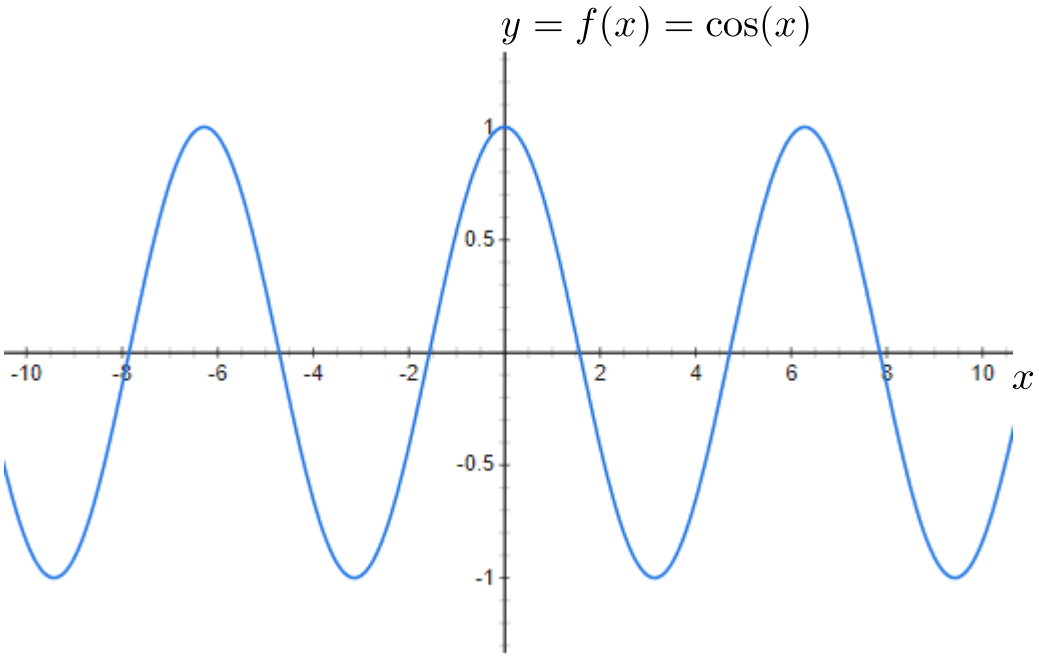

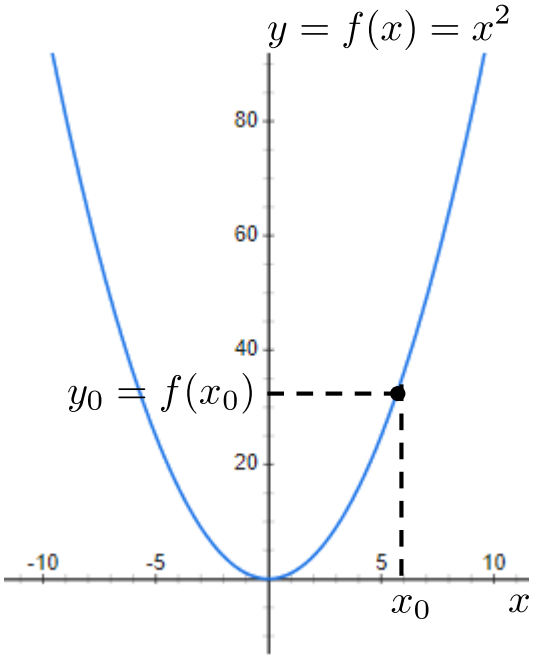

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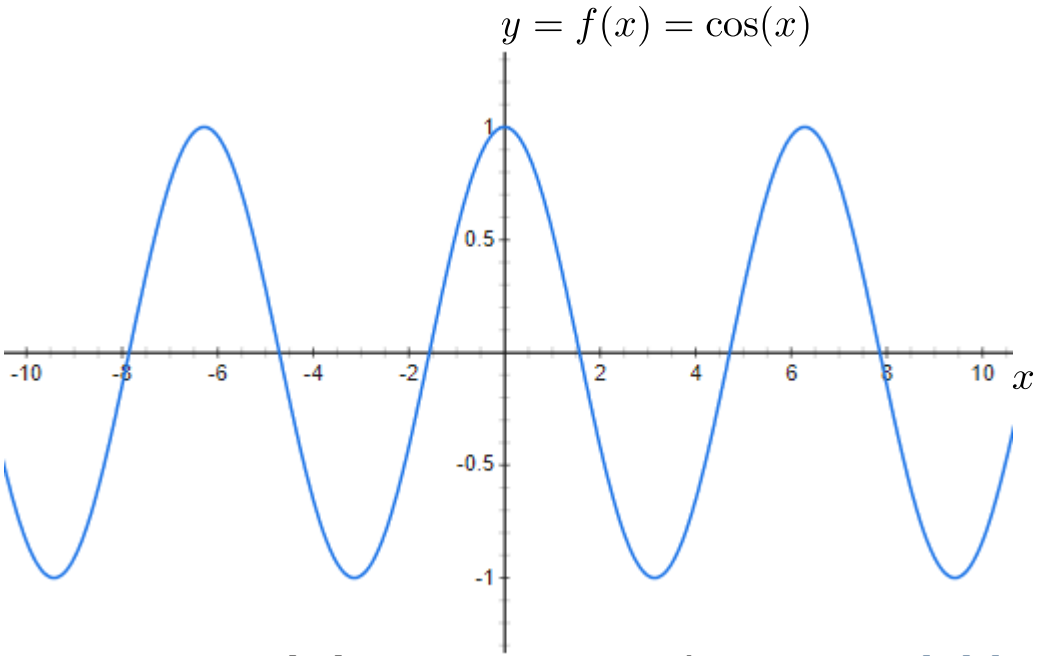

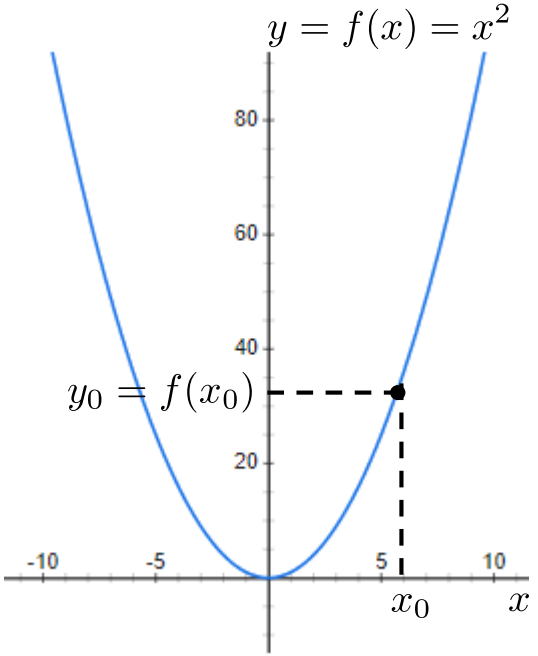

- **Important subtlety:** Here, x and y are ***variables*** that ***depend*** on each other, x_0 and y_0 are ***constants***, and f is a ***function*** that can be ***applied*** to a variable or a constant.

Multivalued functions

- This straightforwardly generalizes to **multivalued** functions $f : \mathbb{R} \to \mathbb{R}^N$:

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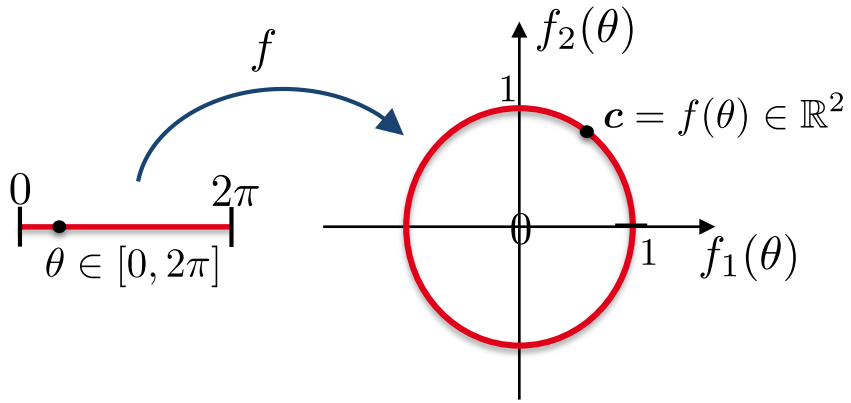

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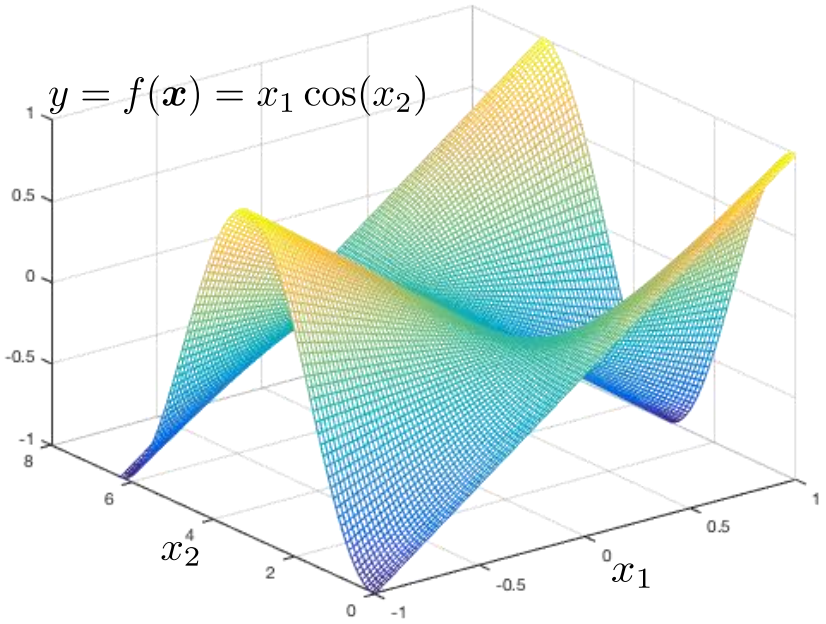

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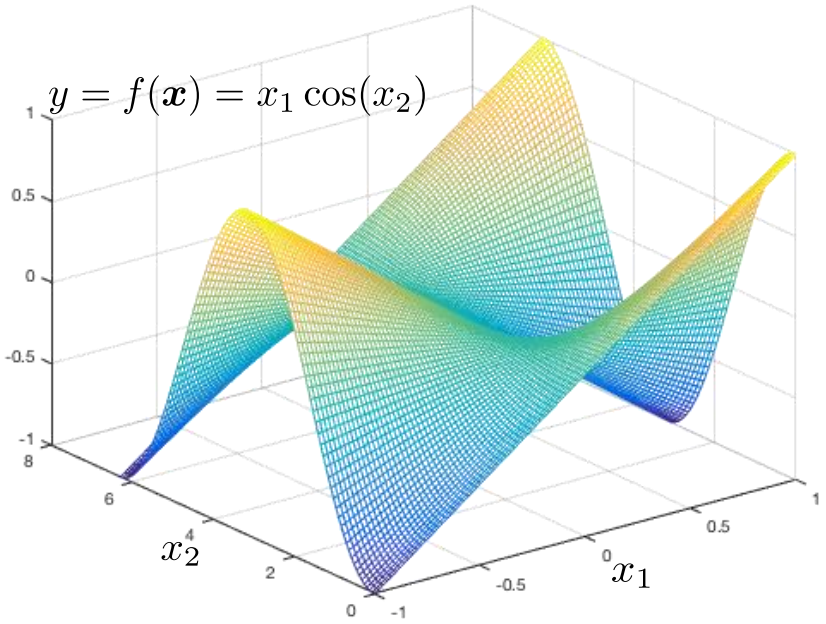

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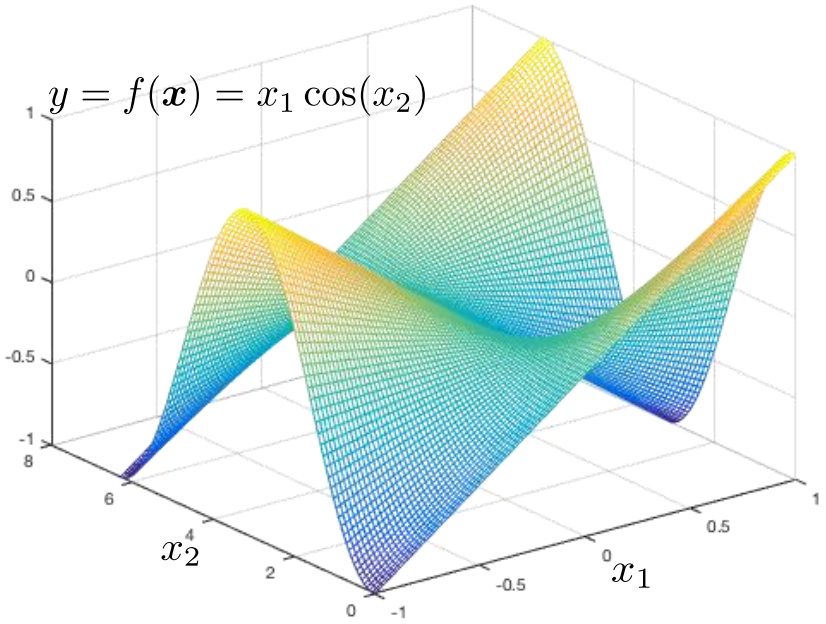

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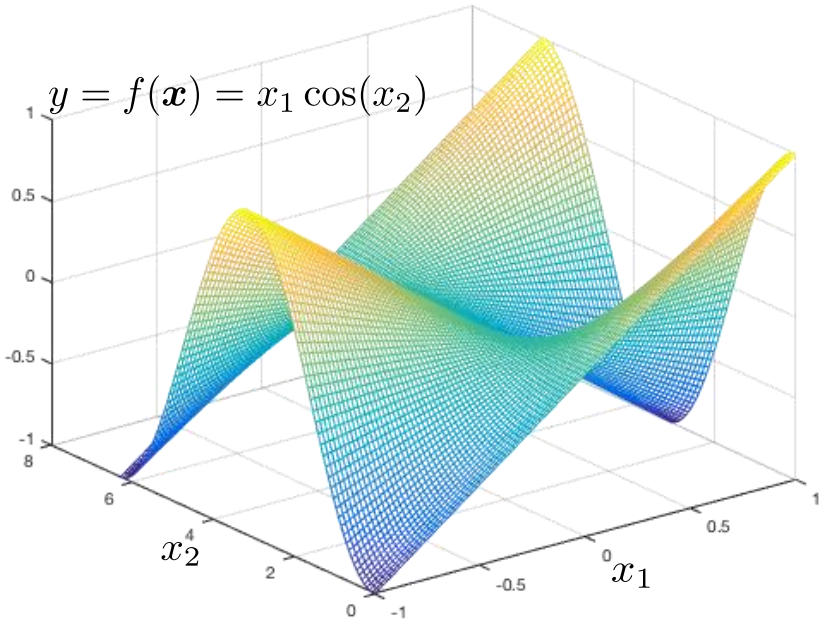

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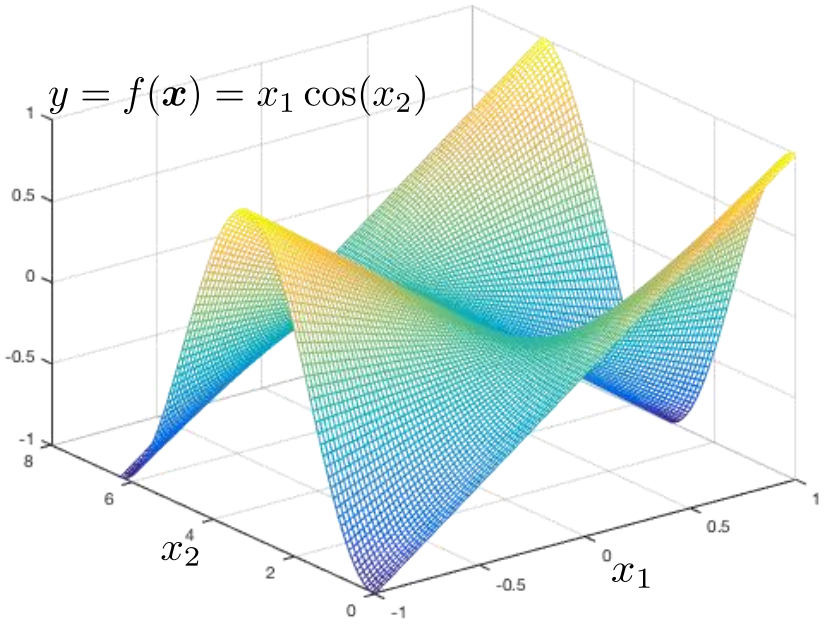

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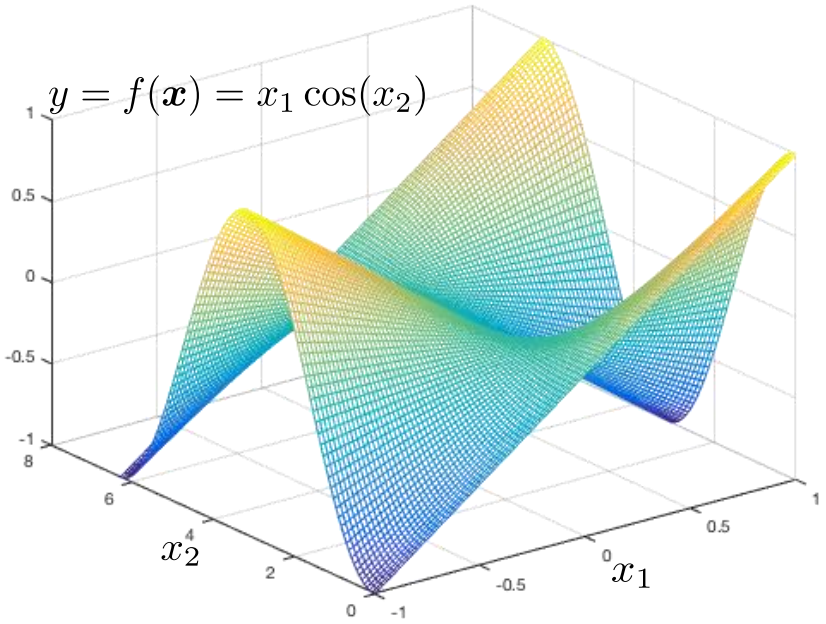

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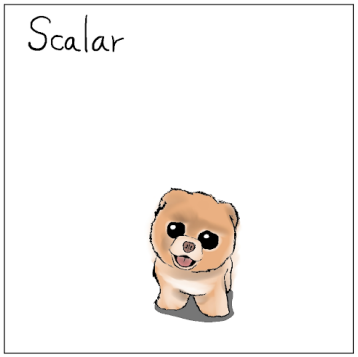

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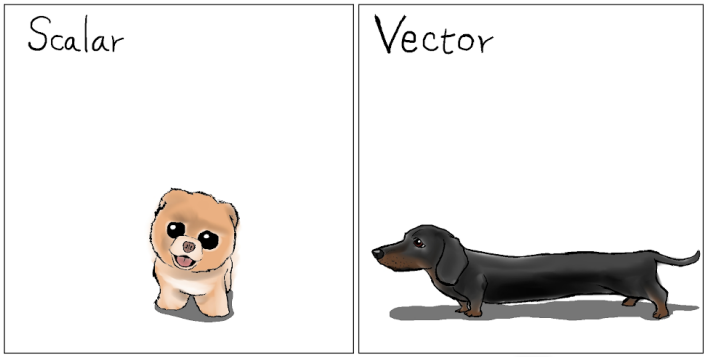

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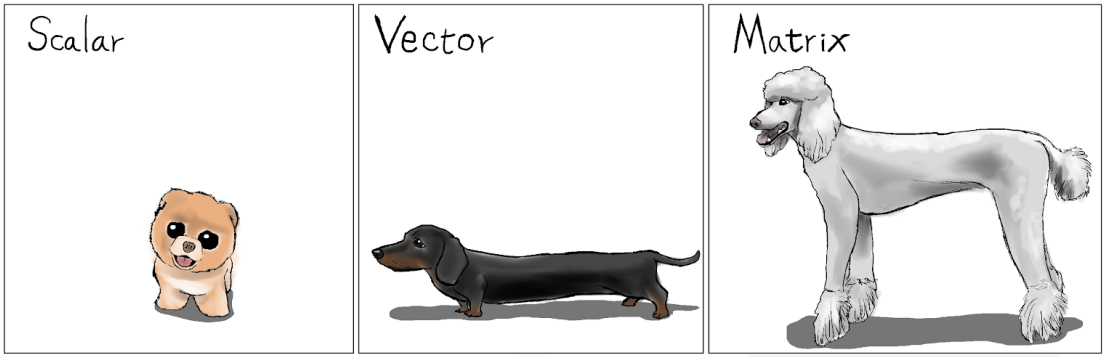

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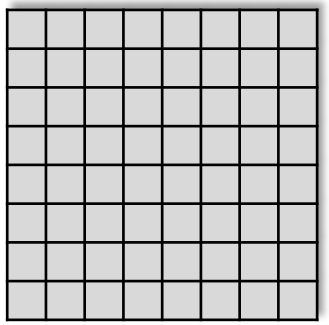
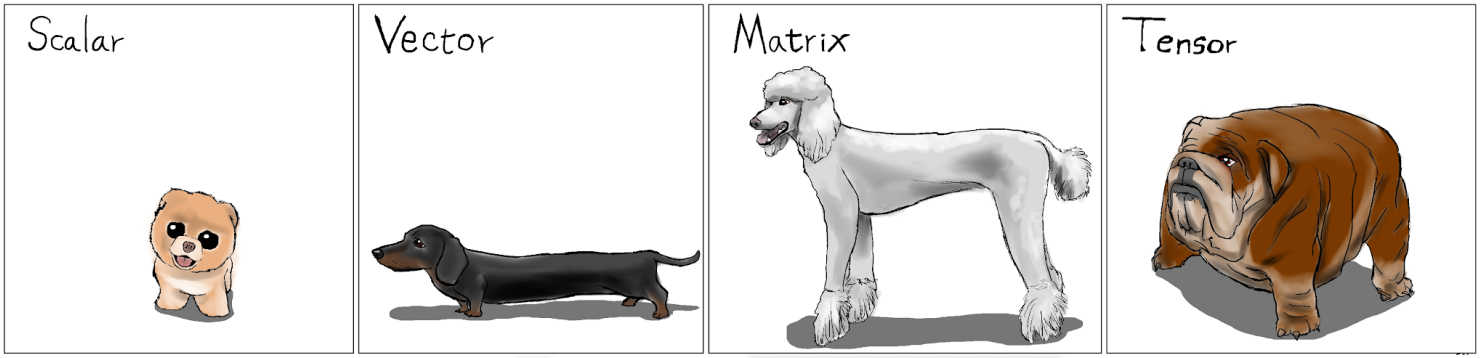


$[a_{ij}]_{ij} \in \mathbb{R}^{I \times J}$

(**2**-way tensor)

Tensors

- Matrices and vectors can be generalized to **Tensors**



$a \in \mathbb{R}$
(**0**-way tensor)

$[a_i]_i \in \mathbb{R}^I$
(**1**-way tensor)

$[a_{ij}]_{ij} \in \mathbb{R}^{I \times J}$
(**2**-way tensor)

$[a_{ijk}]_{ijk} \in \mathbb{R}^{I \times J \times K}$
(**3**-way tensor)

...

...

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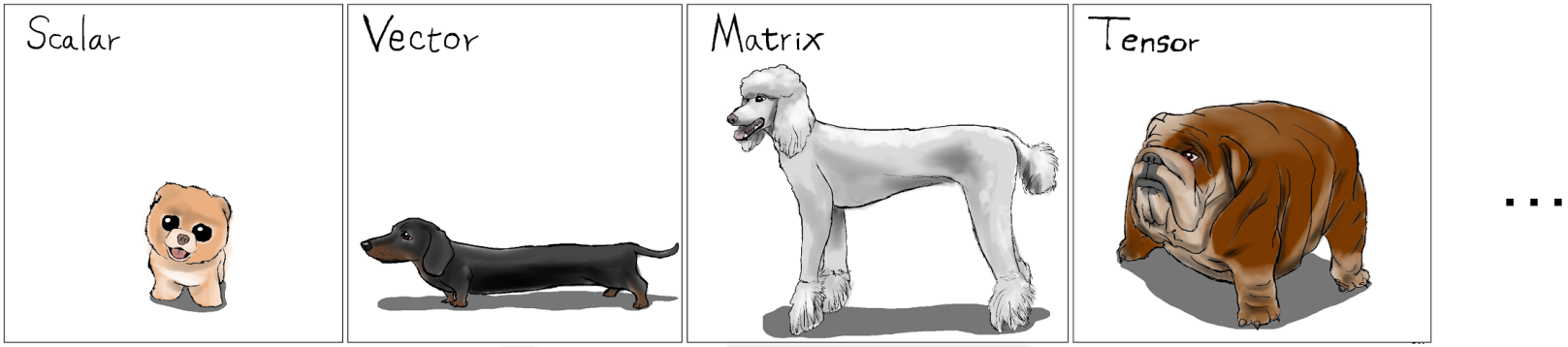

$a \in \mathbb{R}$
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$[a_i]_i \in \mathbb{R}^I$
(**1**-way tensor)

rows

$[a_{ij}]_{ij} \in \mathbb{R}^{I \times J}$
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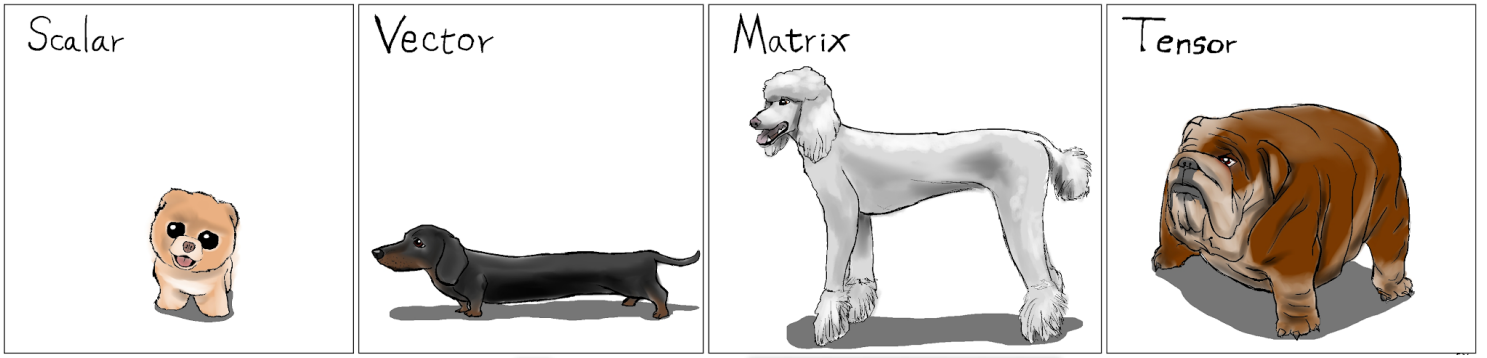

$a \in \mathbb{R}$
(**0**-way tensor)

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rows
columns

$[a_{ij}]_{ij} \in \mathbb{R}^{I \times J}$
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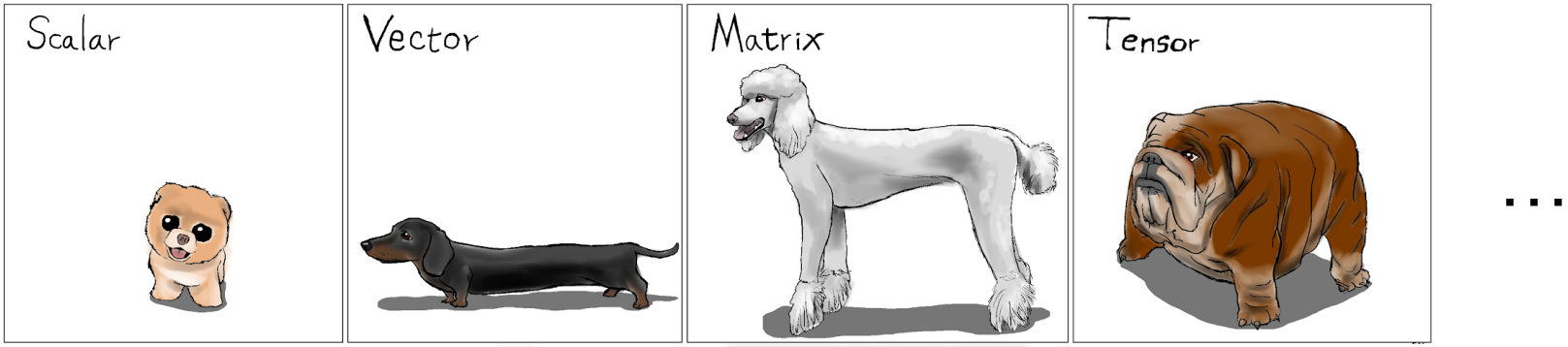

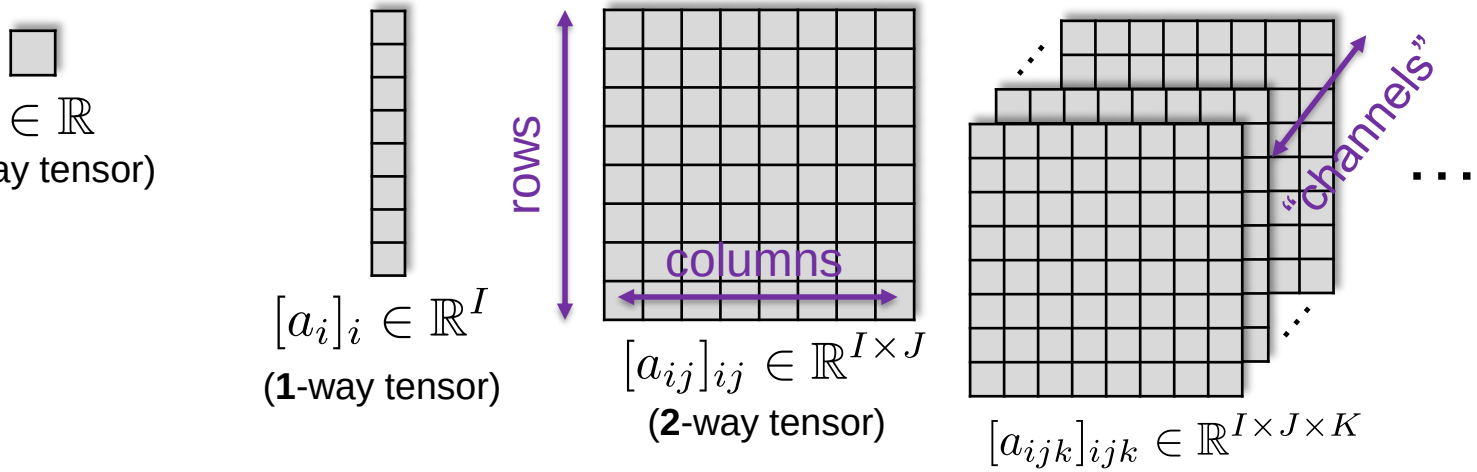

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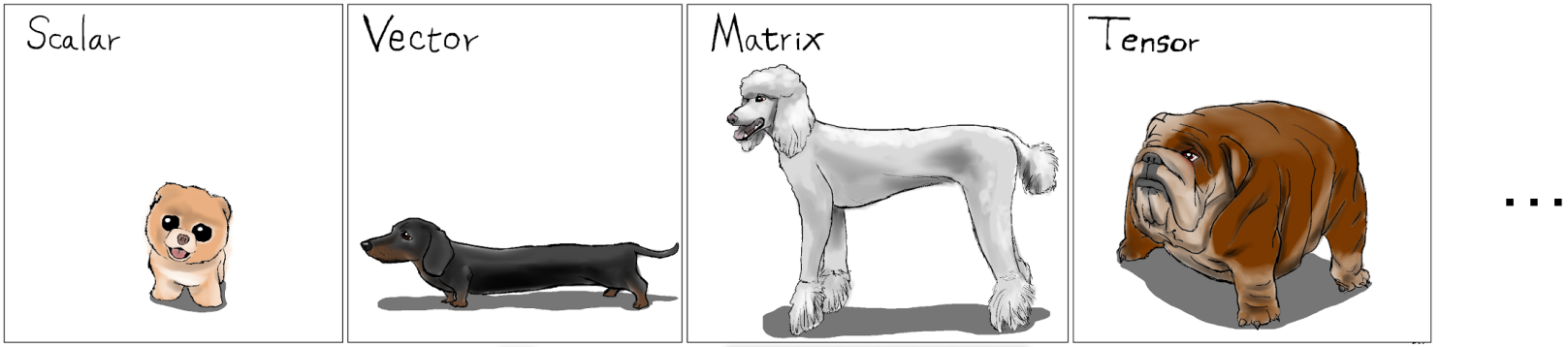

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(**0**-way tensor)

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rows
columns

$[a_{ij}]_{ij} \in \mathbb{R}^{I \times J}$
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"channels"
...

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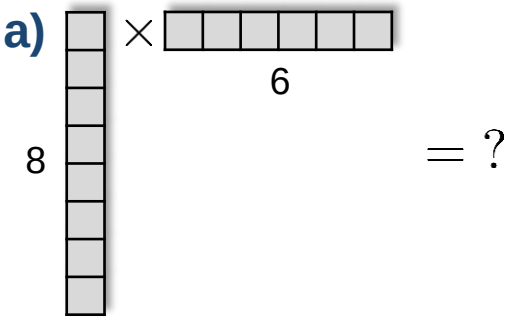
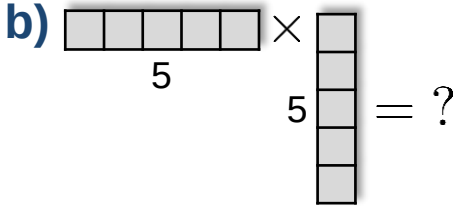
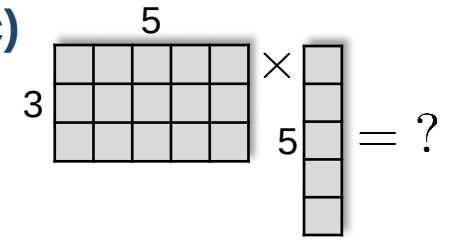
Tensors are the ubiquitous way to represent data in modern deep learning frameworks

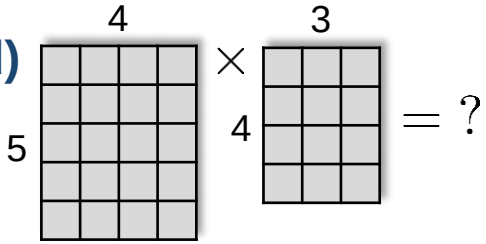
Tensors

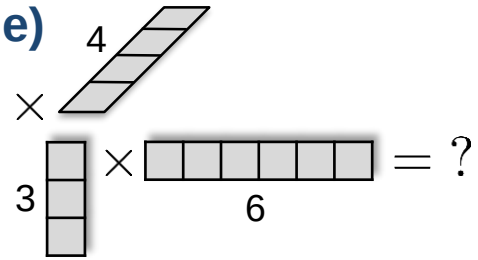
- Some vector/matrix/tensor operations:

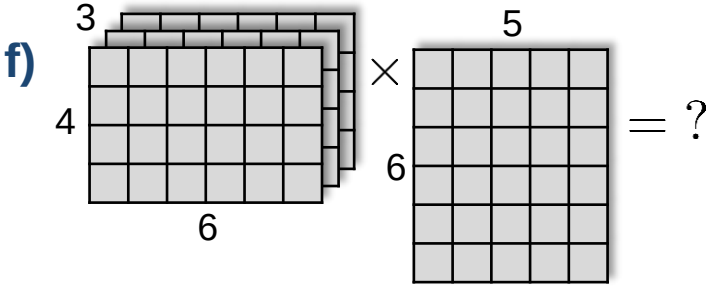
a) 

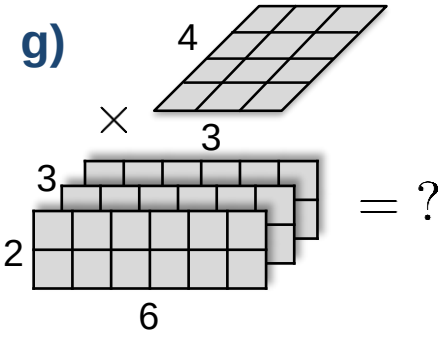
b) 

c) 

d) 

e) 

f) 

g) 

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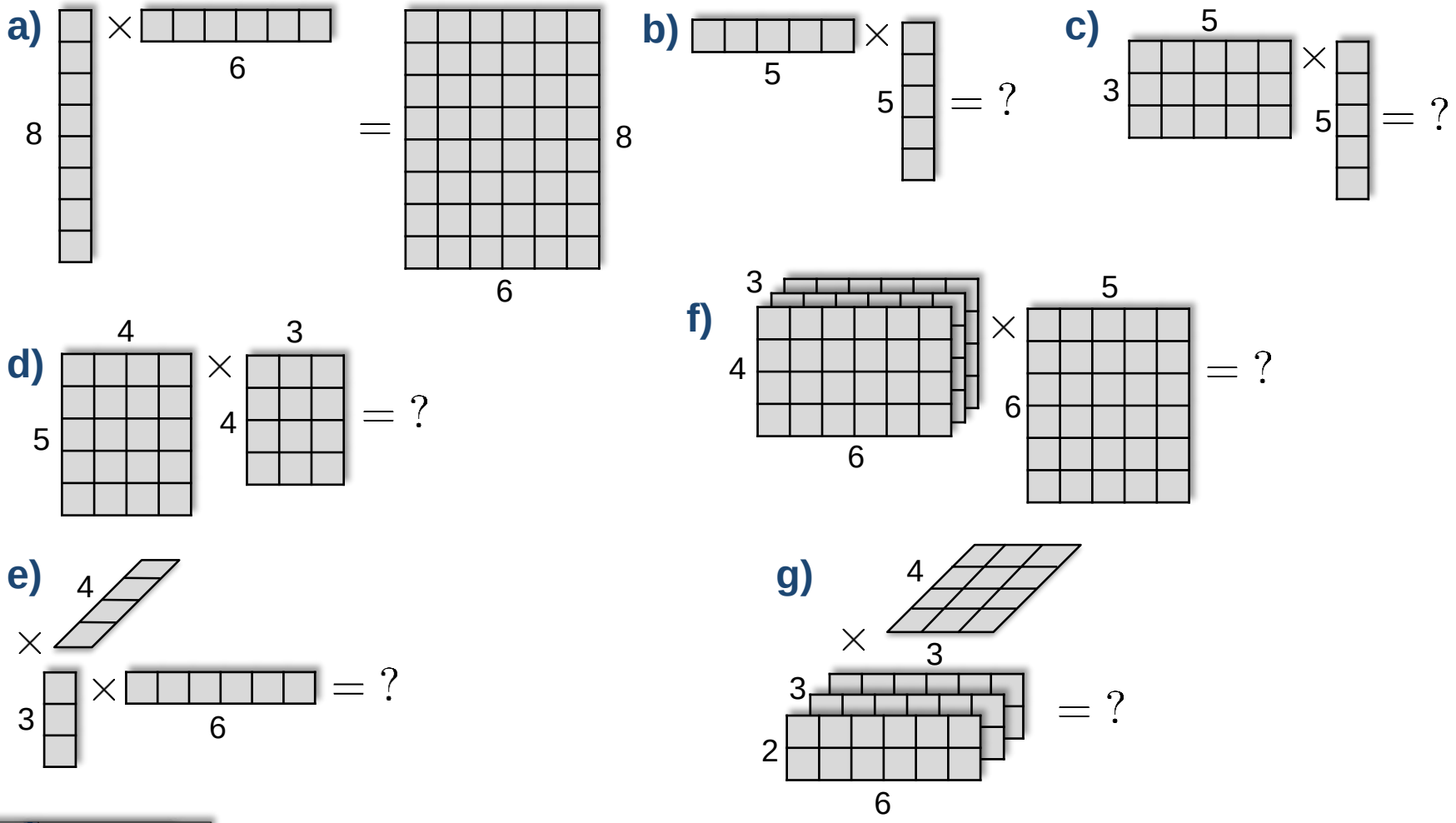

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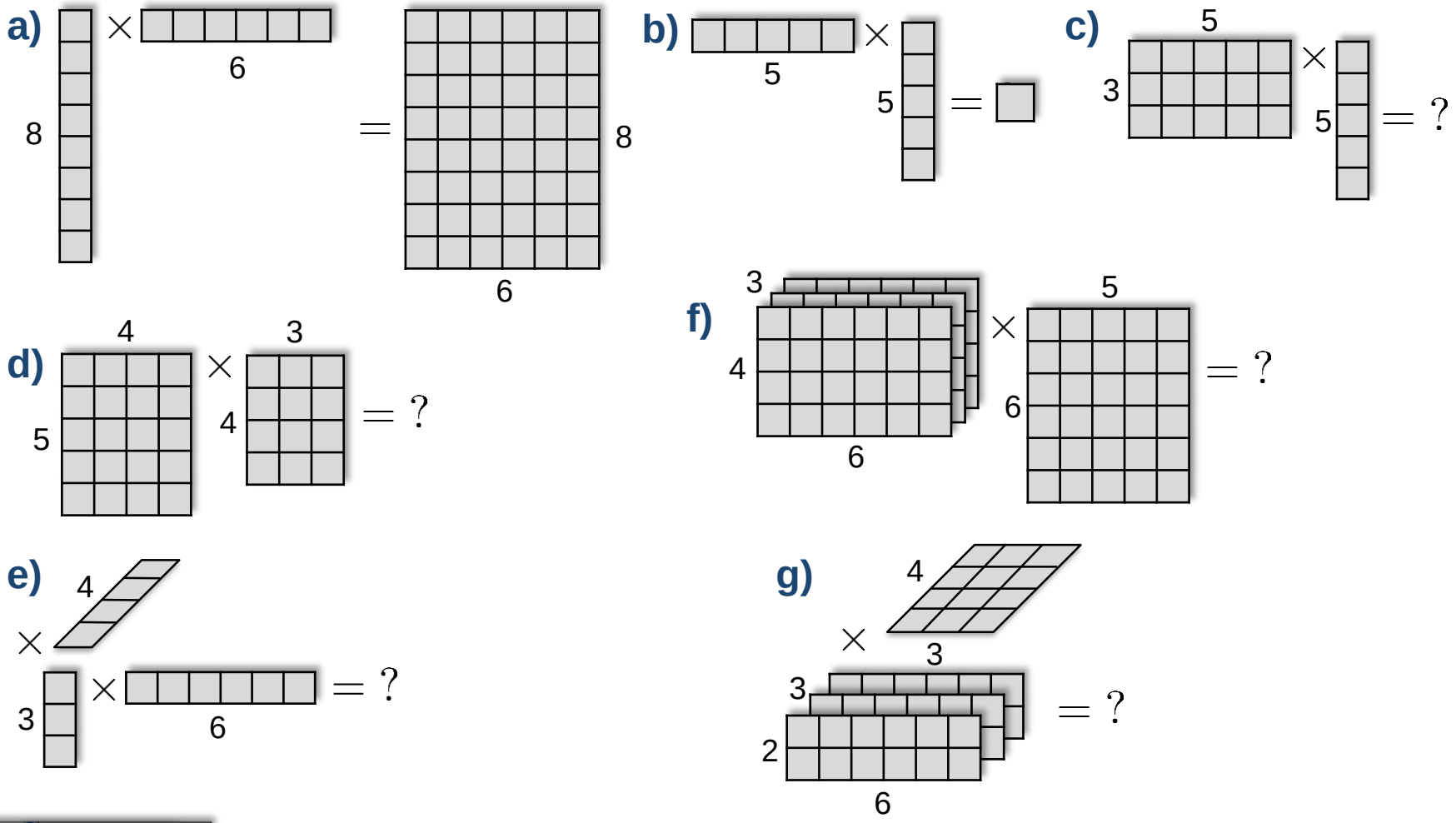

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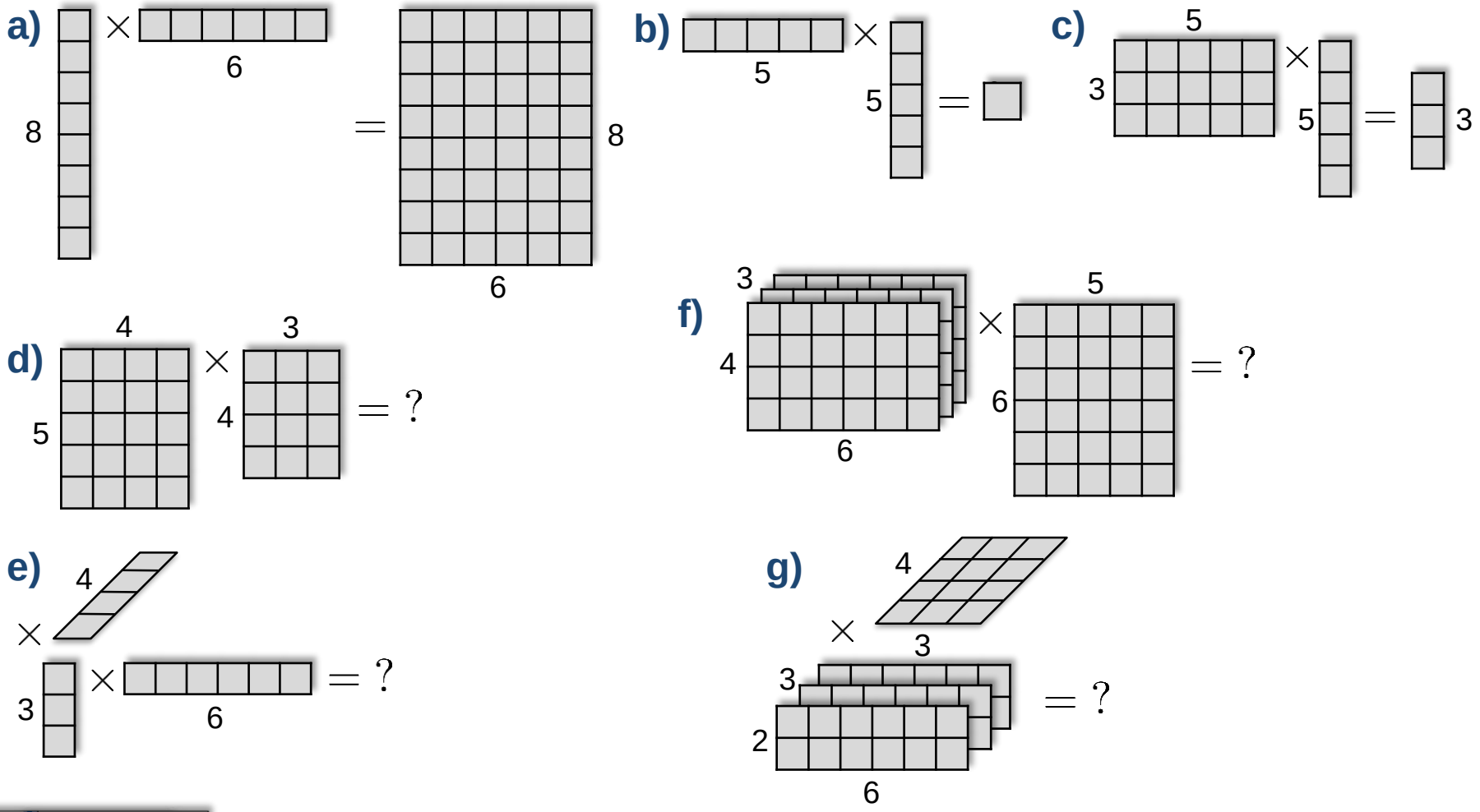

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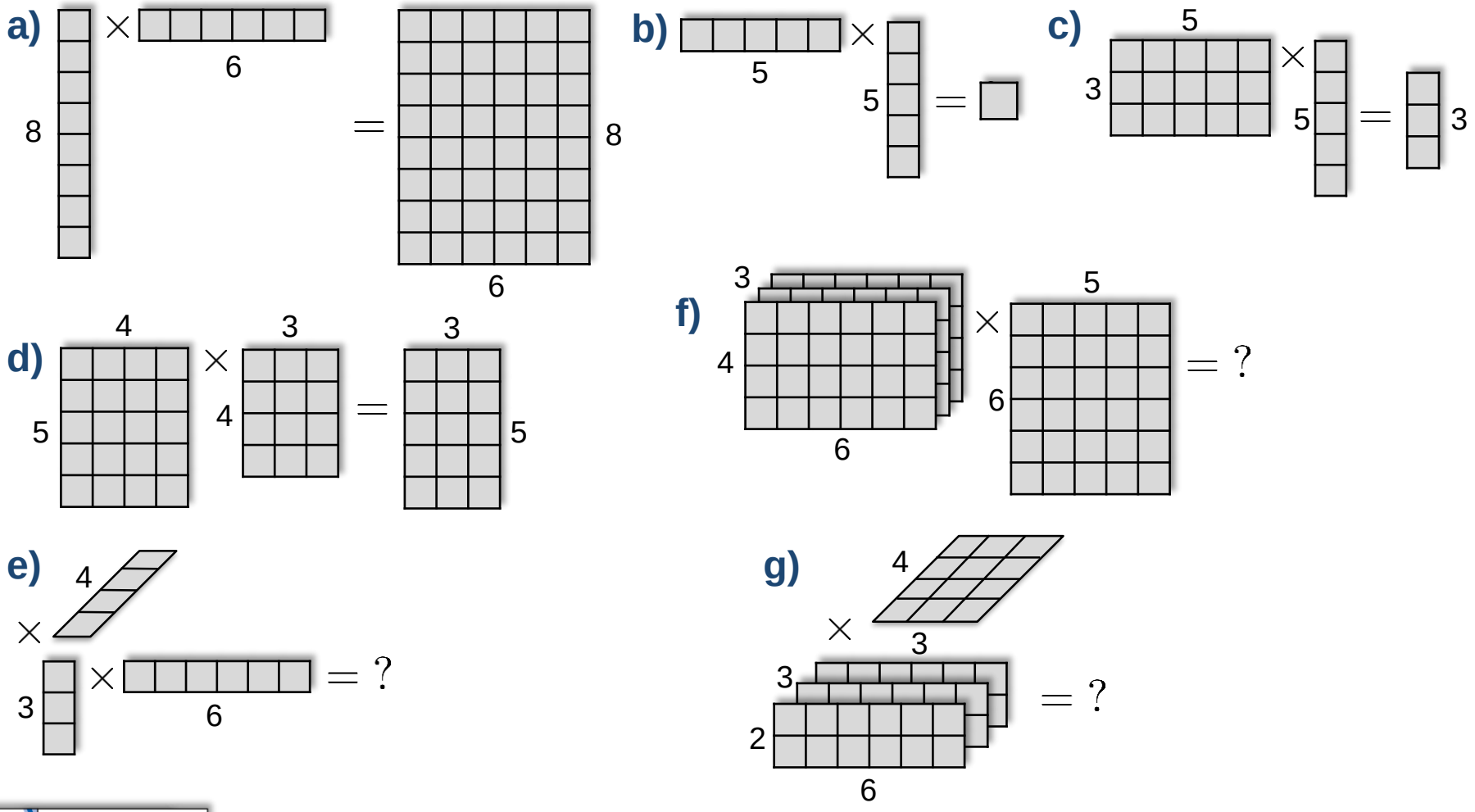

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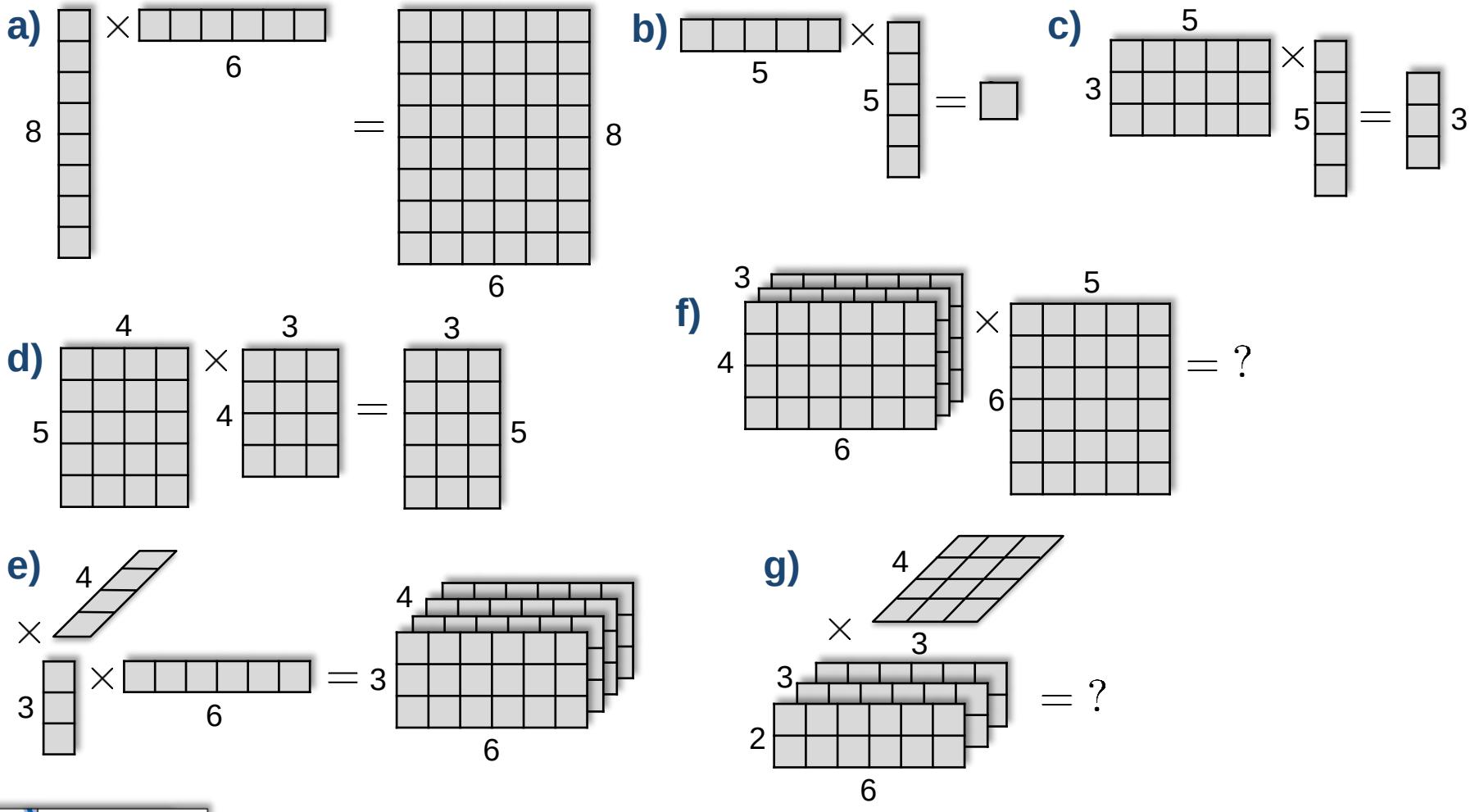

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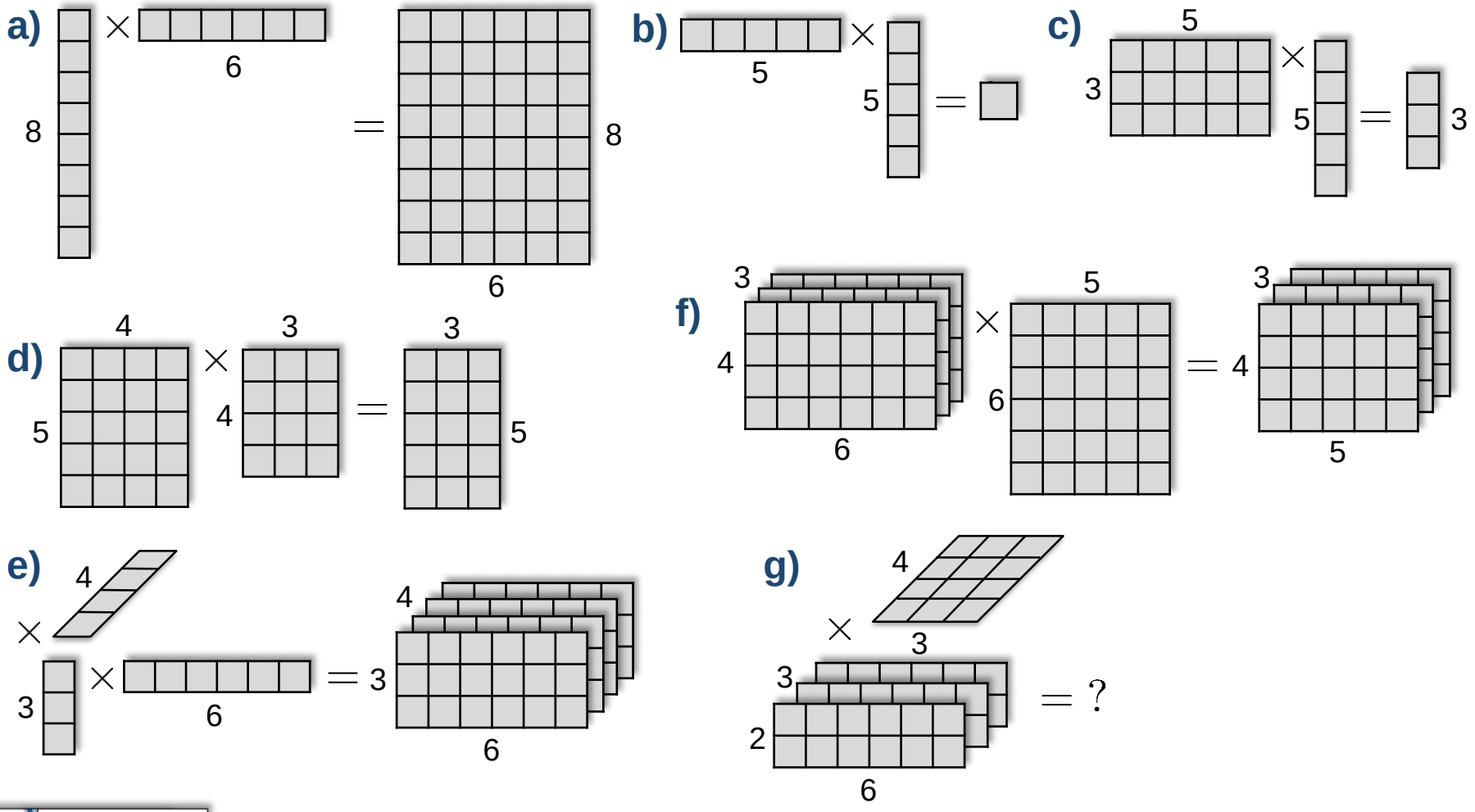

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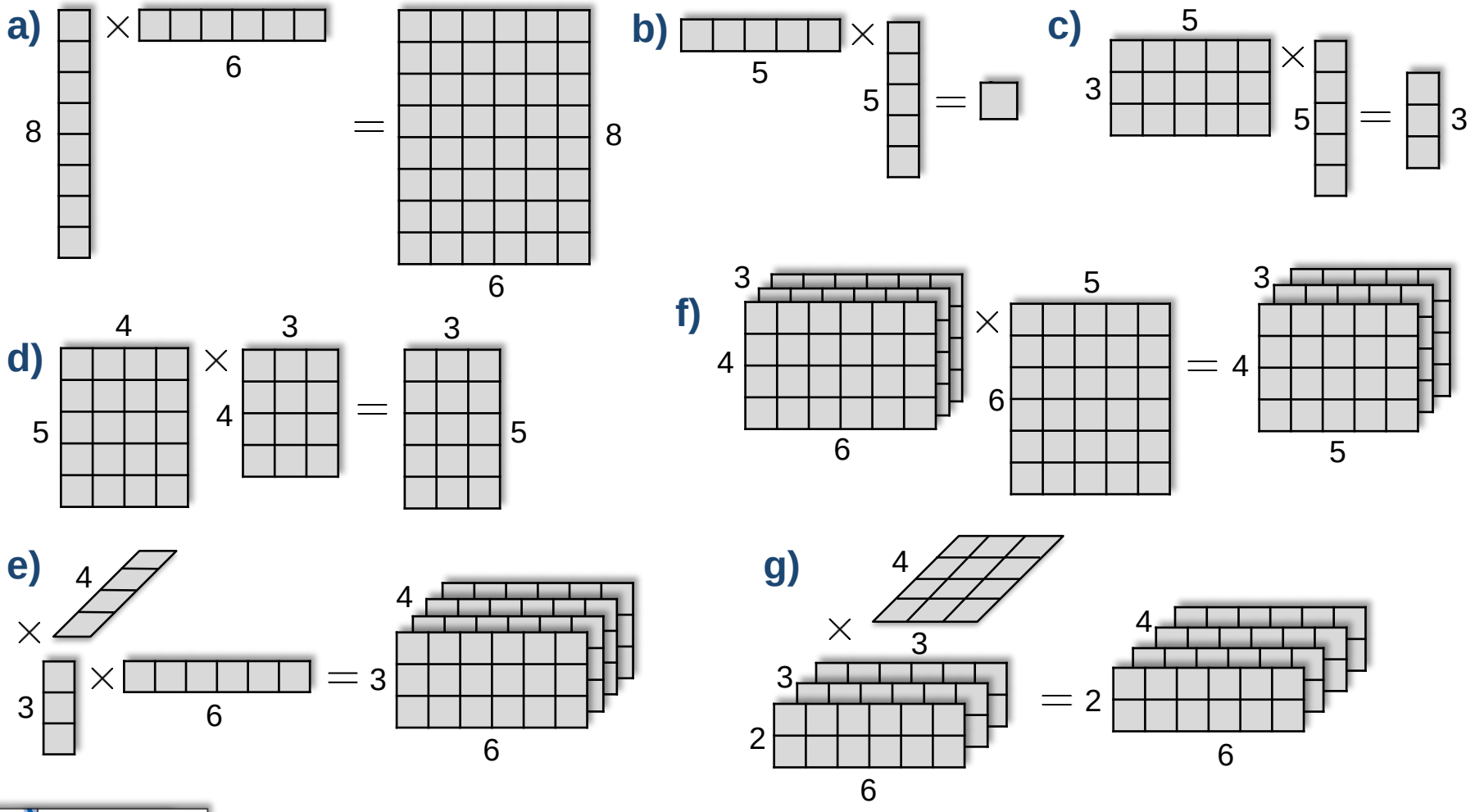

Generalizing the derivative

- Let $\boldsymbol{y} = f(\boldsymbol{x}), \quad \boldsymbol{x} \in \mathbb{R}^D, \quad \boldsymbol{y} \in \mathbb{R}^N.$ *How can we **define** $\frac{d\boldsymbol{y}}{d\boldsymbol{x}}$?*

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- Let $\delta_{\boldsymbol{x}_0} : \left\{ \begin{array}{ccc} \mathbb{R}^D & \rightarrow & \mathbb{R}^N \\ \boldsymbol{h} & \mapsto & f(\boldsymbol{x}_0 + \boldsymbol{h}) - f(\boldsymbol{x}_0) \end{array} \right.$

"the variation of f when moving by a step $\boldsymbol{h}$ from $\boldsymbol{x}_0$."

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Formulas:

$$\left.\frac{d\boldsymbol{y}}{d\boldsymbol{x}}\right|_{\boldsymbol{x}_0} = \begin{bmatrix} \left.\frac{\partial y_1}{\partial x_1}\right|_{x_{1,0}} & \left.\frac{\partial y_1}{\partial x_2}\right|_{x_{2,0}} & \cdots & \left.\frac{\partial y_1}{\partial x_D}\right|_{x_{D,0}} \\ \left.\frac{\partial y_2}{\partial x_1}\right|_{x_{1,0}} & \left.\frac{\partial y_2}{\partial x_2}\right|_{x_{2,0}} & \cdots & \left.\frac{\partial y_2}{\partial x_D}\right|_{x_{D,0}} \\ \vdots & \vdots & \ddots & \vdots \\ \left.\frac{\partial y_N}{\partial x_1}\right|_{x_{1,0}} & \left.\frac{\partial y_N}{\partial x_2}\right|_{x_{2,0}} & \cdots & \left.\frac{\partial y_N}{\partial x_D}\right|_{x_{D,0}} \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1(x_{1,0})}{\partial x_1} & \frac{\partial f_1(x_{2,0})}{\partial x_2} & \cdots & \frac{\partial f_1(x_{D,0})}{\partial D_1} \\ \frac{\partial f_2(x_{1,0})}{\partial x_1} & \frac{\partial f_2(x_{2,0})}{\partial x_2} & \cdots & \frac{\partial f_2(x_{D,0})}{\partial D_1} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_N(x_{1,0})}{\partial x_1} & \frac{\partial f_N(x_{2,0})}{\partial x_2} & \cdots & \frac{\partial f_N(x_{D,0})}{\partial D_1} \end{bmatrix} = \mathbf{J}_{\boldsymbol{x}}[f](\boldsymbol{x}_0)$$

Special Case 1

What does it give for a simple function $f : \mathbb{R}^1 \rightarrow \mathbb{R}^1$?

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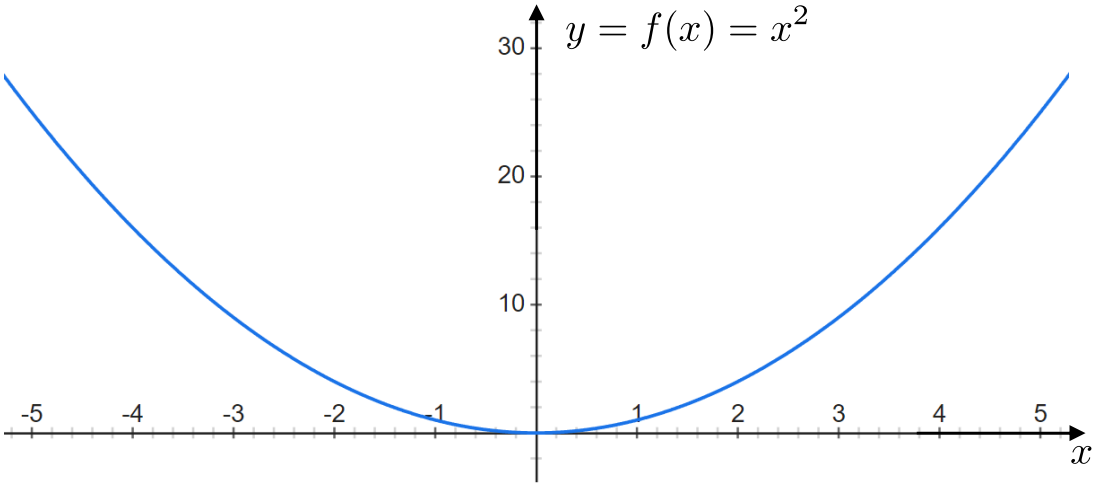

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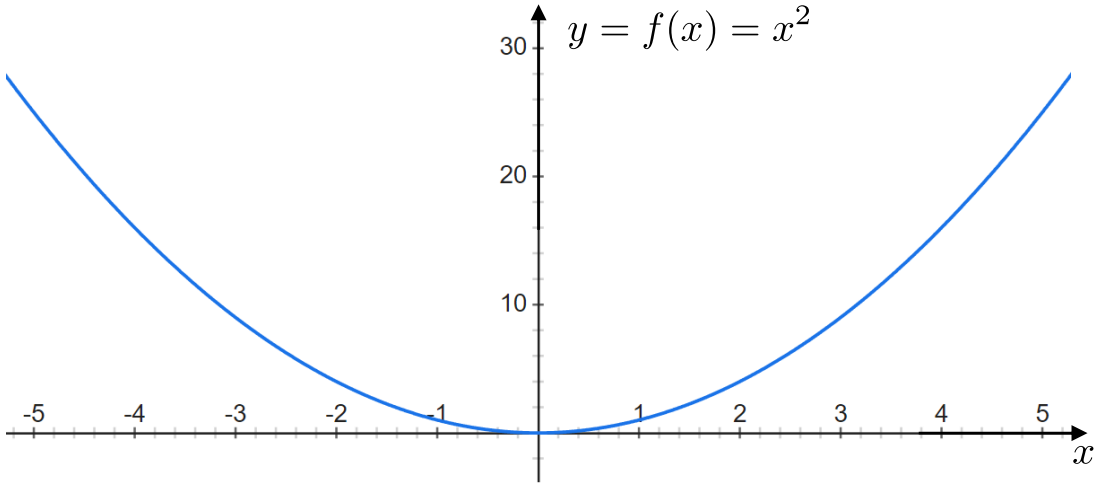

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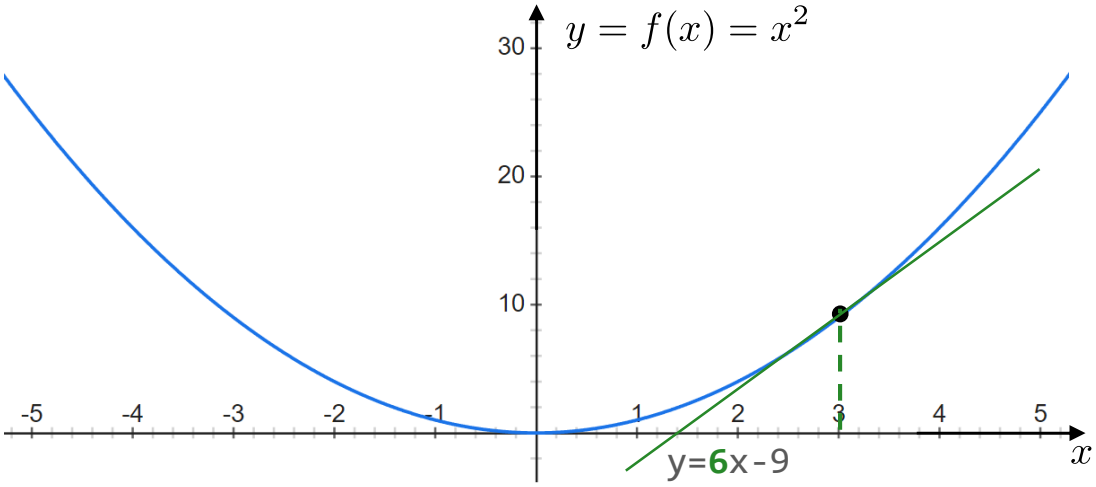

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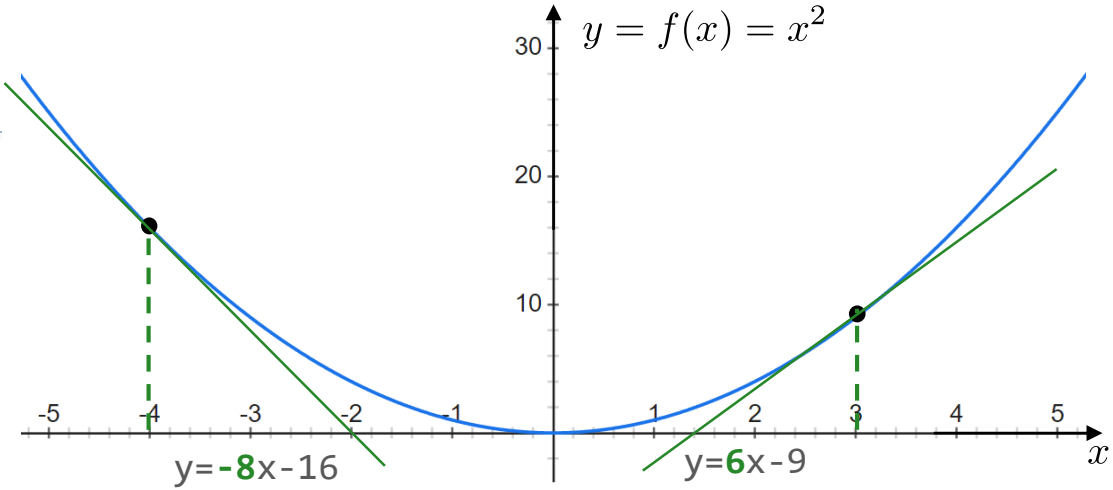

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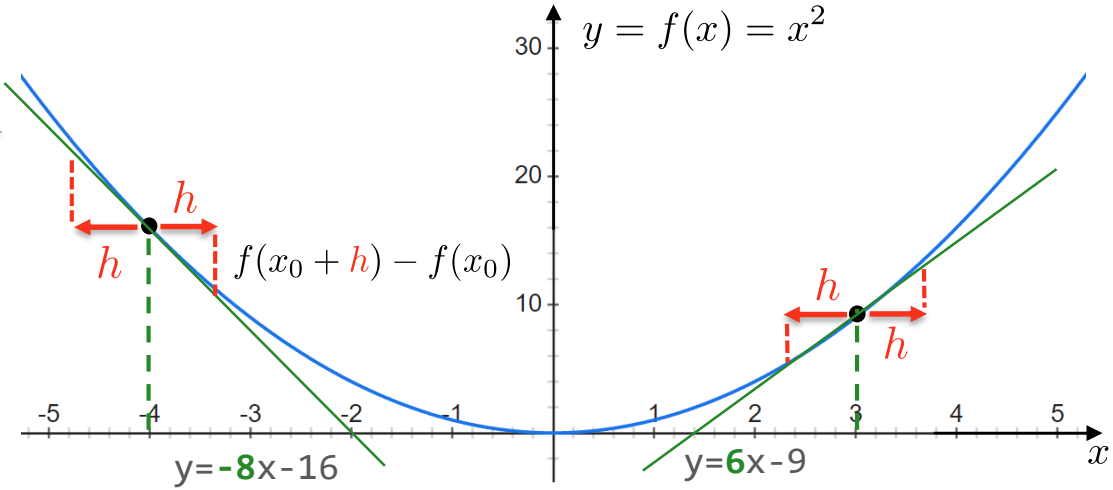

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$$\delta_{x_0} : \begin{cases} \mathbb{R}^1 & \to & \mathbb{R}^1 \\ h & \mapsto & f(x_0 + h) - f(x_0) \end{cases}$$, for h ***infinitesimally*** small.

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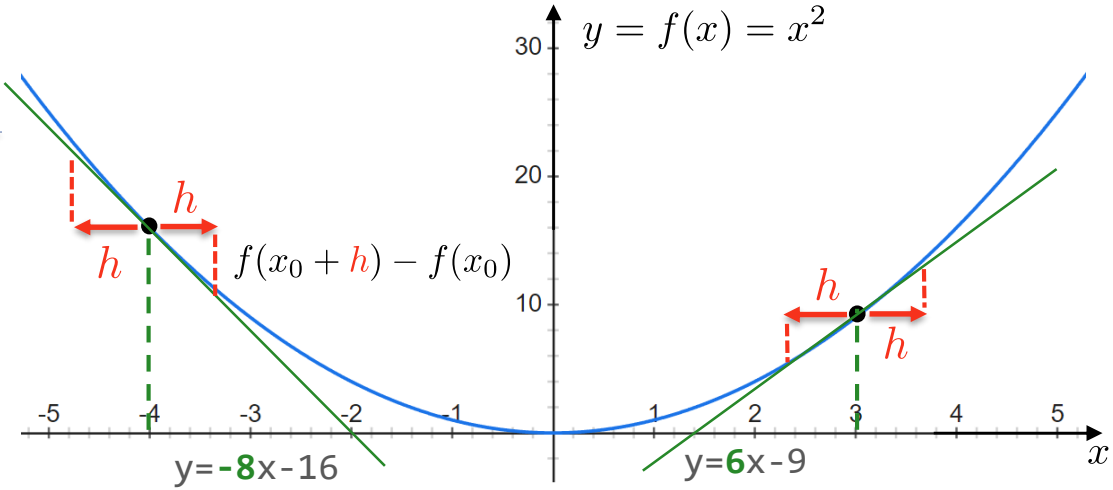

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Indeed: $\lim_{h \to \infty} f(x_0 + h) - f(x_0) = f'(x_0)h$!

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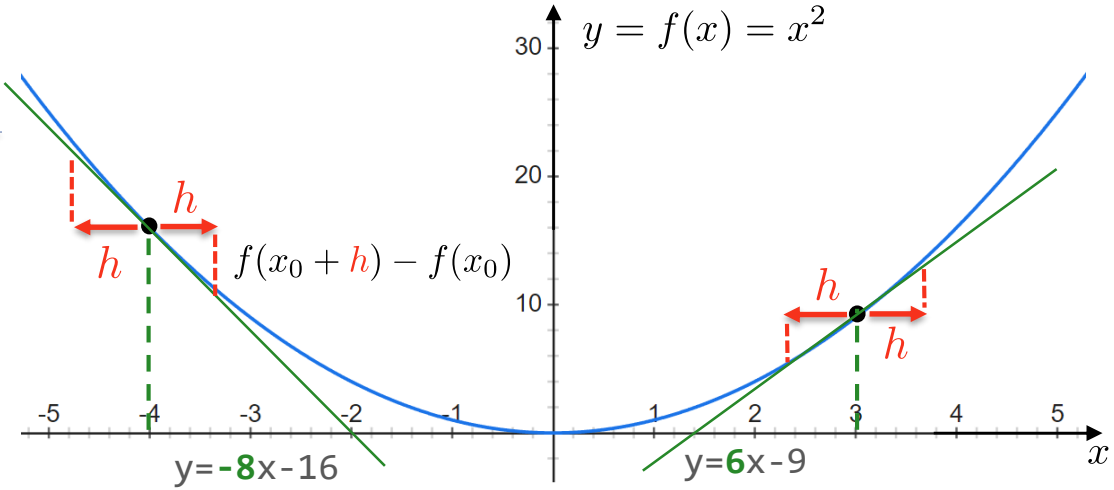

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"the variation of f when moving to the left or to the right from x_0."

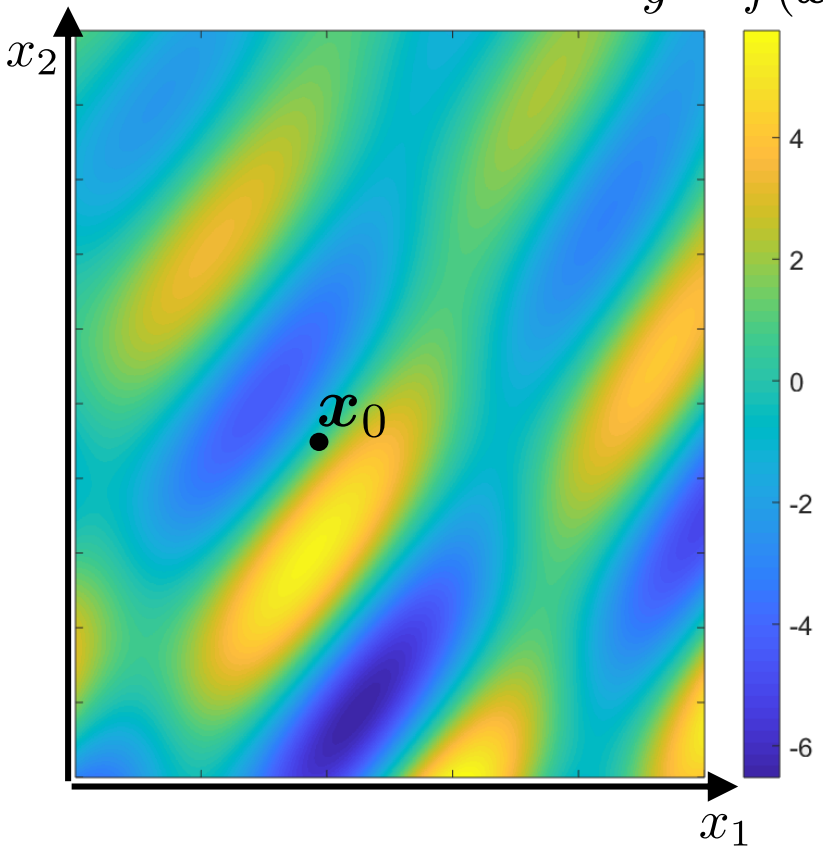
Indeed: $\lim_{h \to \infty} f(x_0 + h) - f(x_0) = f'(x_0)h$!

- As a **linear map**, $f'(x_0)$ can be seen as a **0-way tensor**, in $\mathbb{R}^{1 \times 1}$!

Special Case 2

What about a **real-valued** multivariate function $f : \mathbb{R}^D \rightarrow \mathbb{R}^1$?

$$y = f(\boldsymbol{x})$$



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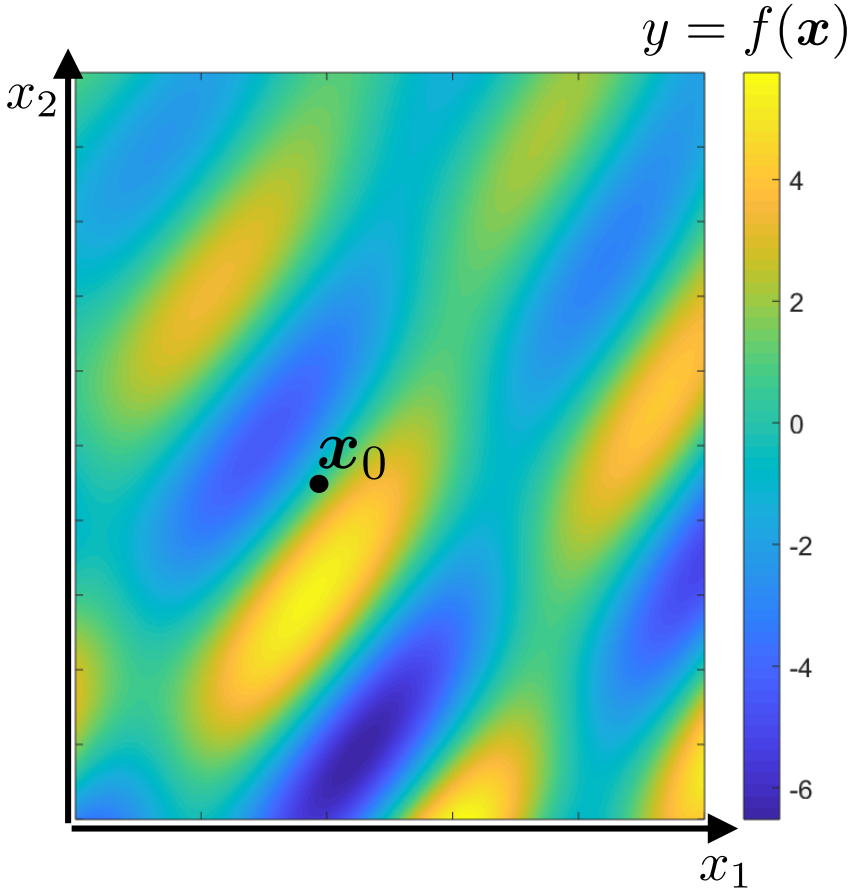

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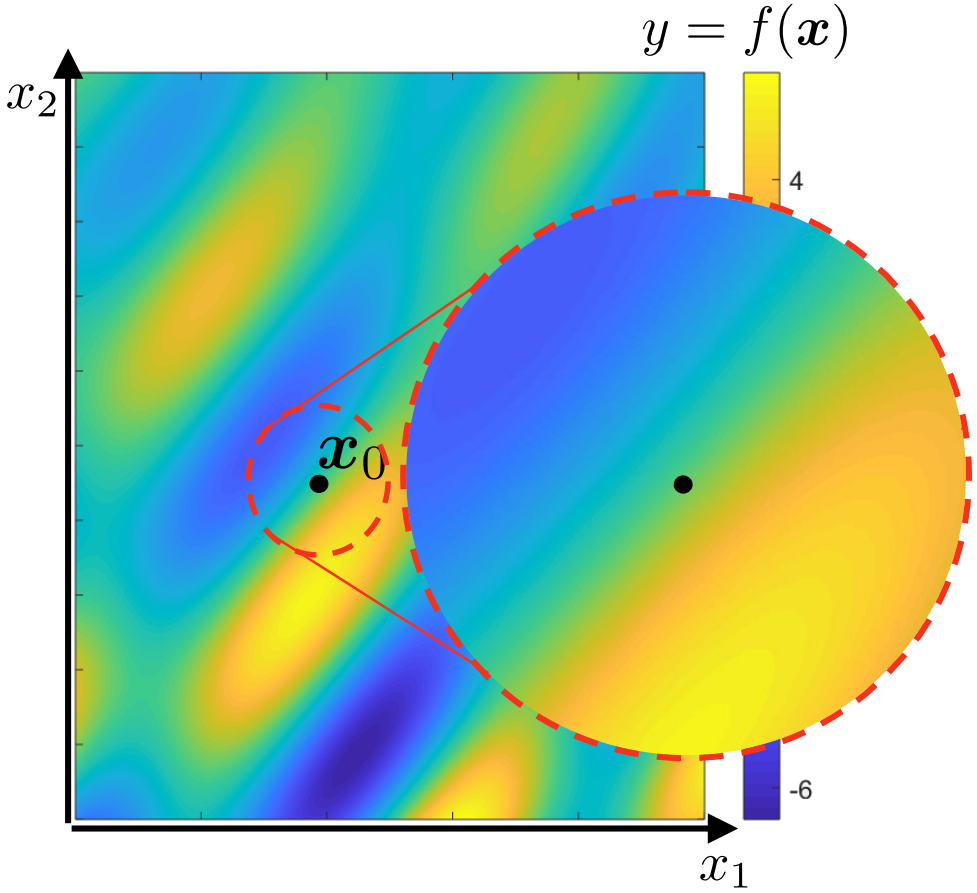

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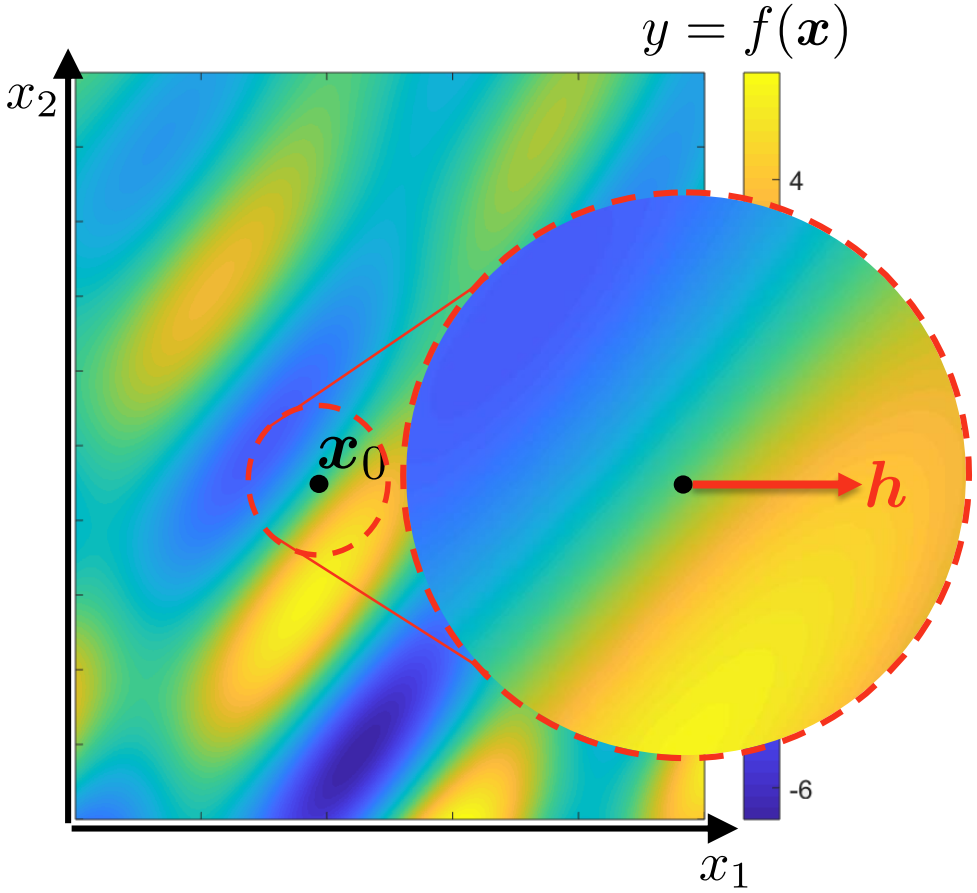

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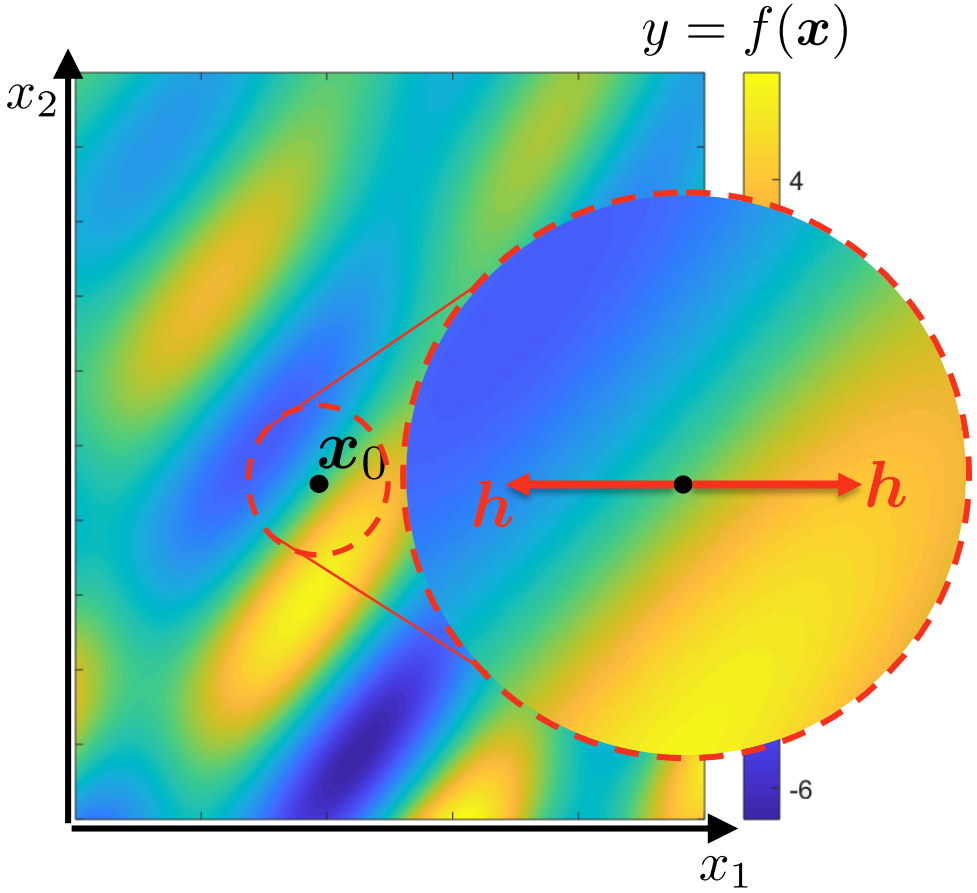

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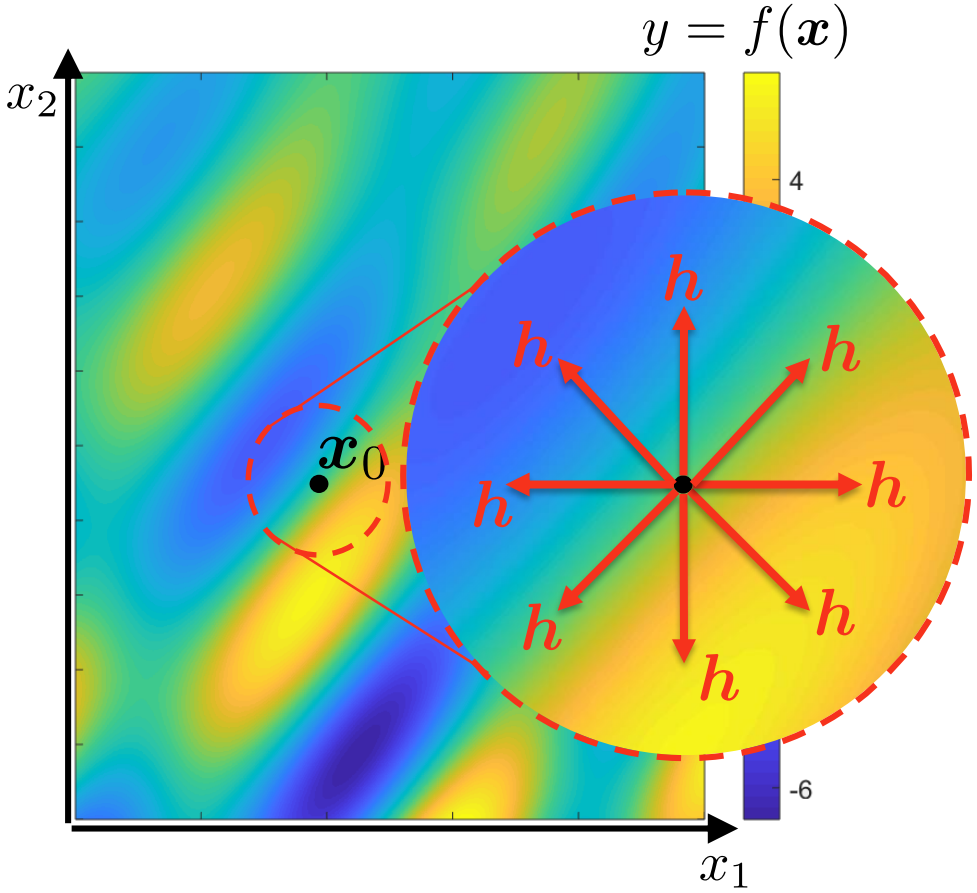

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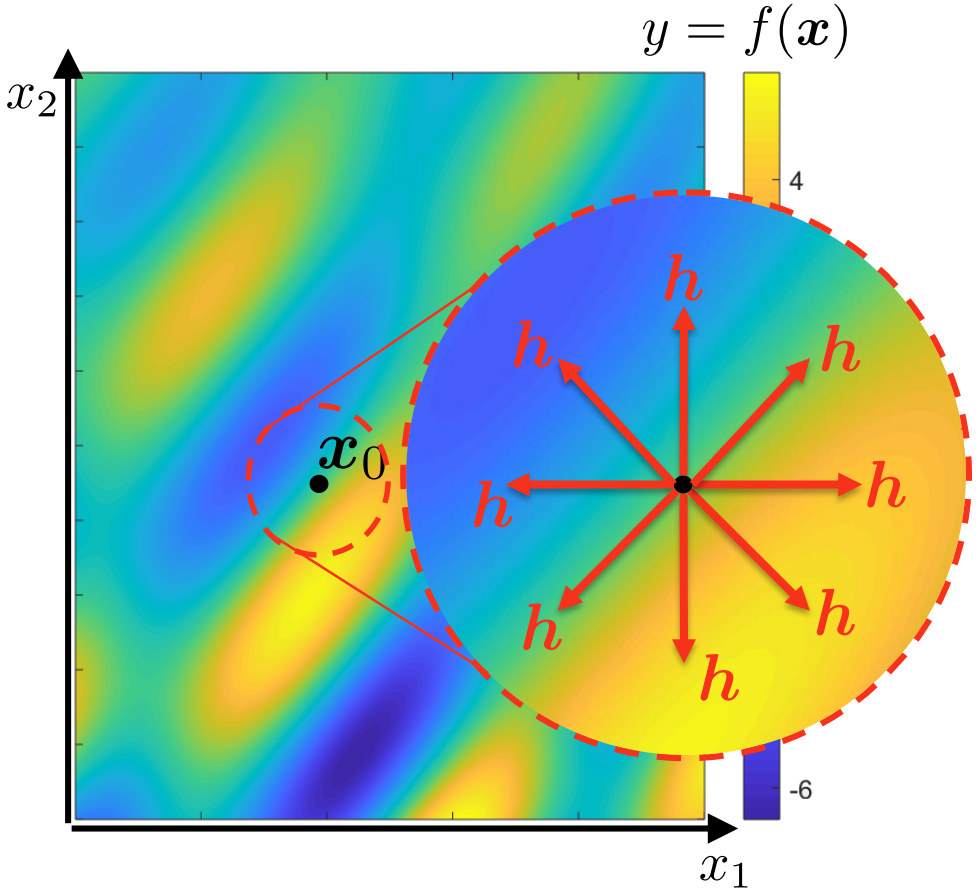

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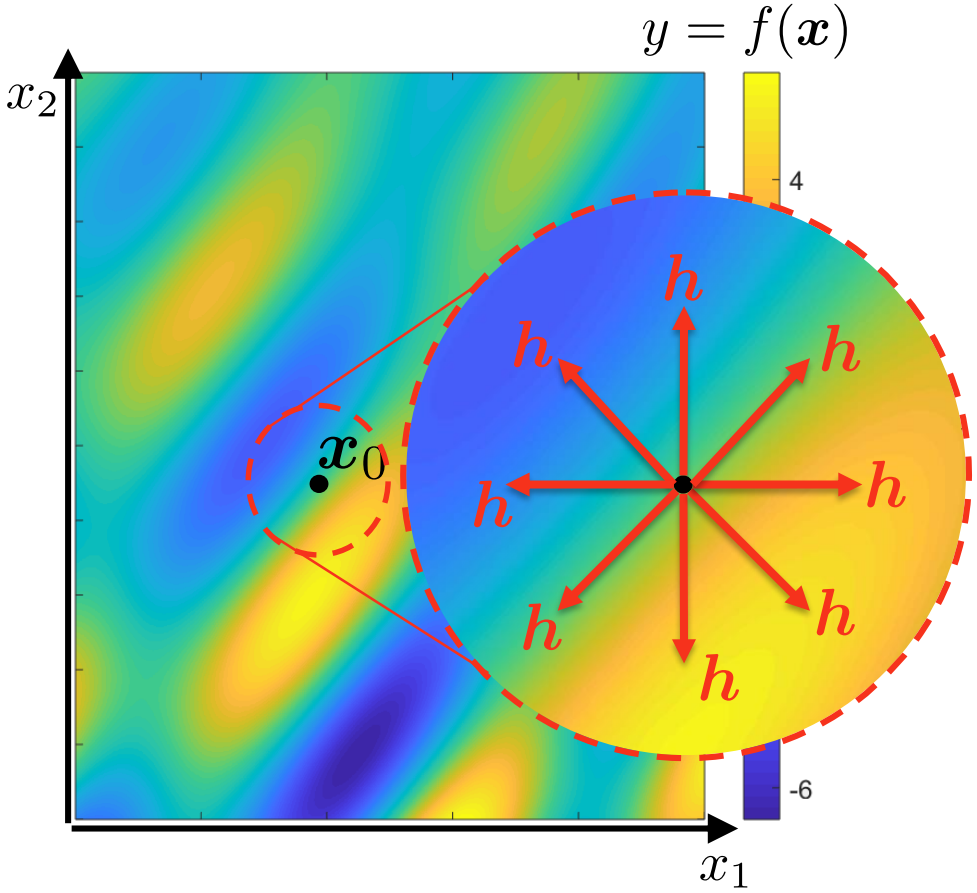

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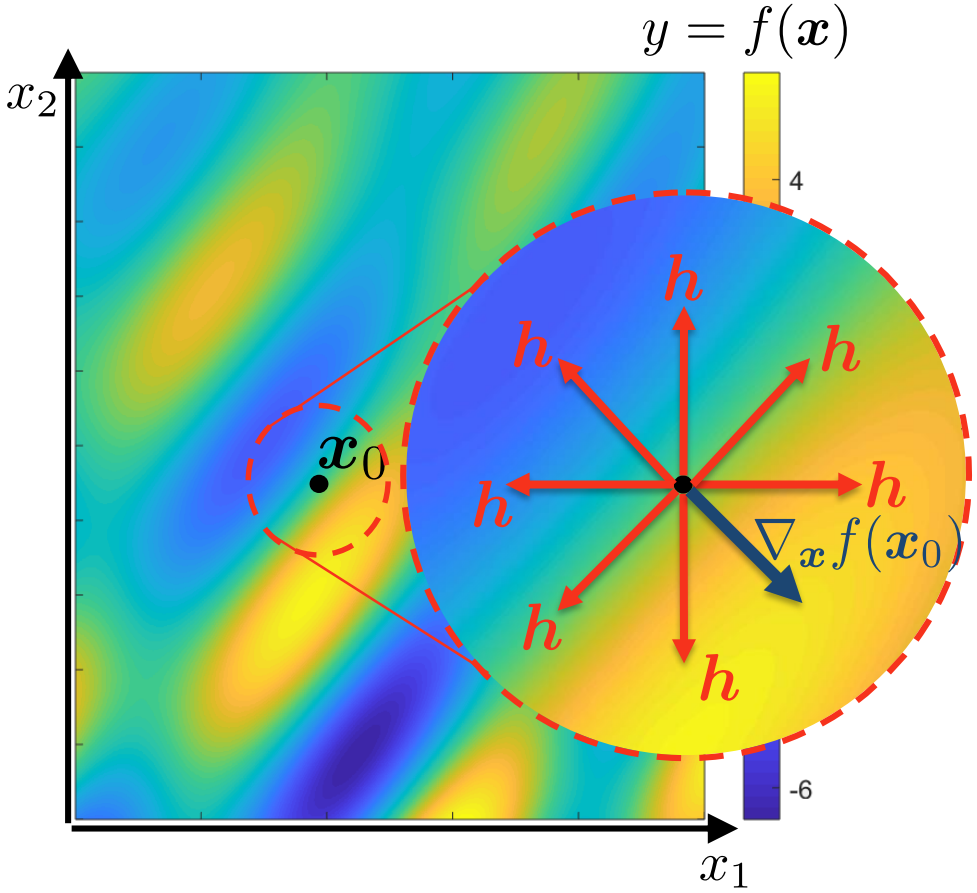

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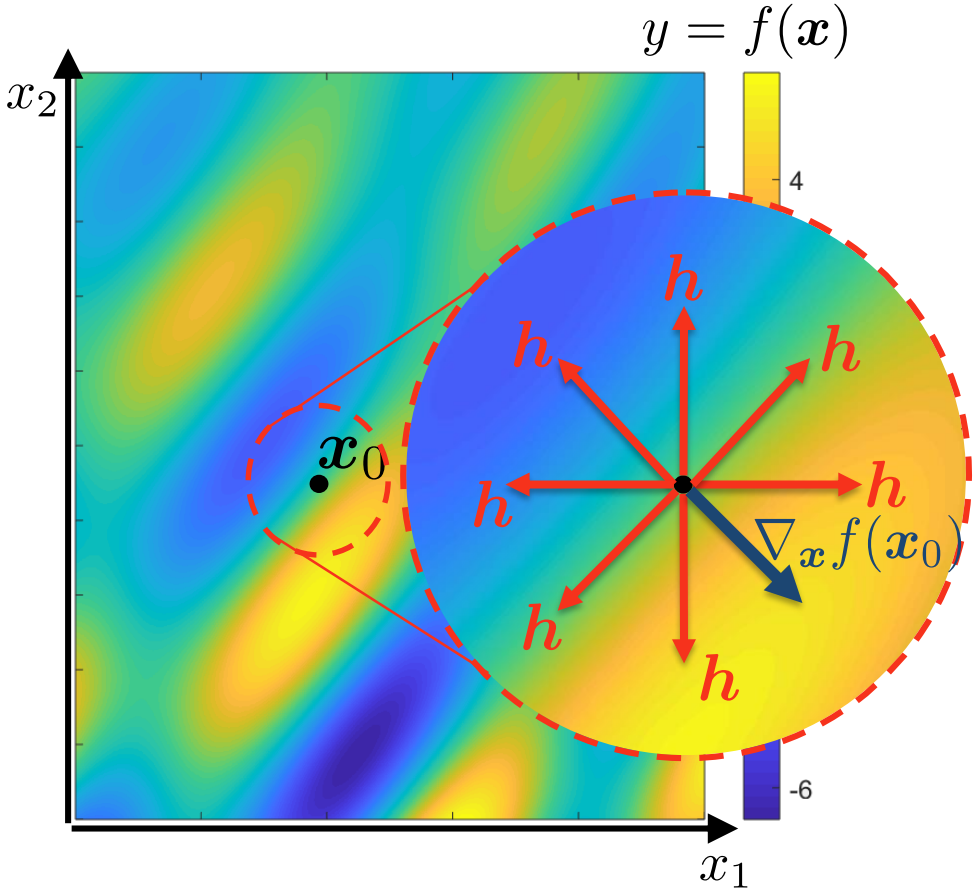

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Note: All of this generalizes naturally to **matrix** or **tensor variables**, yielding **tensor total derivatives**.

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with $\mathbb{R}^{M\times D}$, $\mathbb{R}^{M\times N}$, $\mathbb{R}^{N\times 1}$, $\mathbb{R}^{1\times D}$ respectively.

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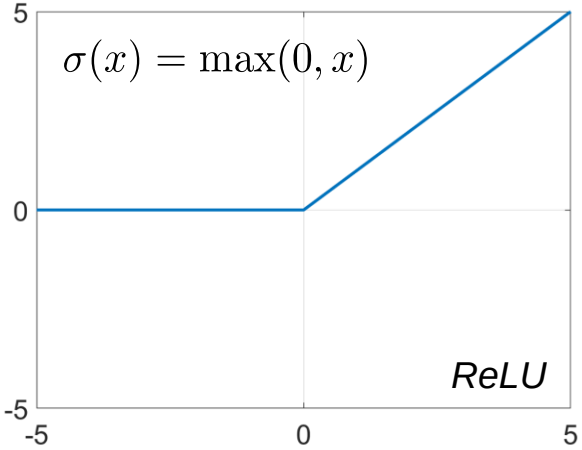
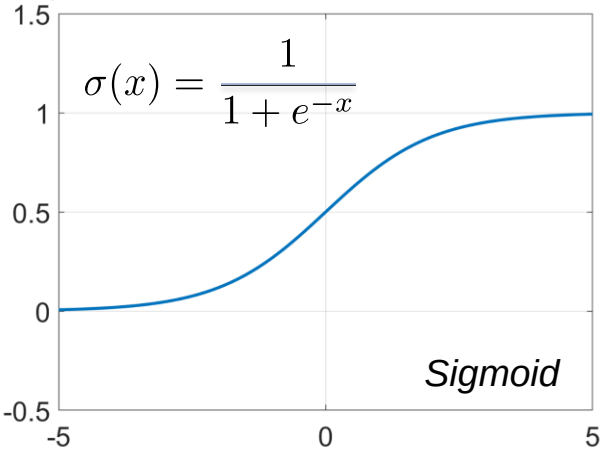
Exercises: *Simple Derivatives*

- Calculate the derivatives of the following functions:



$\sigma'(x) = ?$



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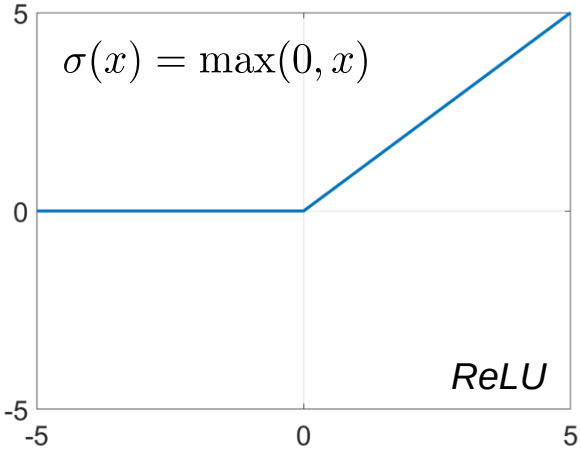
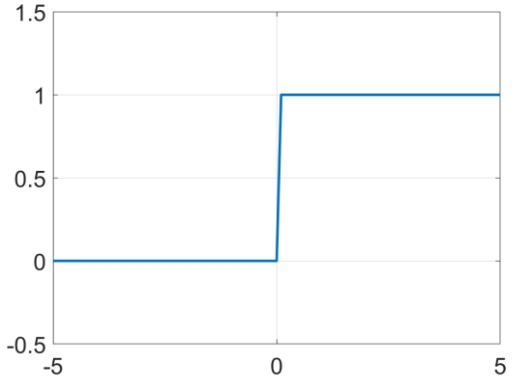
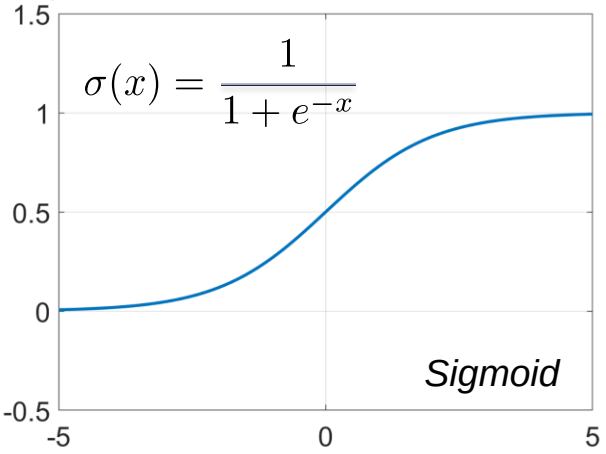


$$\sigma'(x) = \begin{cases} 0 \text{ for } x < 0 \\ 1 \text{ for } x < 1 \end{cases} = H(x)$$





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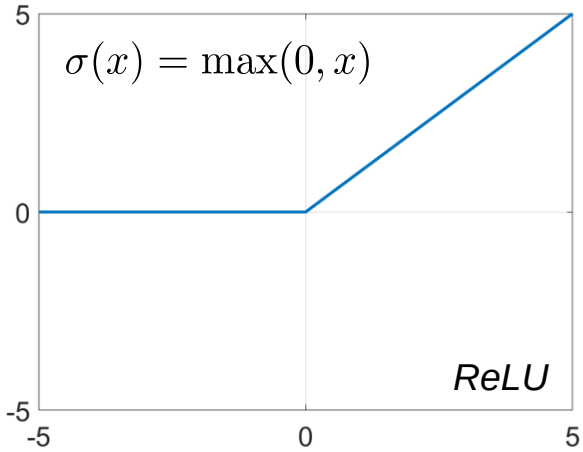

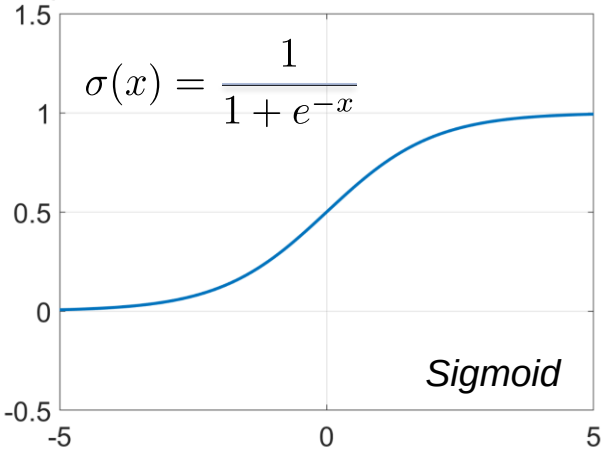

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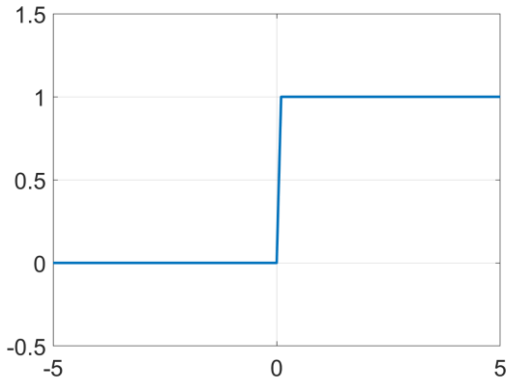
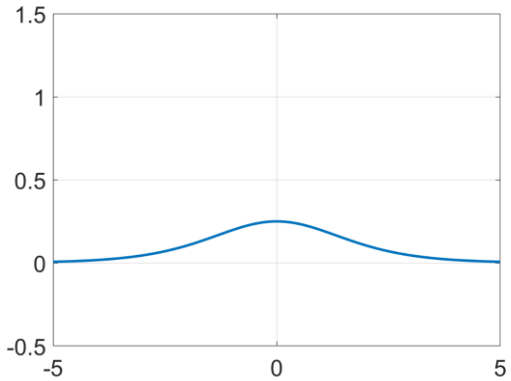
$$\sigma'(x) = \frac{e^{-x}}{(1+e^{-x})^2} = \sigma(x)(1-\sigma(x))$$





Exercises: *Gradients*

$$\nabla_{\boldsymbol{x}} f(\boldsymbol{x}_0) = \left[\frac{\partial f}{\partial x_1}(x_{0,1}), \ldots, \frac{\partial f}{\partial x_D}(x_{0,D}) \right]^\top$$

- Calculate the gradient of the following functions at $\boldsymbol{x}_0$:

$$f(\boldsymbol{x}) = b + \boldsymbol{w}^\top \boldsymbol{x} \qquad \Rightarrow \nabla_{\boldsymbol{x}} f(\boldsymbol{x}_0) = ?$$

$$g(\boldsymbol{x}) = \|\boldsymbol{x}\|_2^2 \qquad \Rightarrow \nabla_{\boldsymbol{x}} g(\boldsymbol{x}_0) = ?$$

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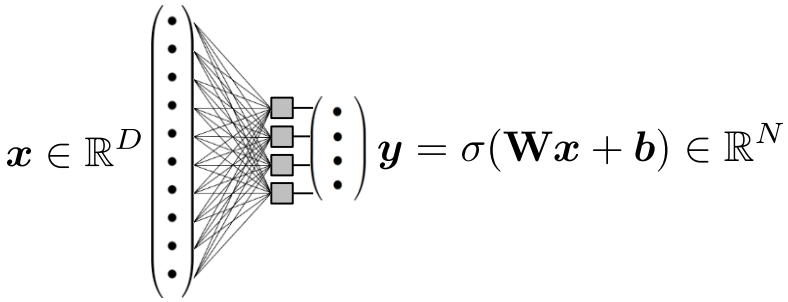
Exercises: *Jacobians*

Remember: *the* ***simple perceptron***



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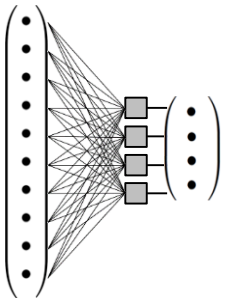

$\boldsymbol{x} \in \mathbb{R}^D$

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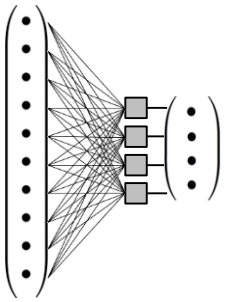

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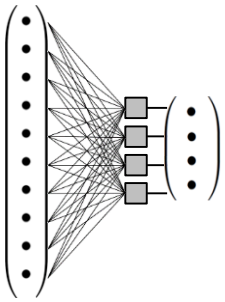

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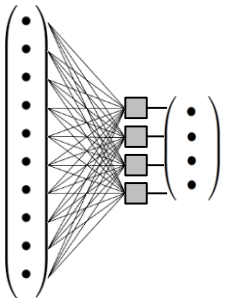

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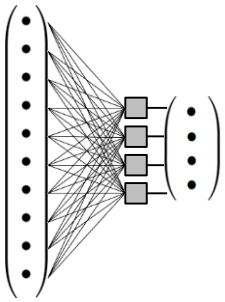

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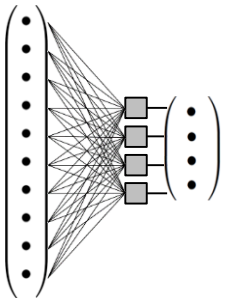

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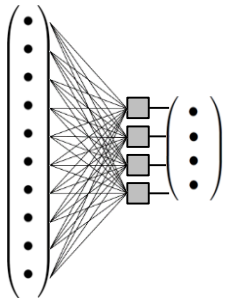

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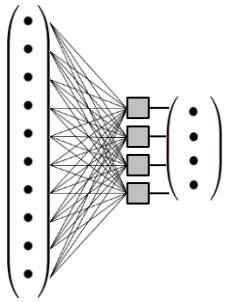

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Exercises: *Jacobians*

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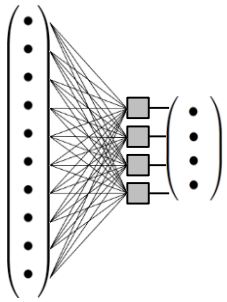

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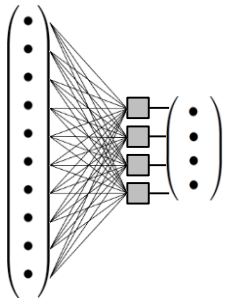

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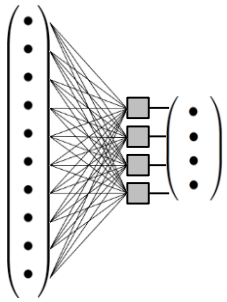

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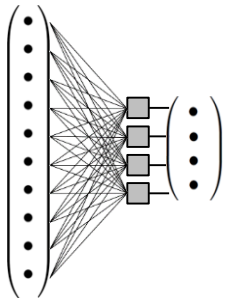

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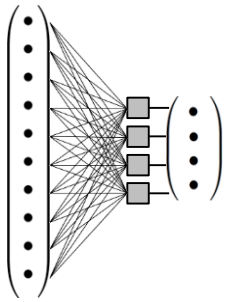

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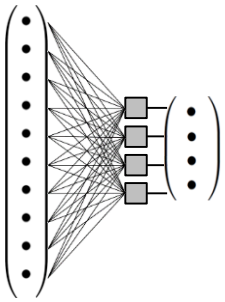

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Exercise: *Least Squares*

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- Calculate the gradient of $f(\boldsymbol{\theta}) = \|\mathbf{W}\boldsymbol{\theta} - \boldsymbol{y}\|_2^2$ where $\mathbf{W} \in \mathbb{R}^{N \times D}$

Hint: Let $\boldsymbol{a} = \mathbf{W}\boldsymbol{\theta} - \boldsymbol{y}$, $g(\boldsymbol{a}) = \|\boldsymbol{a}\|_2^2$ and $r = g(\boldsymbol{a}) = f(\boldsymbol{\theta})$.

$$\nabla_\theta f(\boldsymbol{\theta}_0)^\top = \frac{\partial r}{\partial \boldsymbol{\theta}}\Big|_{\boldsymbol{\theta}_0} = \frac{\partial r}{\partial \boldsymbol{a}}\Big|_{\mathbf{W}\boldsymbol{\theta}_0 - \boldsymbol{y}} \times \frac{\partial \boldsymbol{a}}{\partial \boldsymbol{\theta}}\Big|_{\boldsymbol{\theta}_0}$$

$$= \nabla_{\boldsymbol{a}} g(\mathbf{W}\boldsymbol{\theta}_0 - \boldsymbol{y})^\top \times \mathbf{W}$$

We have: $\nabla_{\boldsymbol{a}} g(\boldsymbol{a}_0) = 2\boldsymbol{a}_0$

$$= 2(\mathbf{W}\boldsymbol{\theta}_0 - \boldsymbol{y})^\top \mathbf{W}$$

$$\nabla_\theta f(\boldsymbol{\theta}_0) = 2\mathbf{W}^\top(\mathbf{W}\boldsymbol{\theta}_0 - \boldsymbol{y})$$