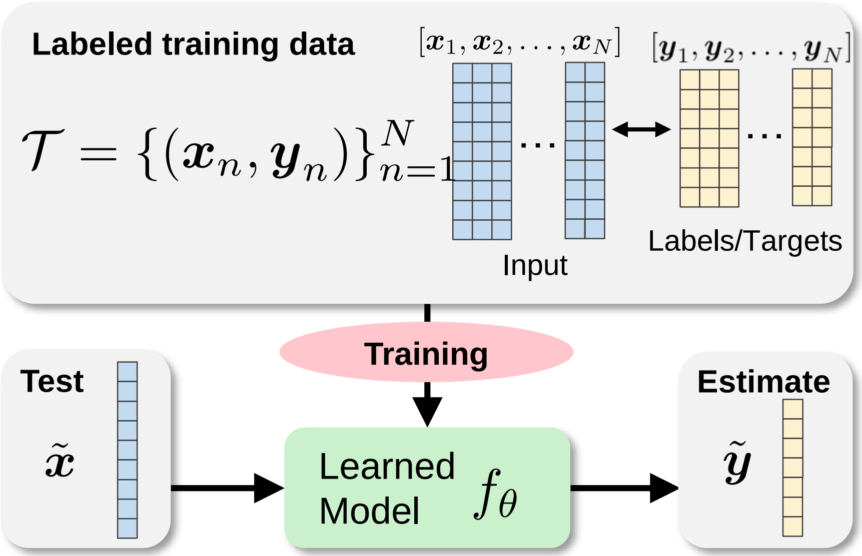
OUTLINE

I. ~~Introduction~~
II. ~~Background~~
III. ~~Fitting a Model~~
IV. ~~Supervised Learning~~
V. Unsupervised Learning

VI. Convolutional Neural Networks

OUTLINE

Supervised Learning



1. Discrete case: («one-hot»)
- Classification

Ex. application: *dog breed*

$\boldsymbol{y} =$ 1

13: German Shepherd

2. Continuous case:
- Regression

Ex. application: *head pose*

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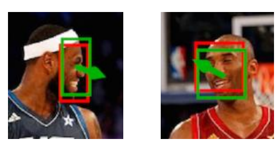
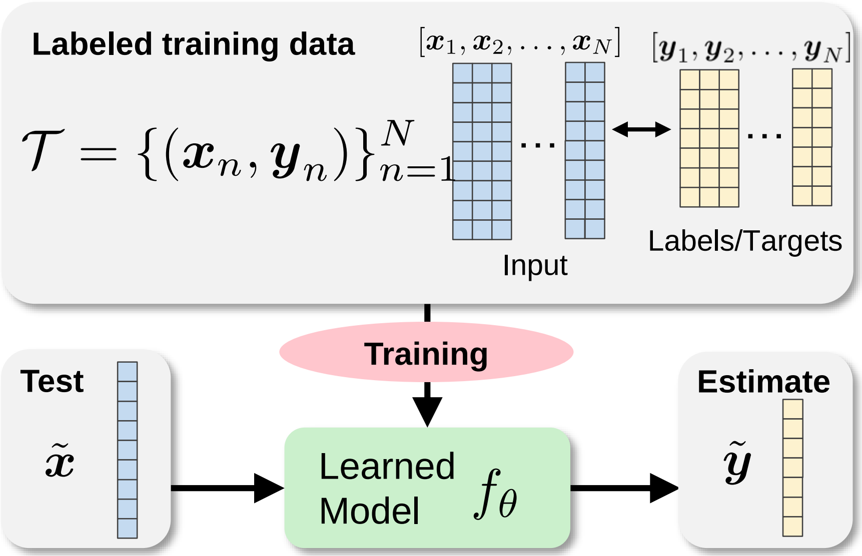


3. Sparse case:
- Multi-classification

Ex. application: *image labelling*

$\boldsymbol{y} =$ 0.1, 0.6, 0.3

Supervised Learning



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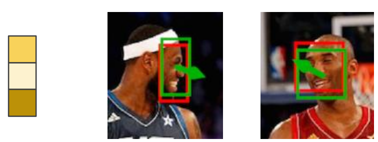

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Unsupervised Learning

Supervised Learning

Labeled training data

$$\mathcal{T} = \{(\boldsymbol{x}_n, \boldsymbol{y}_n)\}_{n=1}^{N}$$

$[\boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_N]$ Input ⟷ $[\boldsymbol{y}_1, \boldsymbol{y}_2, \dots, \boldsymbol{y}_N]$ Labels/Targets

Training

Test $\tilde{\boldsymbol{x}}$ → Learned Model f_θ → **Estimate** $\tilde{\boldsymbol{y}}$

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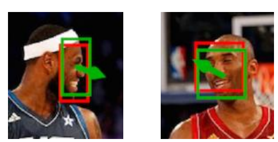

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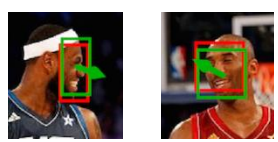

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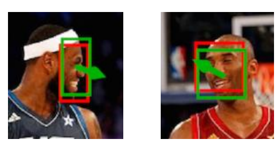

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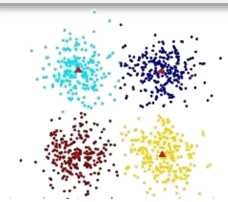

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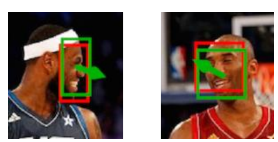

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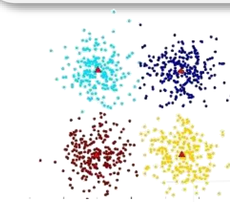

2. Continuous case: ($\dim(\boldsymbol{z}) \ll \dim(\boldsymbol{x})$)
- ▸ Dimensionality Reduction

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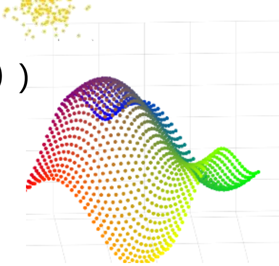

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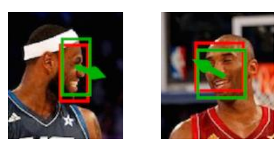

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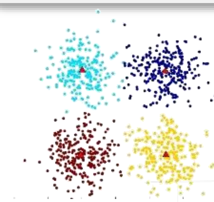

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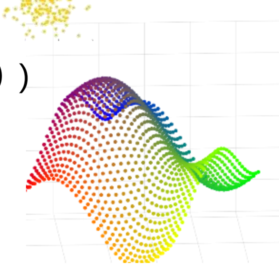

3. Sparse case:
- Dictionary Learning

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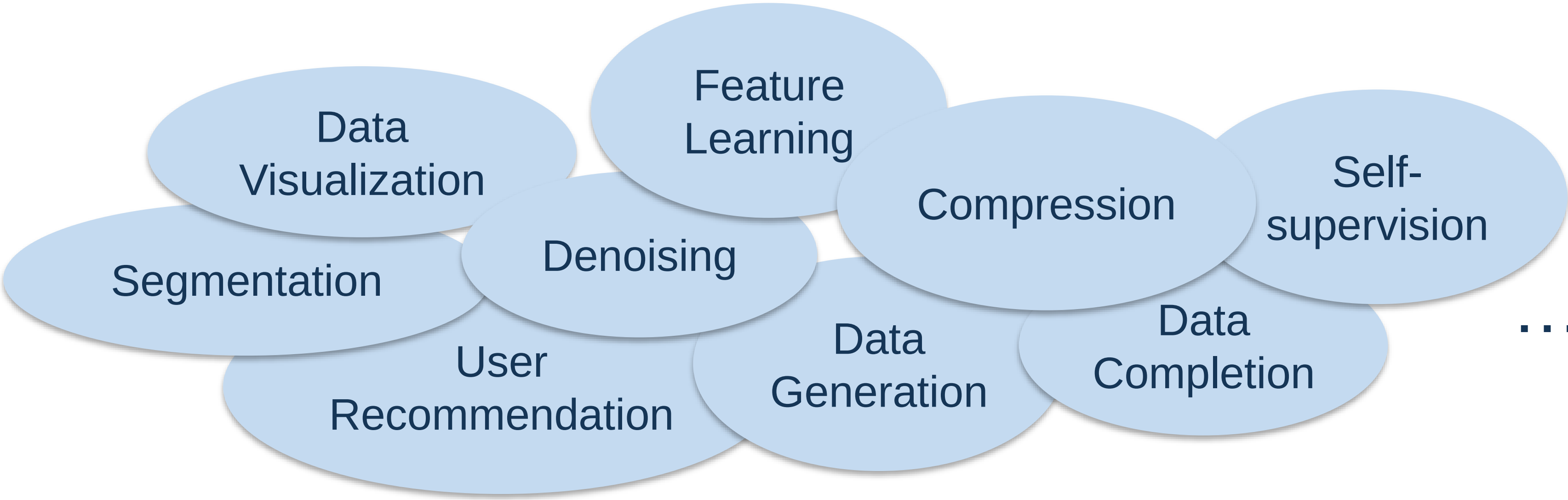
Applications of Unsupervised Learning



Applications of Unsupervised Learning

Feature Learning

Data Visualization

Compression

Self-supervision

Denoising

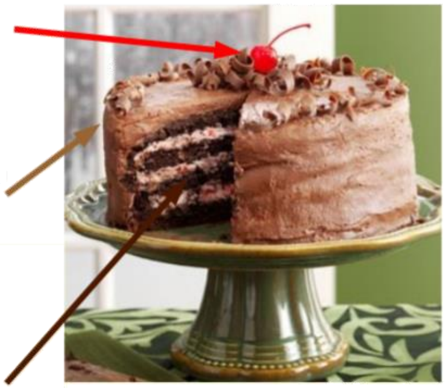
Segmentation

User Recommendation

Data Generation

Data Completion

...



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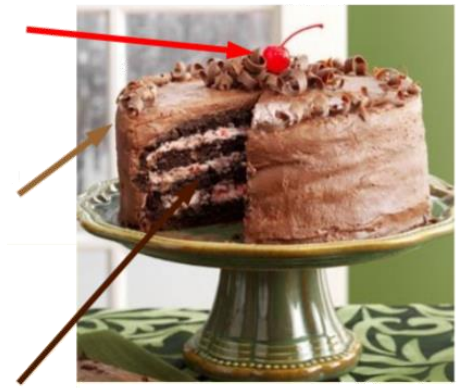
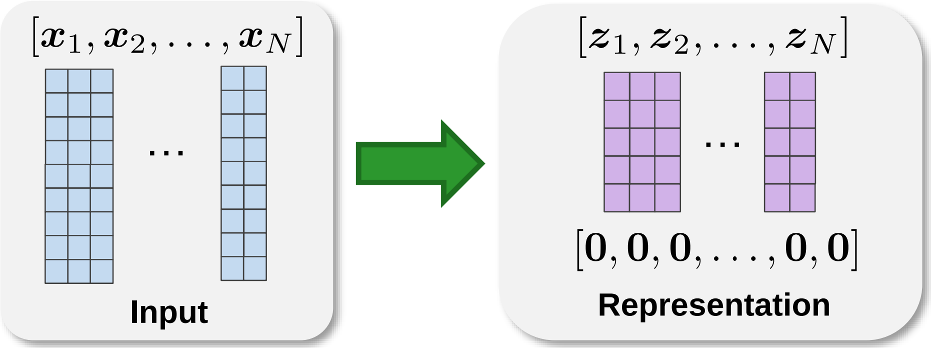


⇨ Potential to learn from massive amount of unlabeled data to generate even more

How does it work?



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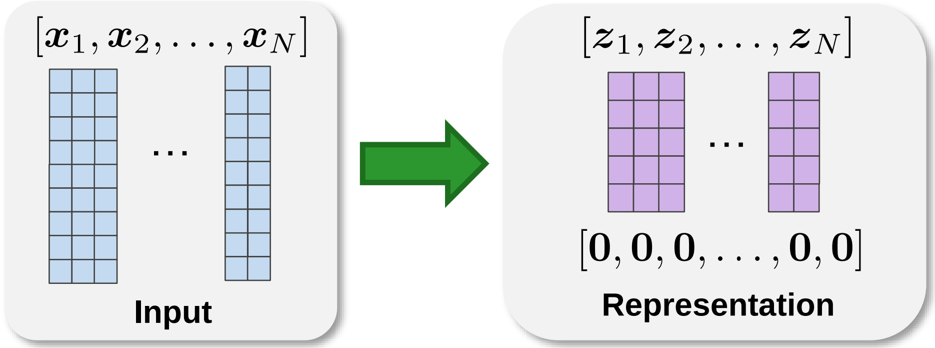

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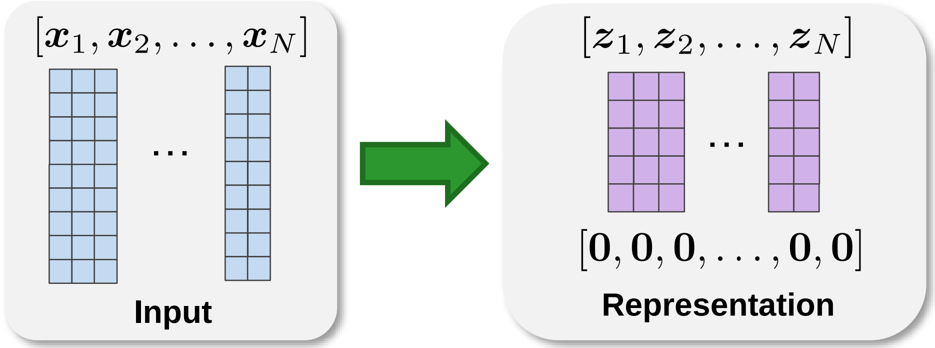

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  - Preserve information from original data
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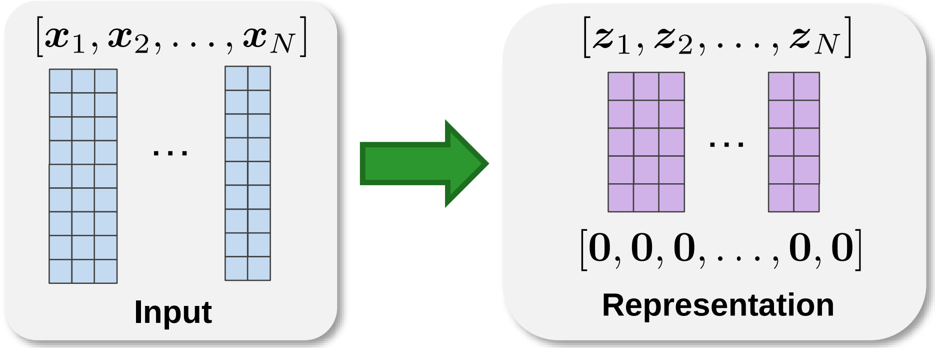

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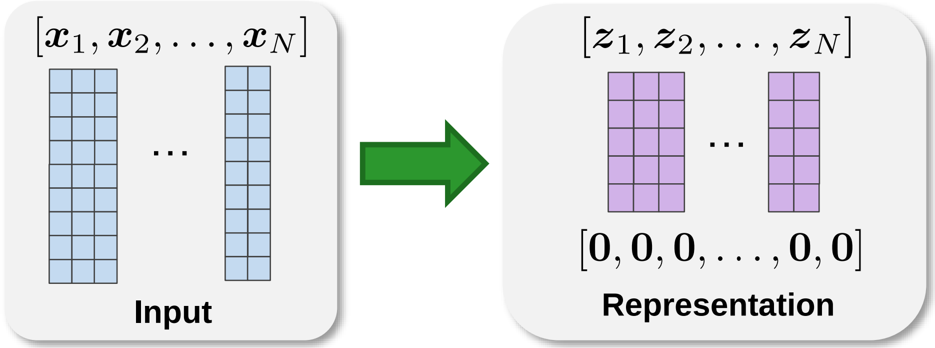

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- **A unifying view :** Generative models

Find $\begin{cases} p_\theta(\boldsymbol{z}) & \text{(as simple as possible)} \\ p_\theta(\boldsymbol{x}|\boldsymbol{z}) & \text{(typically, transformation of } \boldsymbol{z}\text{)} \end{cases}$ so that $p_\theta(\boldsymbol{x}) = \int_{\boldsymbol{z}} p_\theta(\boldsymbol{x}|\boldsymbol{z}) p_\theta(\boldsymbol{z}) d\boldsymbol{z}$

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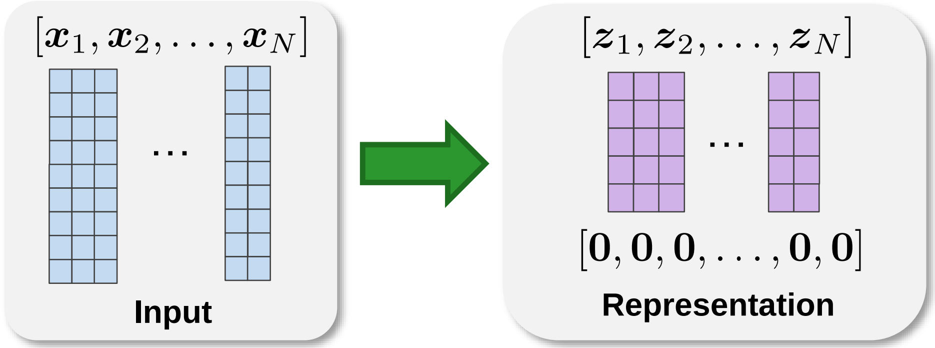

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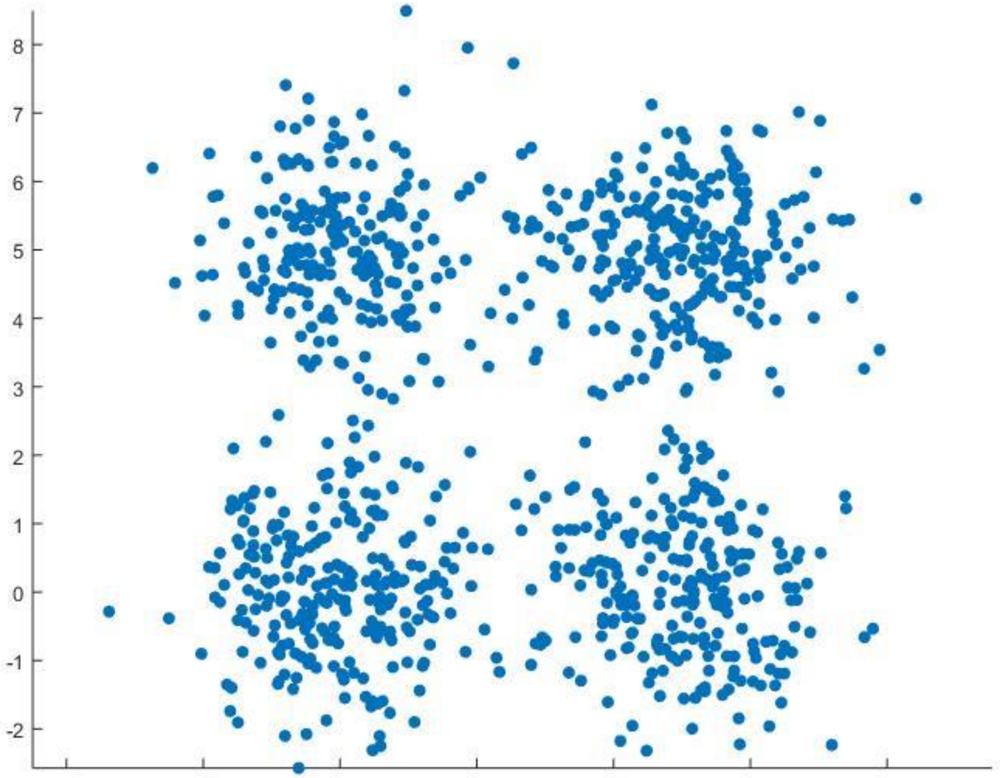
Generation	$p_\theta(\boldsymbol{z}) \rightarrow \boldsymbol{z} \rightarrow \boldsymbol{x}$
Compression	$\boldsymbol{x} \rightarrow p_\theta(\boldsymbol{z}|\boldsymbol{x}) \rightarrow \boldsymbol{z}$
Reconstruction / Completion / Denoising	$\tilde{\boldsymbol{x}} \rightarrow p_\theta(\boldsymbol{z}|\tilde{\boldsymbol{x}}) \rightarrow \boldsymbol{z} \rightarrow p_\theta(\boldsymbol{x}|\boldsymbol{z}) \rightarrow \hat{\boldsymbol{x}}$

How would you cluster/group/separate these 2D points?

$\{\boldsymbol{x}_n\}_{n=1}^{N} \subset \mathbb{R}^2$



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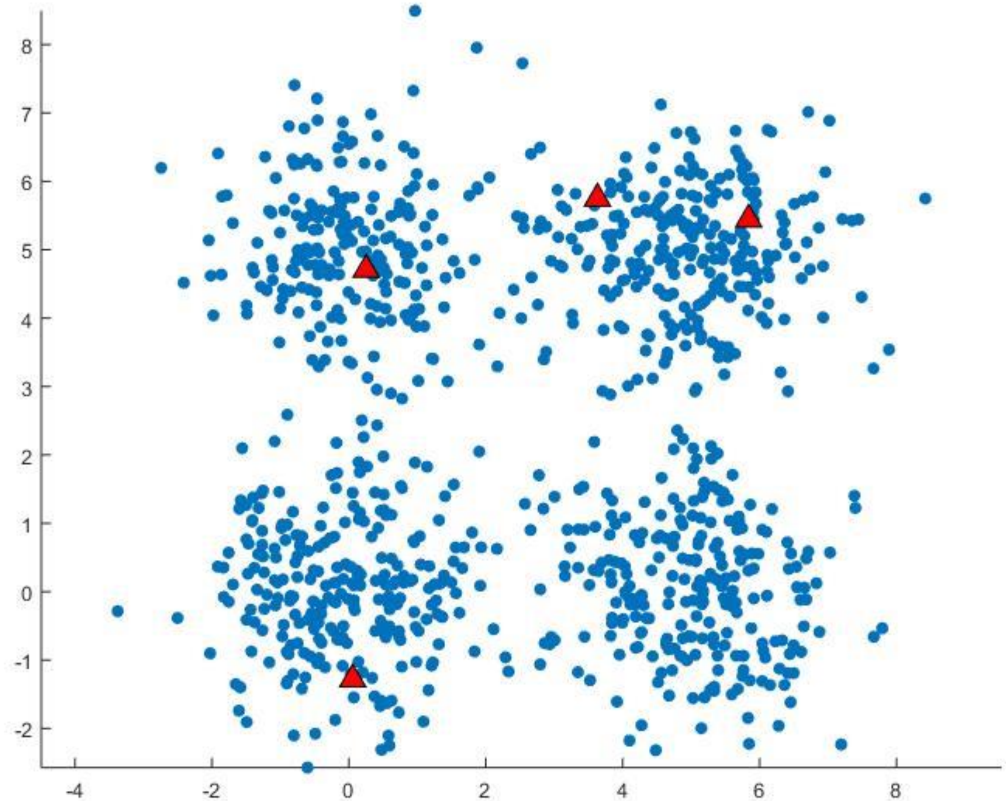

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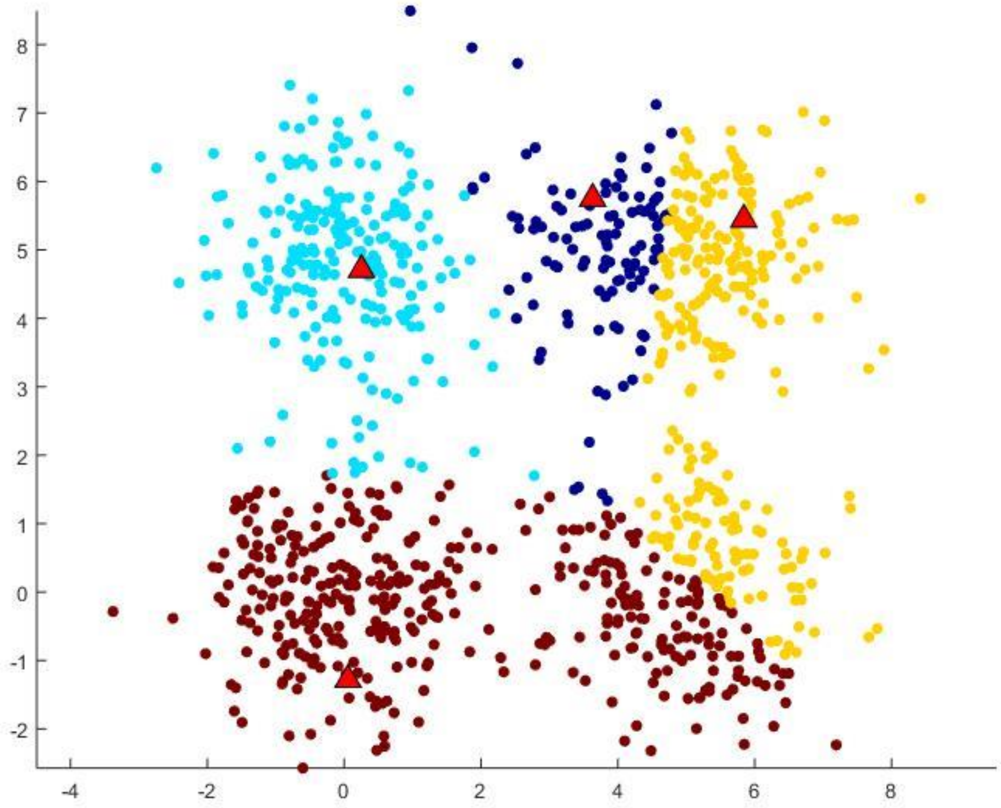

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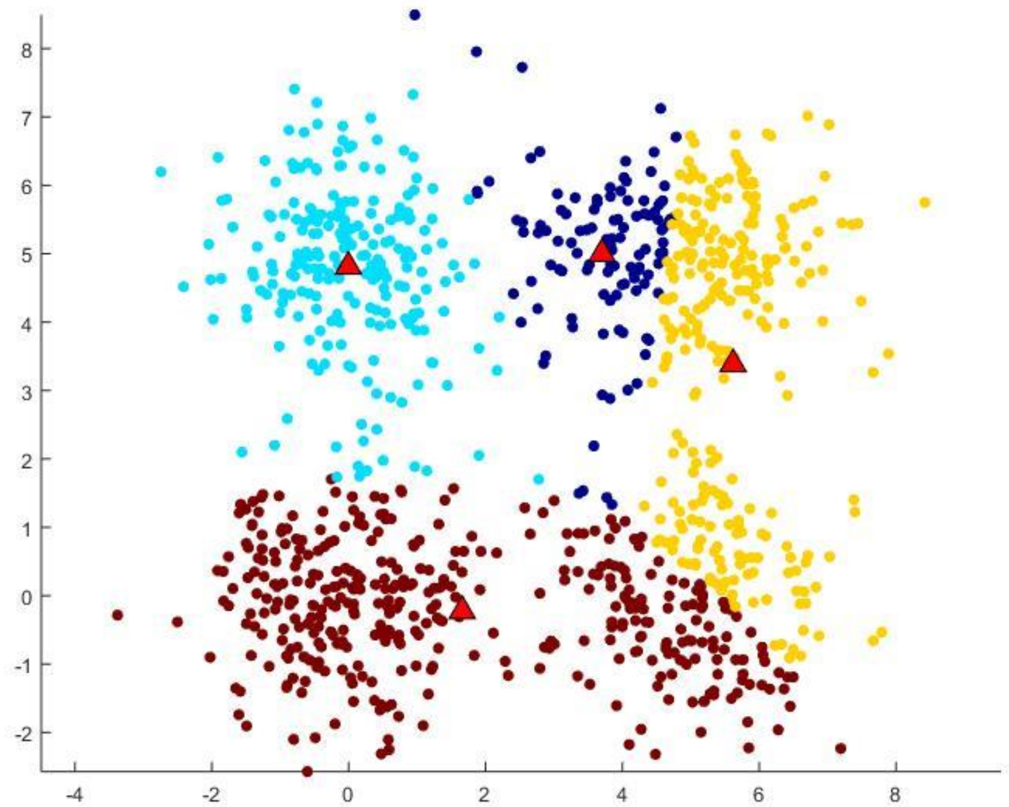

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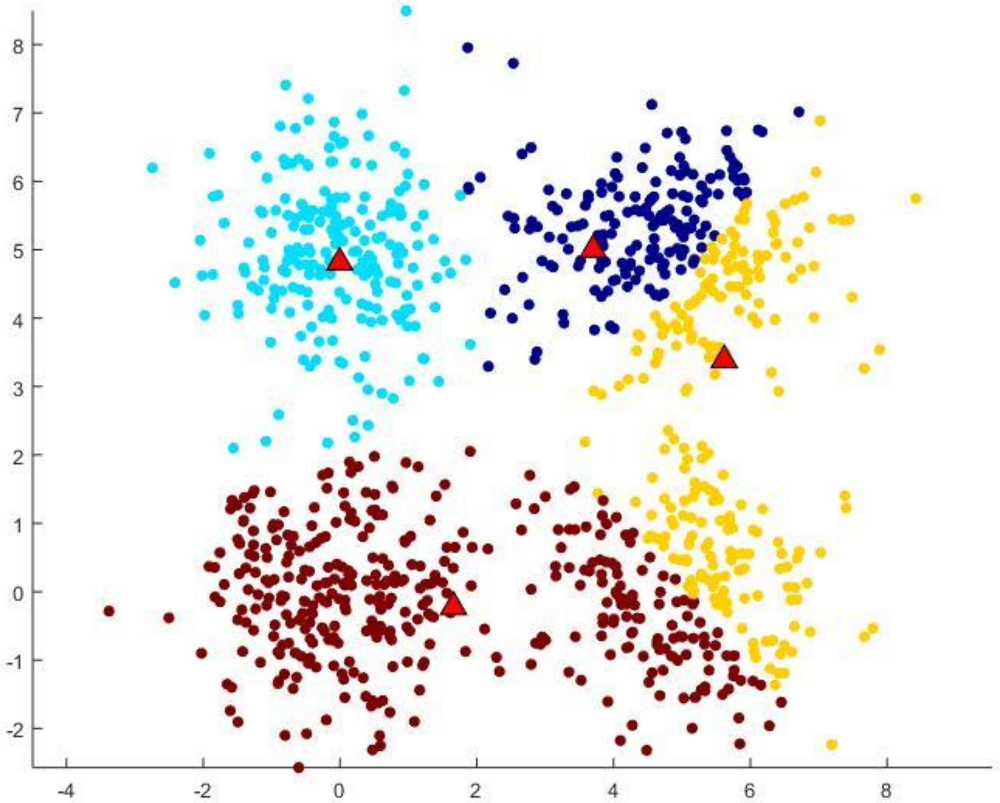

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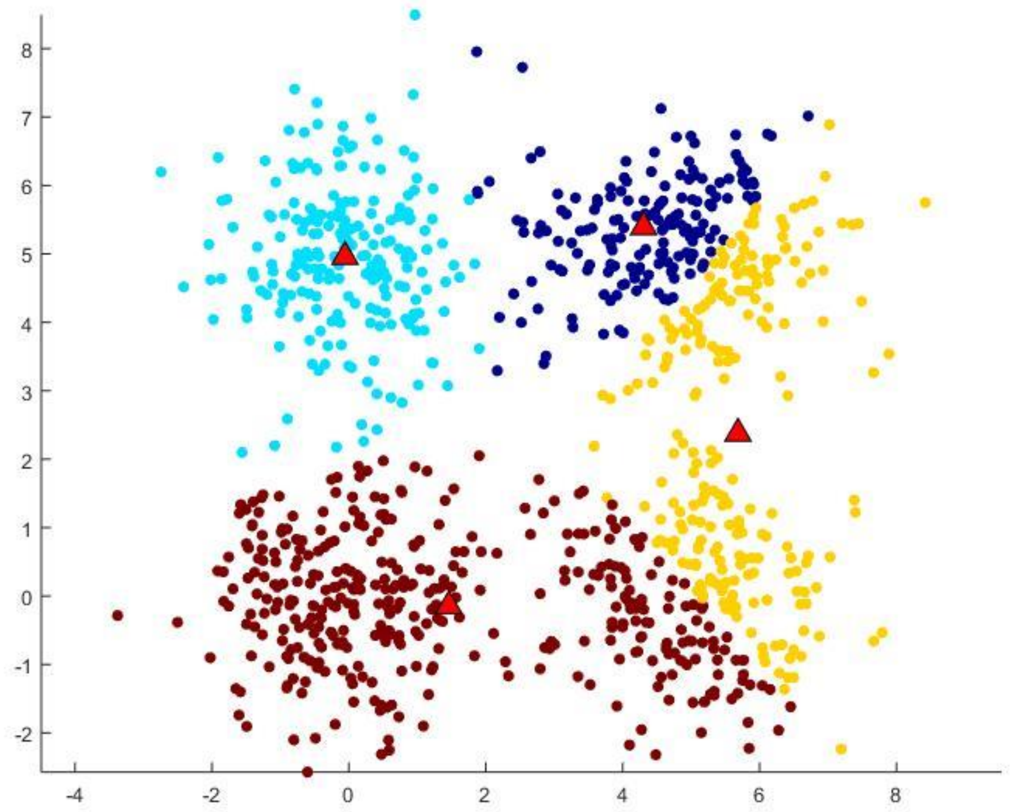

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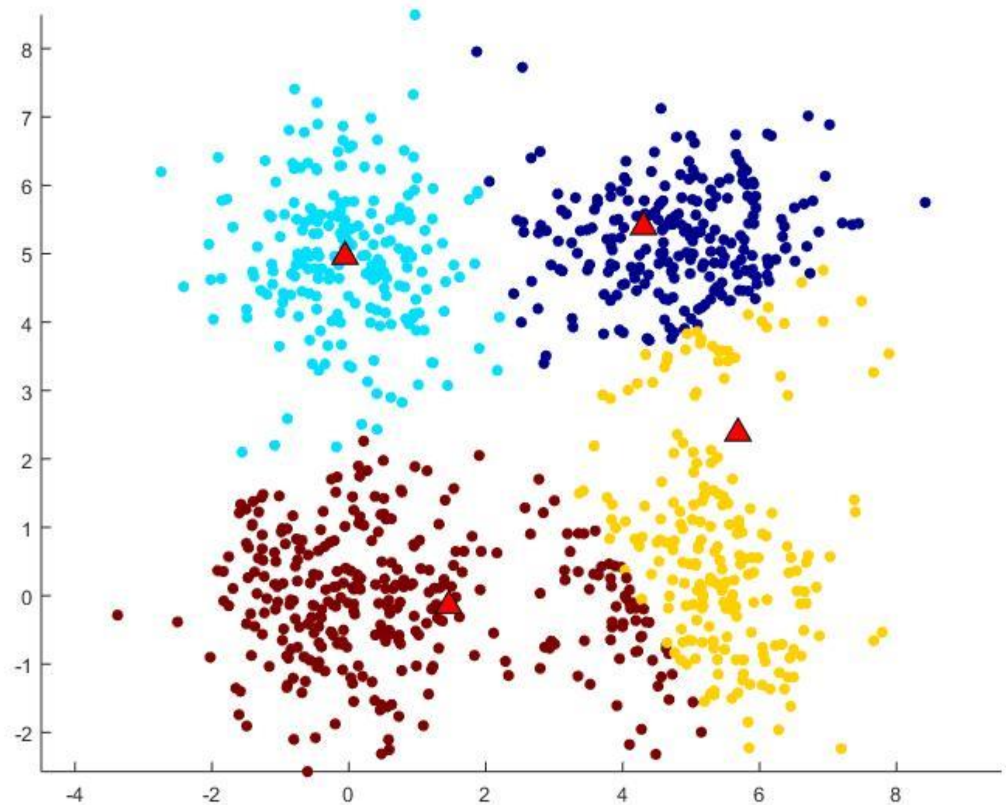

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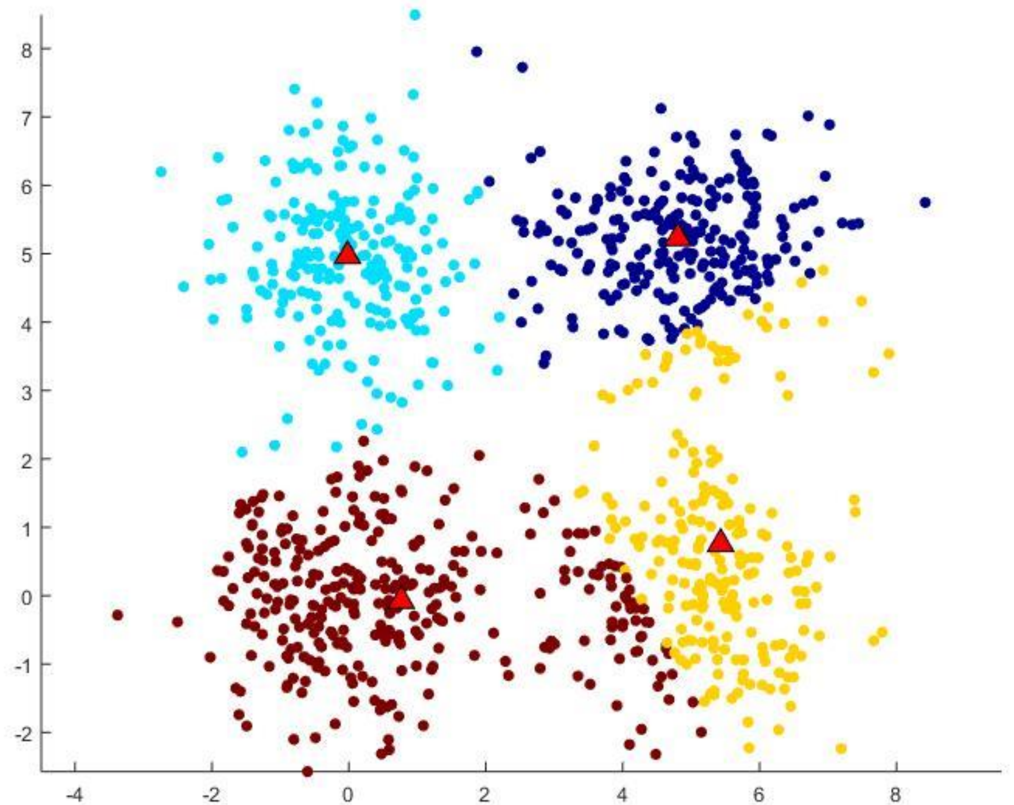

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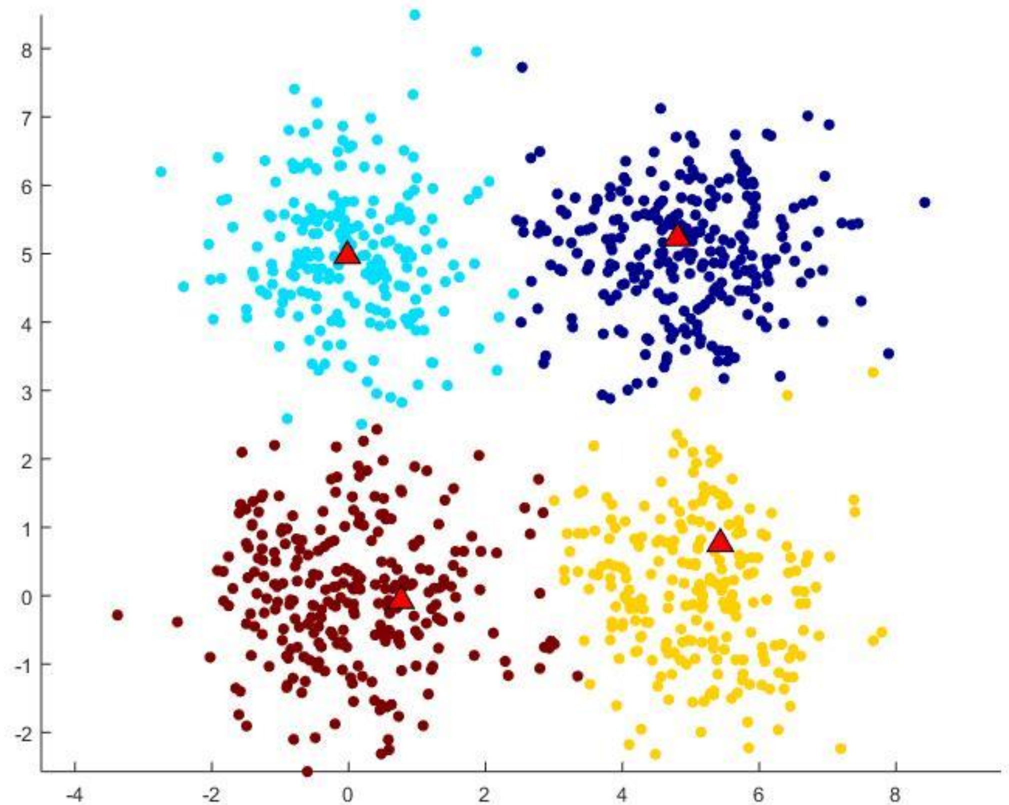

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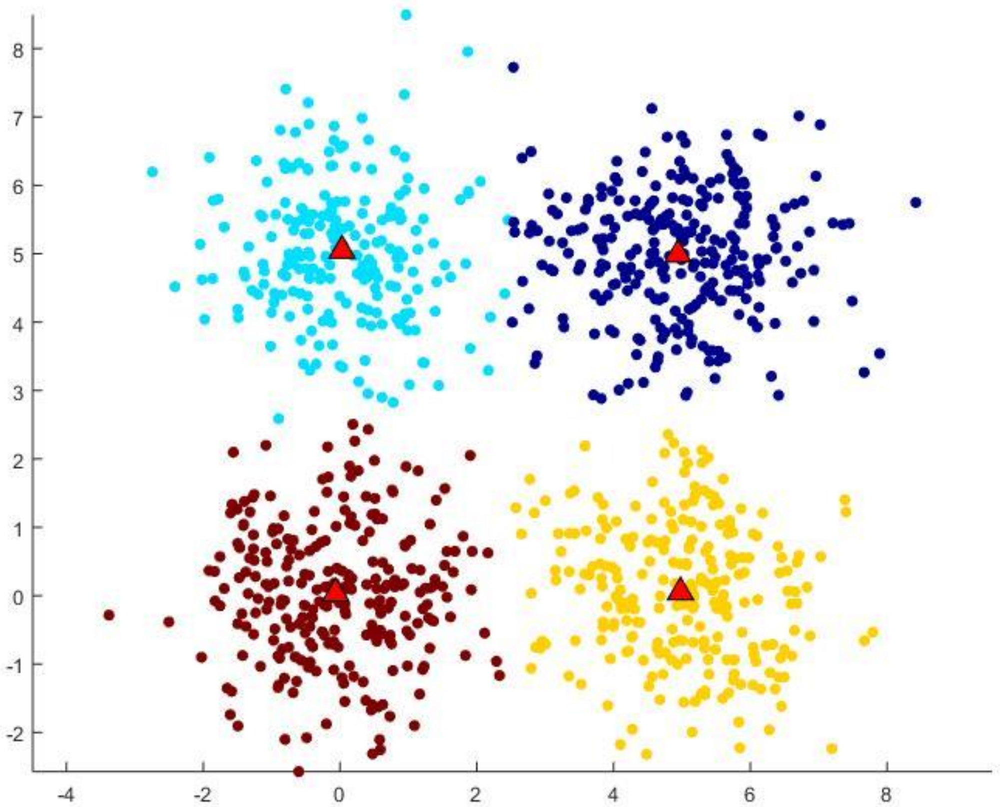

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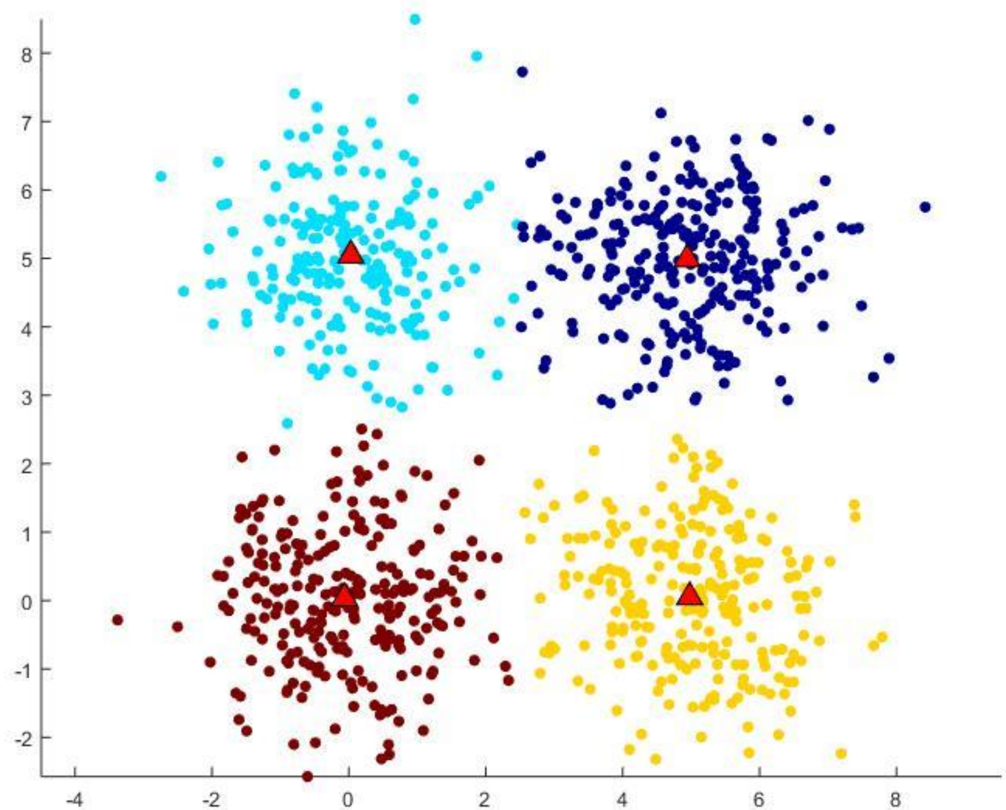

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The K-means Algorithm

K-means: What does it do?

- It decreases this **total loss** at each iteration:

$$\mathcal{C}(\mathbf{X}, \mathcal{G}, \theta) = \sum_{k=1}^{K} \sum_{n \in \mathcal{G}_k} \|\boldsymbol{x}_n - \boldsymbol{c}_k\|^2, \quad \theta = \{\boldsymbol{c}_1, \dots, \boldsymbol{c}_K\}$$

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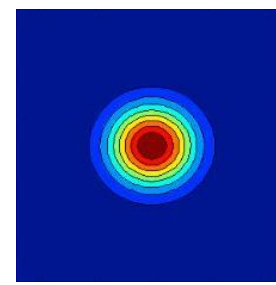

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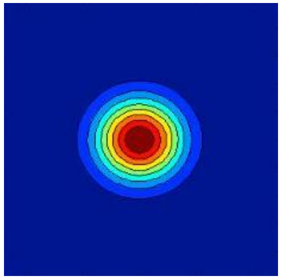

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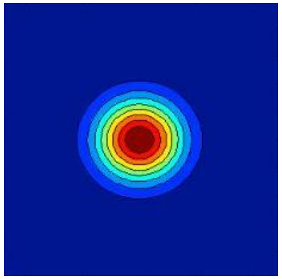

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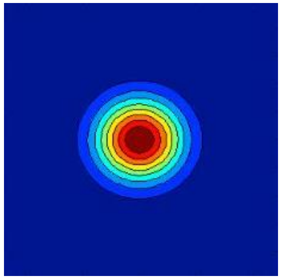

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K-means can be seen as a *maximum a posteriori* (MAP) approach!

Gaussian mixture models: a generalization

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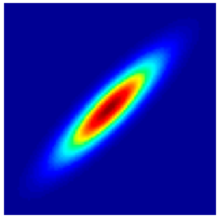

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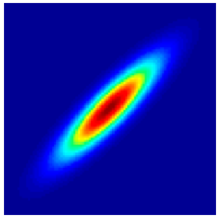

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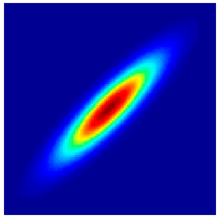

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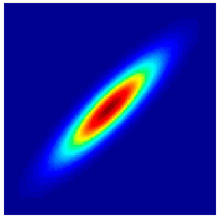

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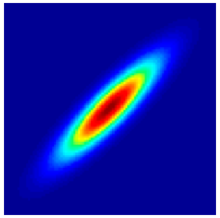

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Repeat until convergence:

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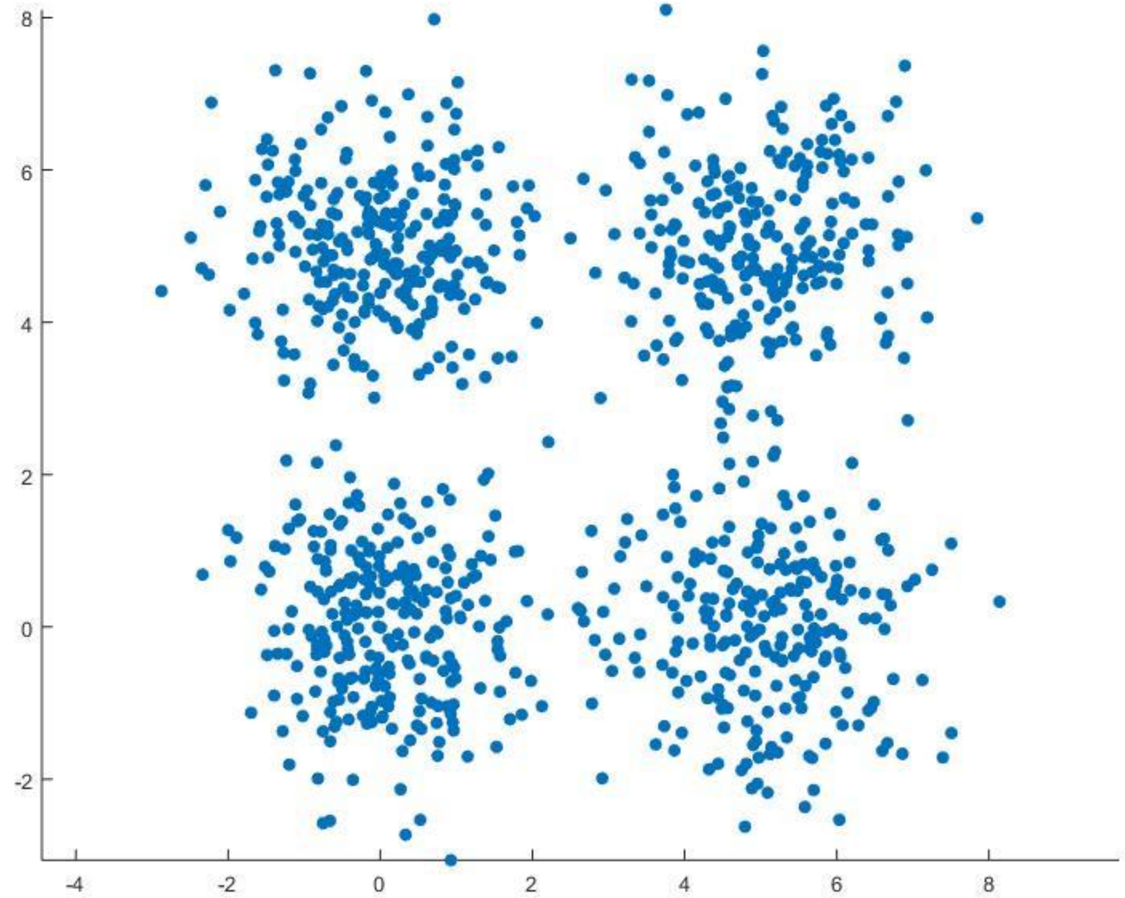
- $\pi_k^{(i)} = \frac{1}{N}\sum_n r_{n,k}^{(i)} = \frac{\bar{r}_k}{N}$
- $\boldsymbol{c}_k^{(i)} = \frac{1}{\bar{r}_k}\sum_n r_{n,k}^{(i)}\boldsymbol{x}_n$
- $\Sigma_k^{(i)} = \frac{1}{\bar{r}_k}\sum_n r_{n,k}^{(i)}(\boldsymbol{x}_n - \boldsymbol{c}_k^{(i)}) \cdot (\boldsymbol{x}_n - \boldsymbol{c}_k^{(i)})^\top$

Gaussian Mixture Model Expectation-Maximization (GMM EM)



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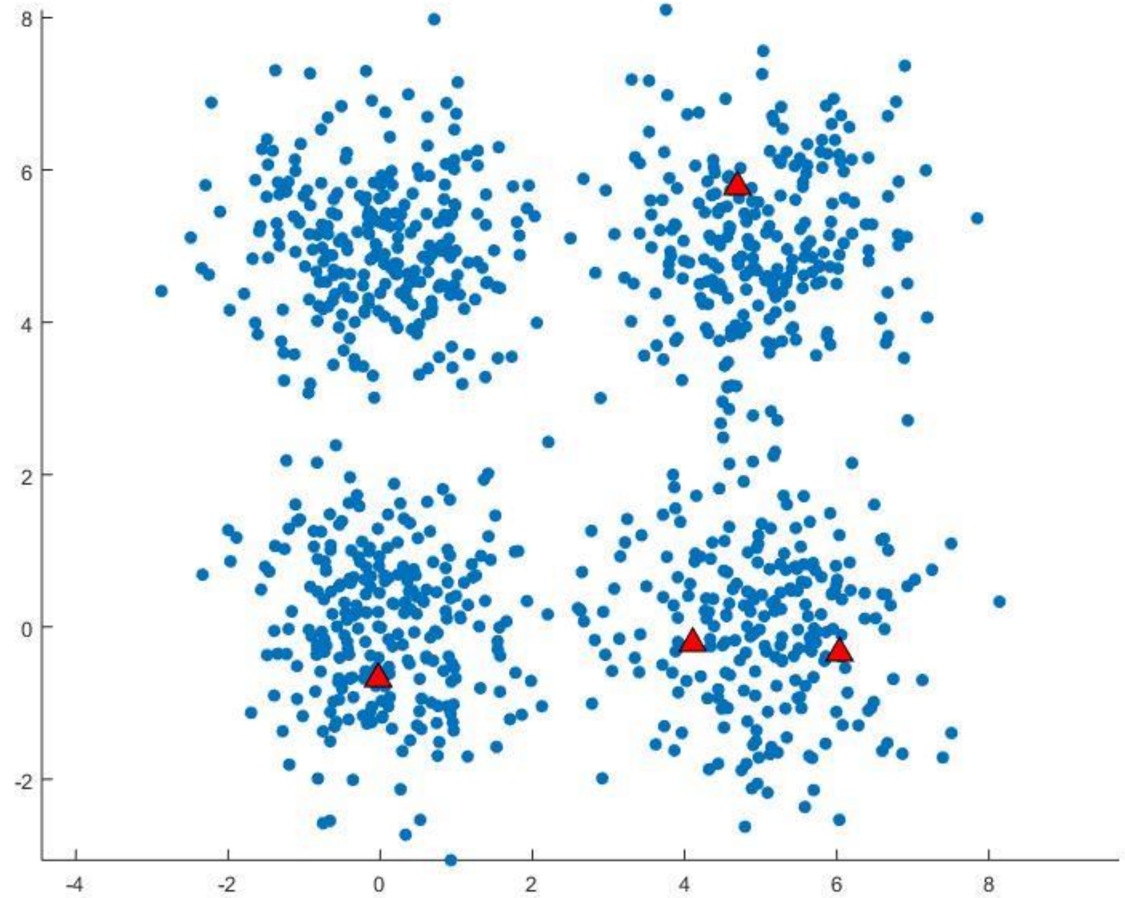

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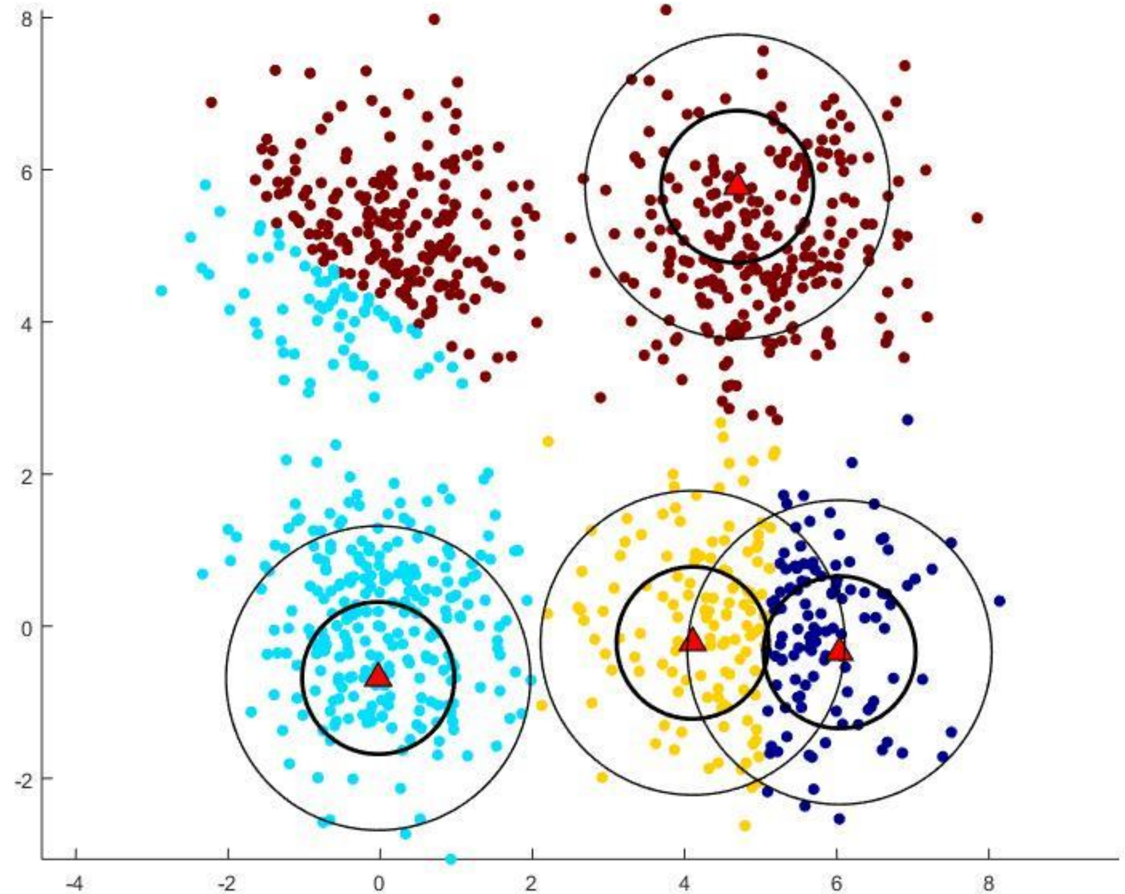

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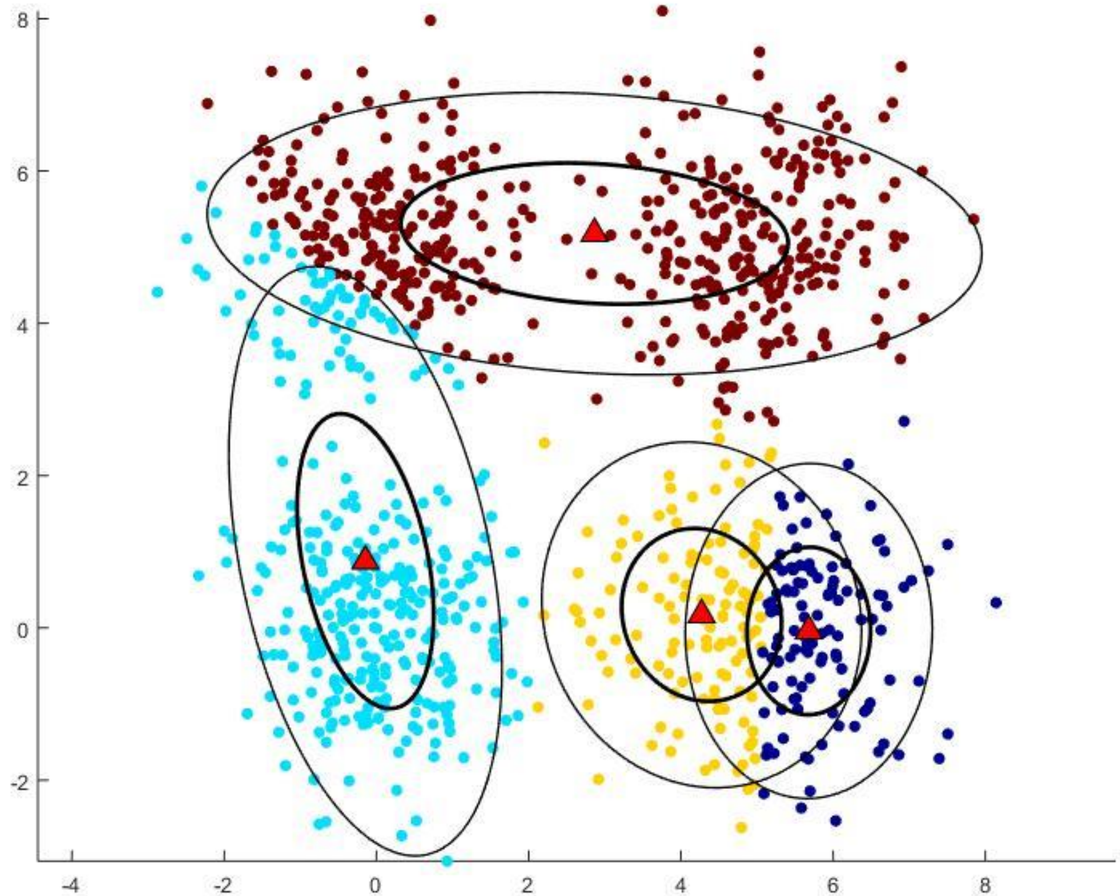

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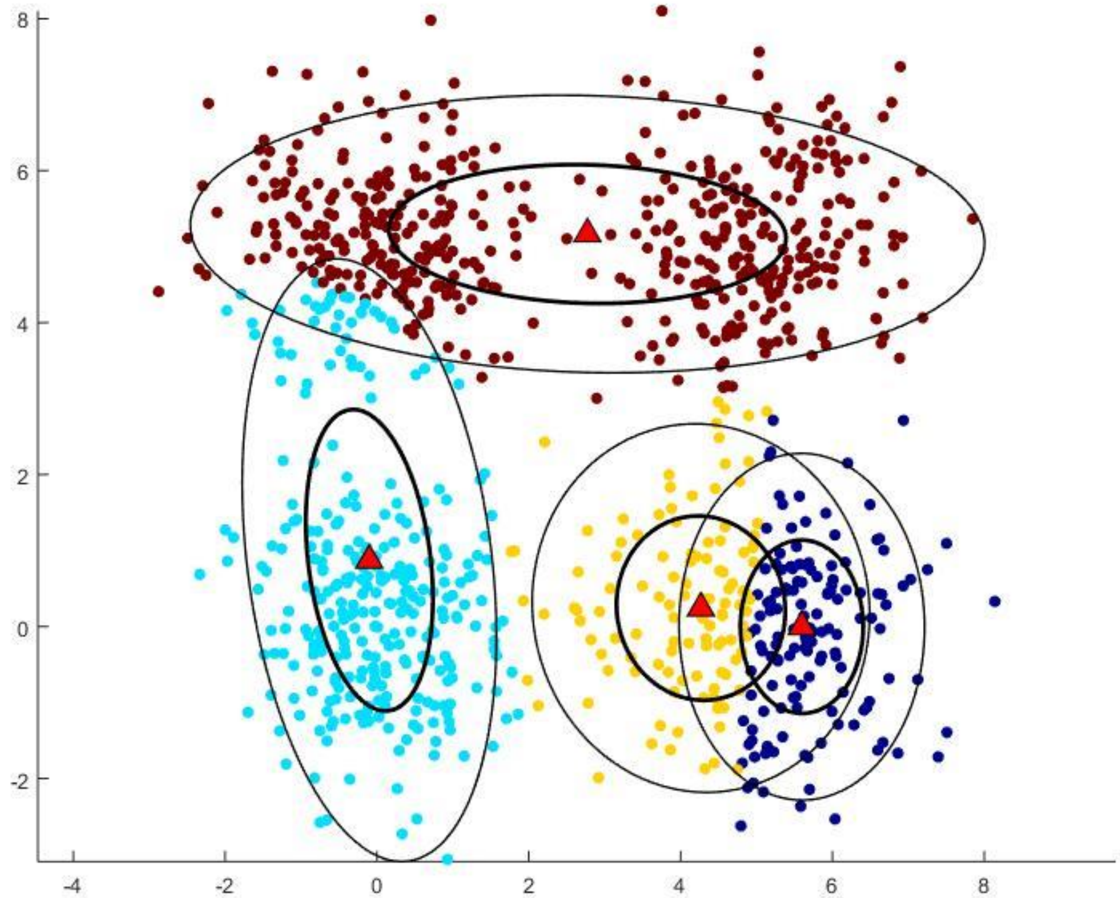

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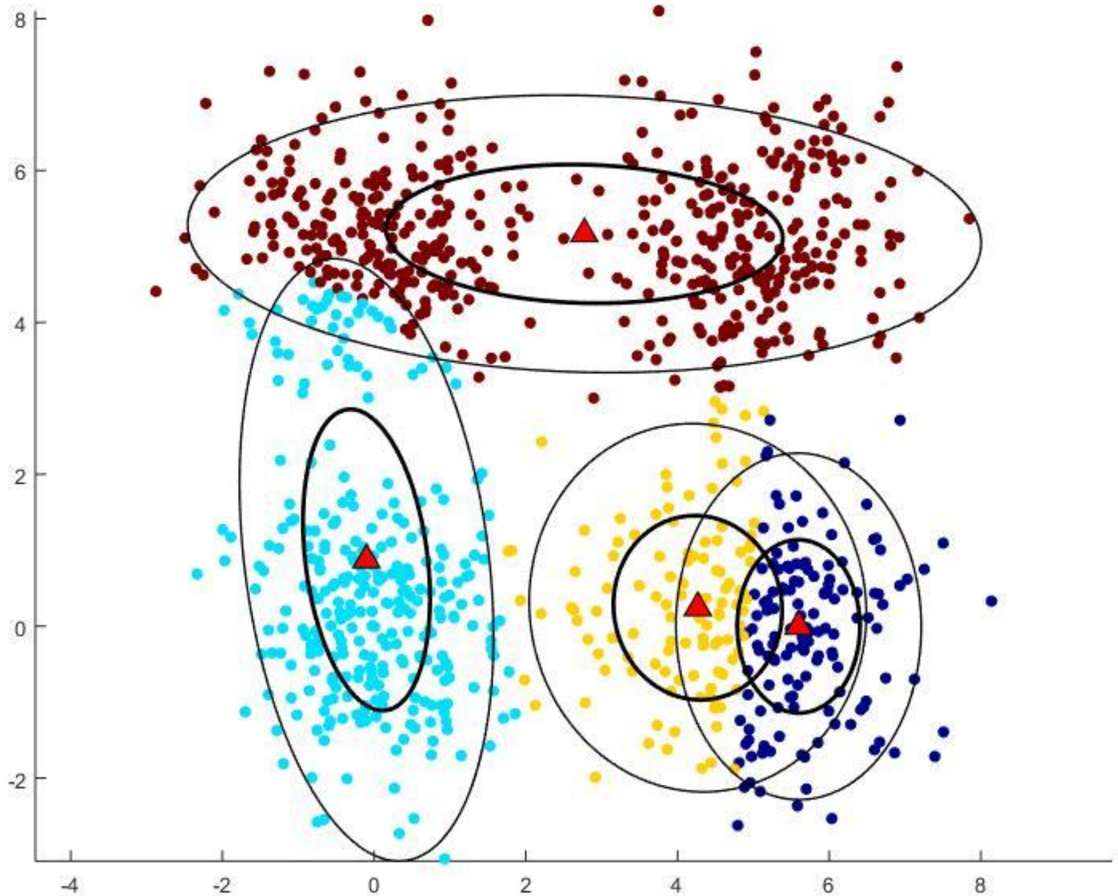

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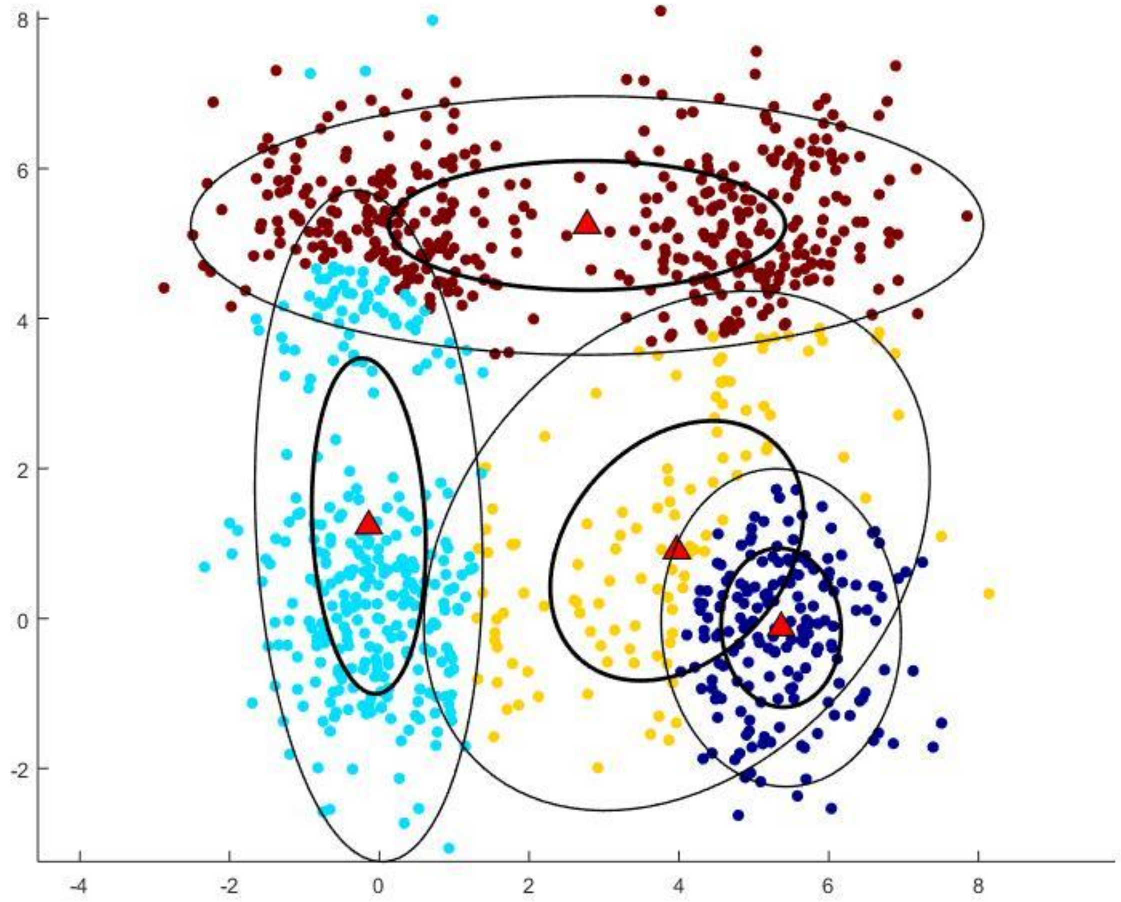

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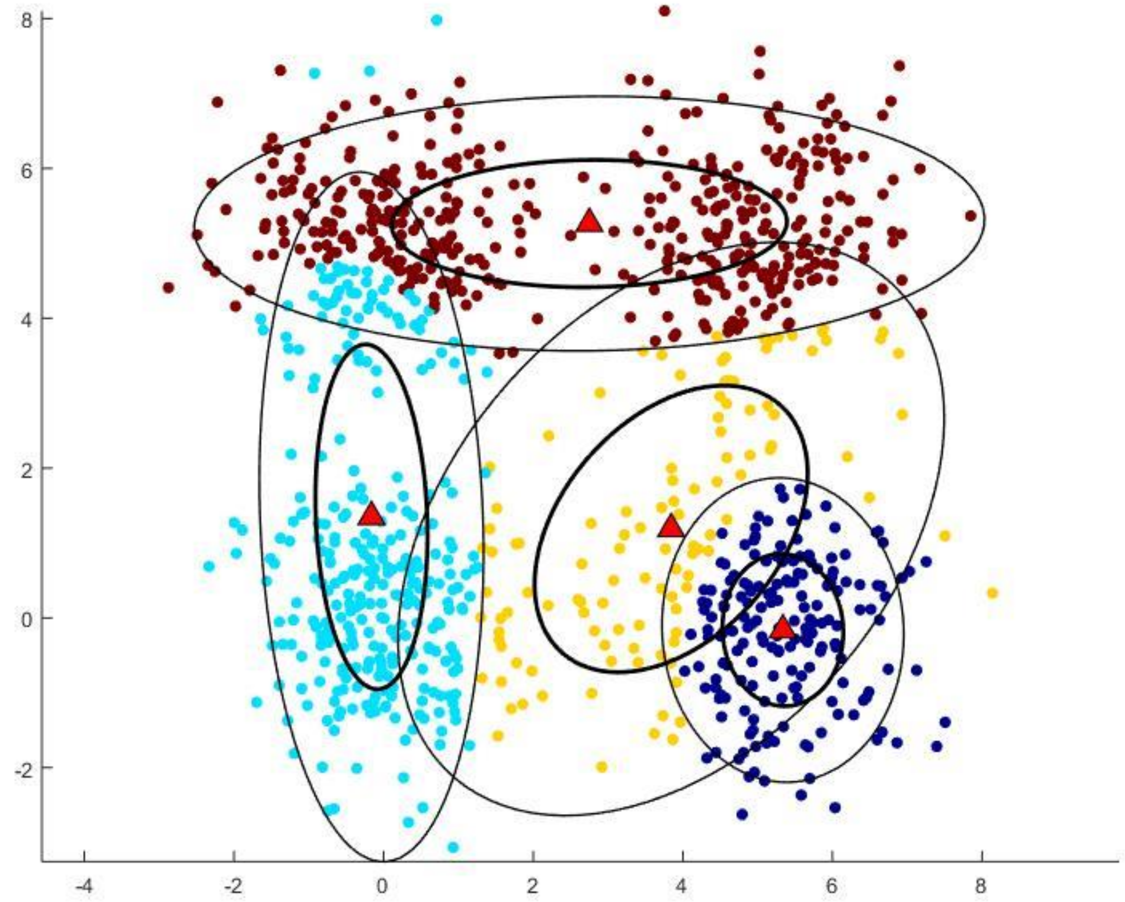

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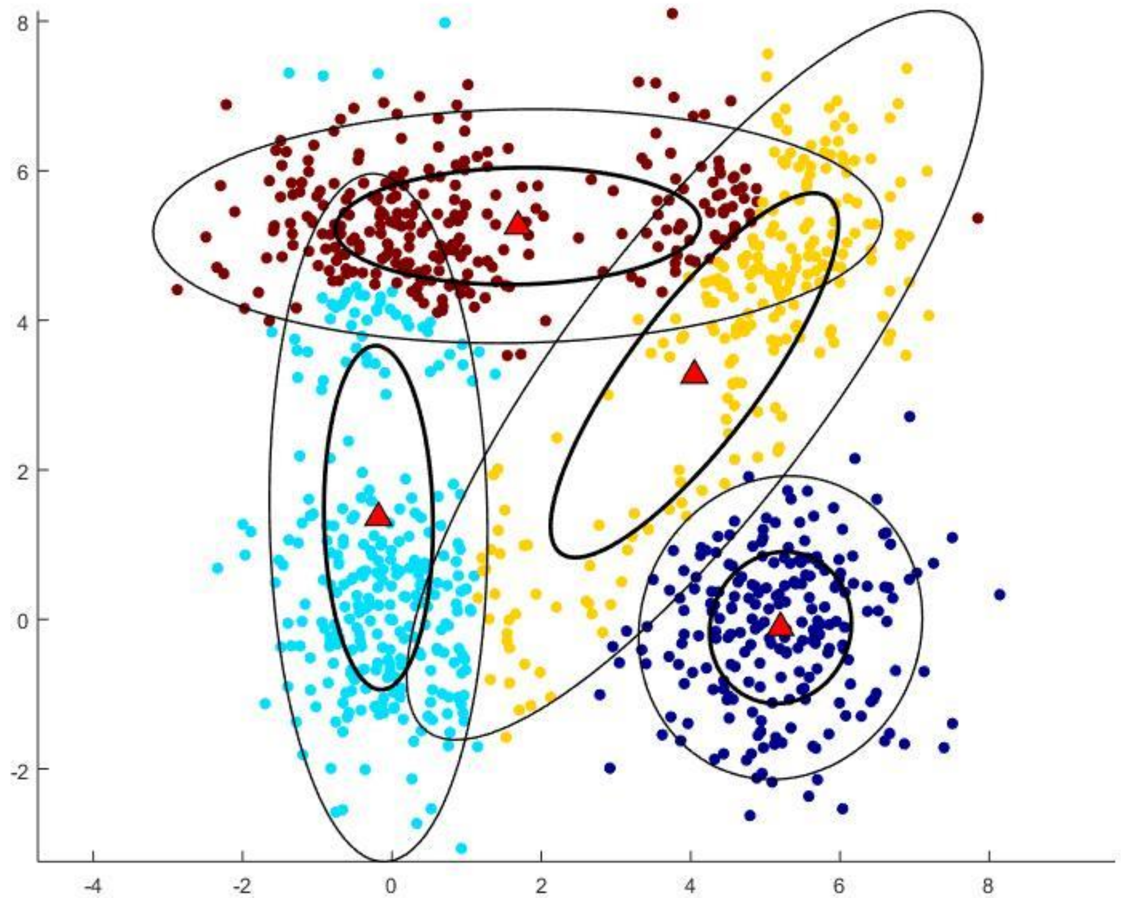

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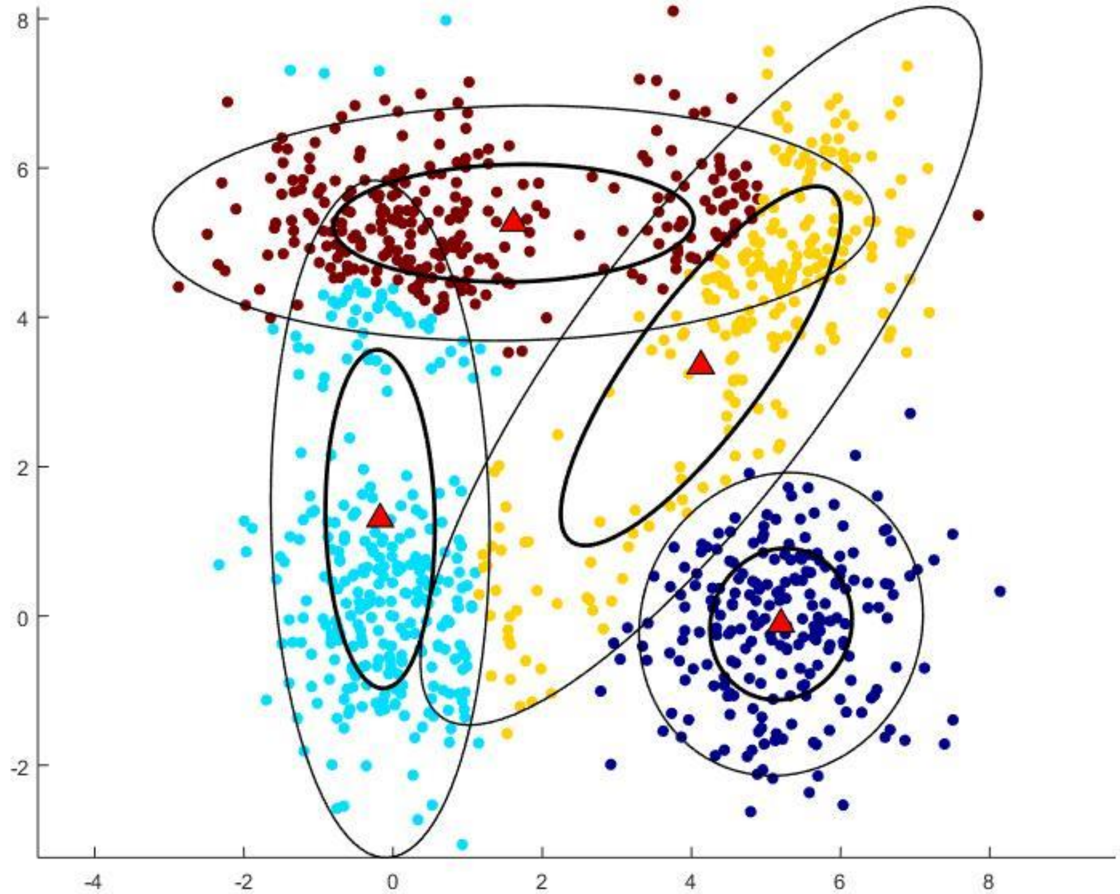

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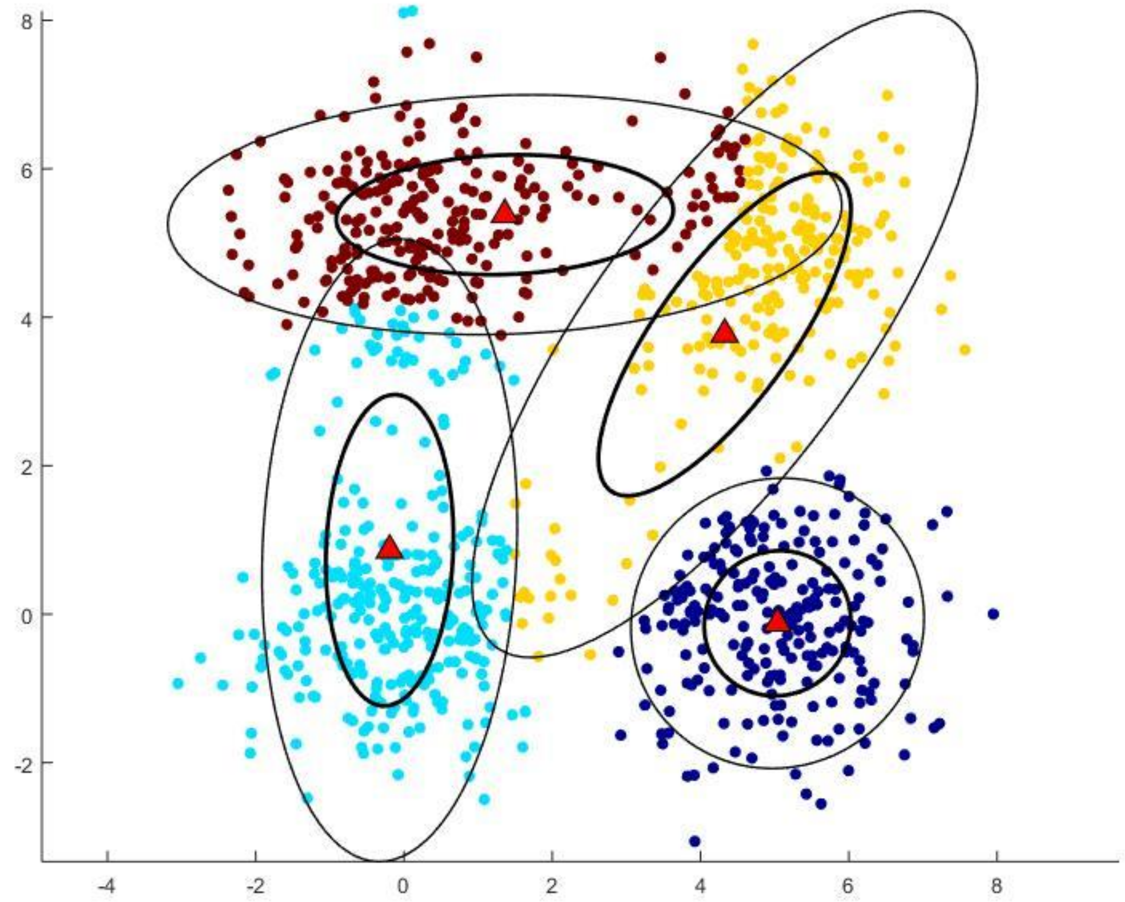

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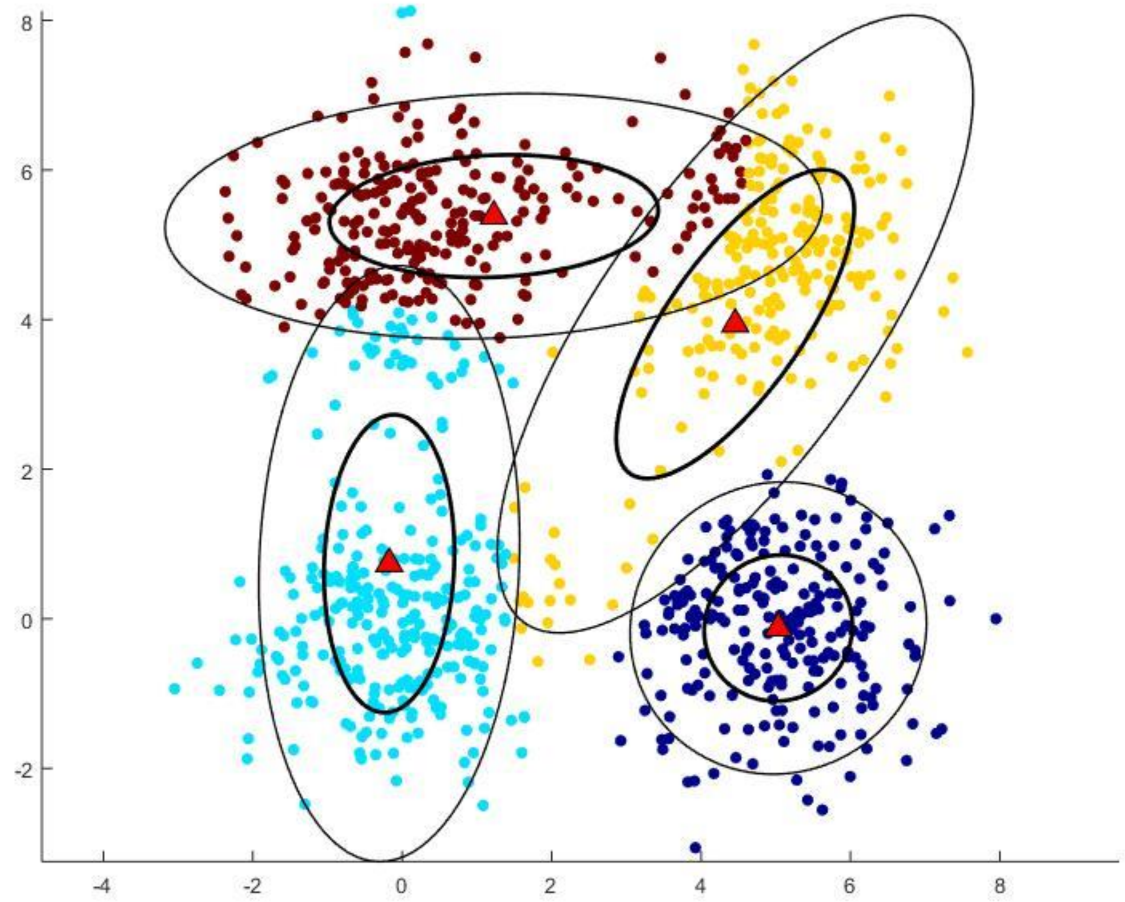

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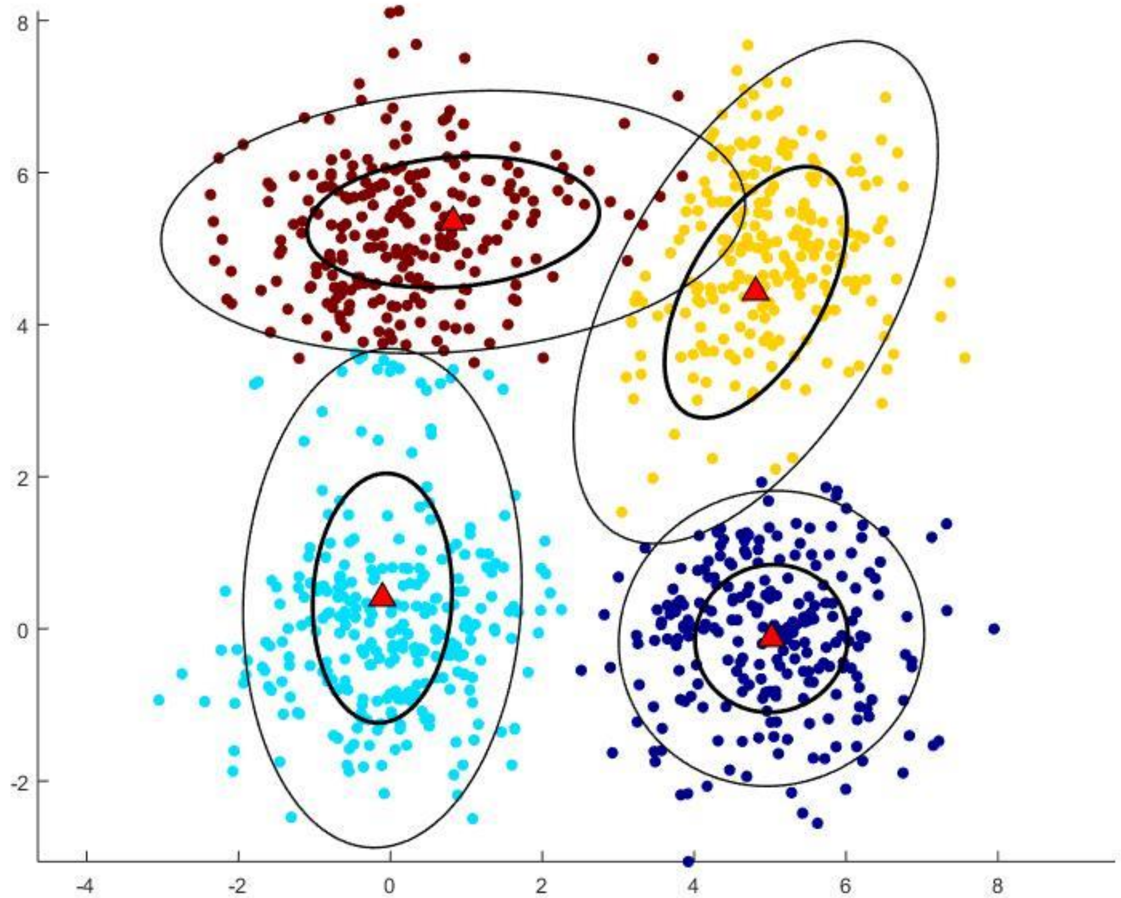

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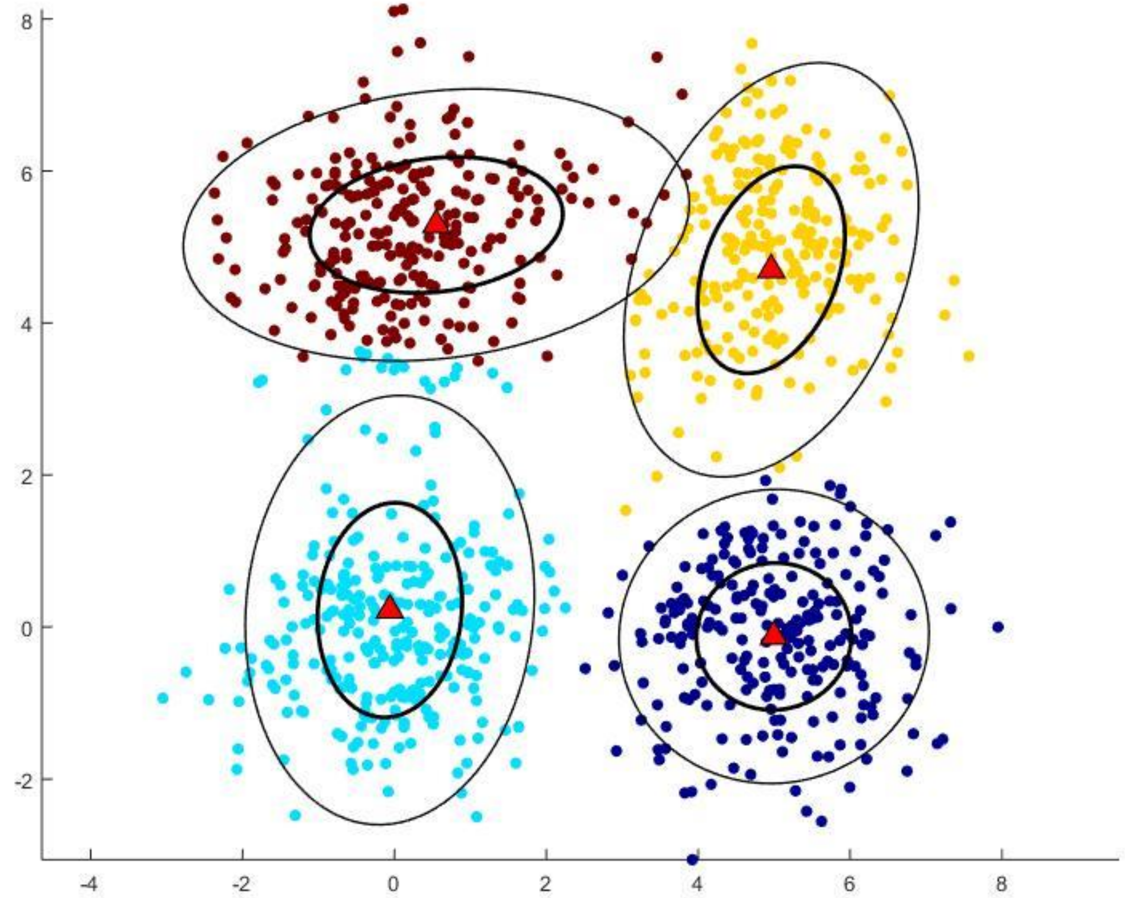

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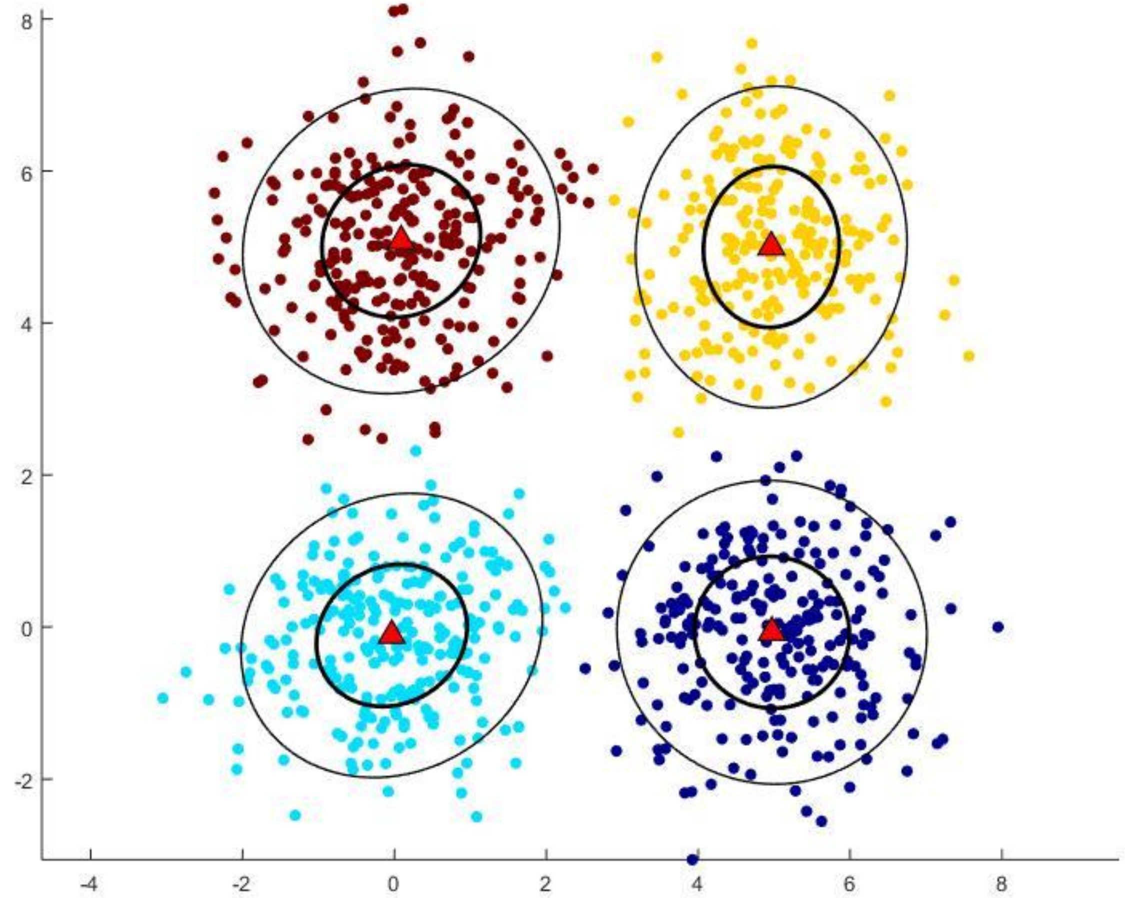

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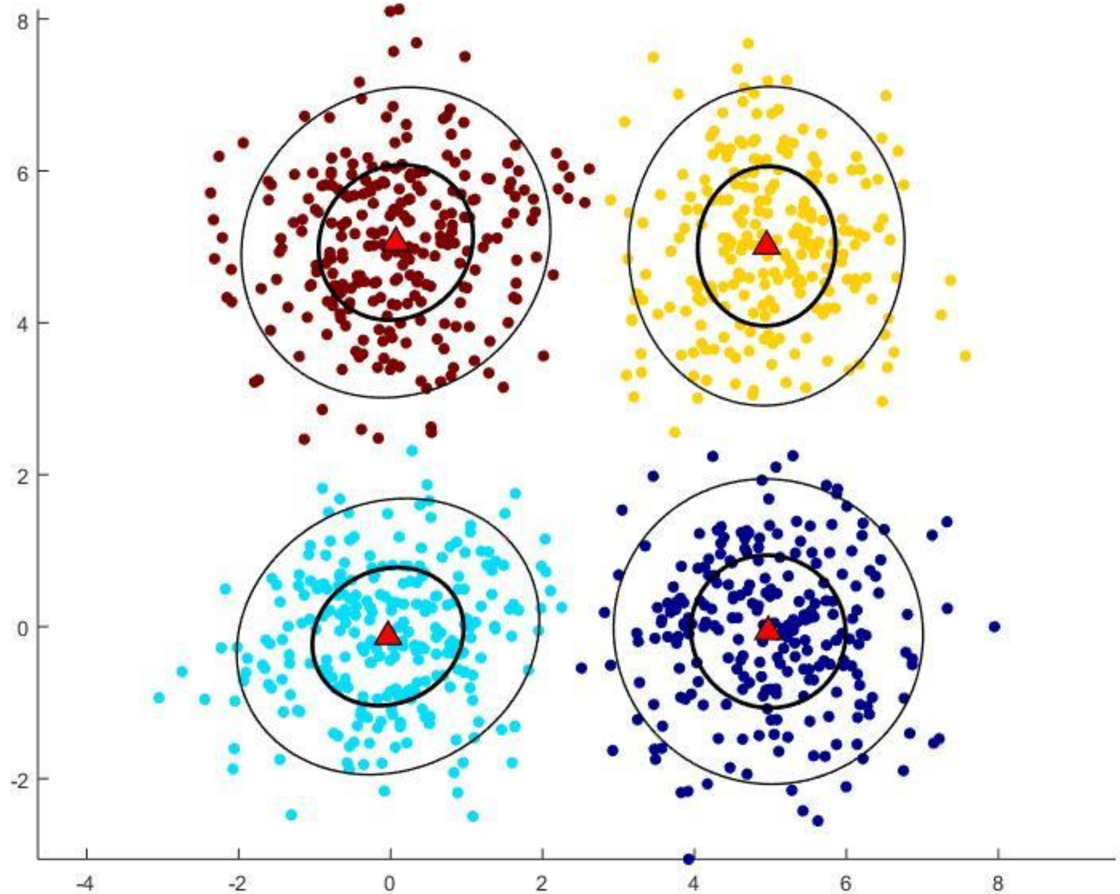

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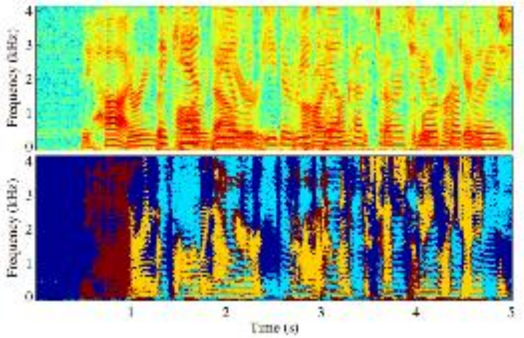

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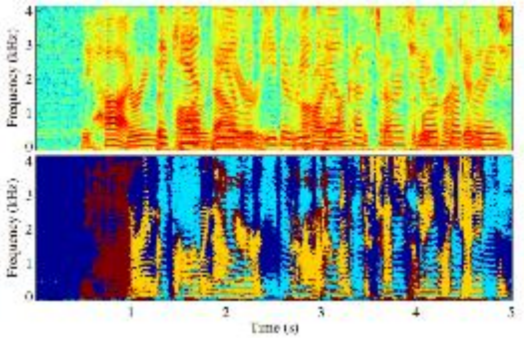
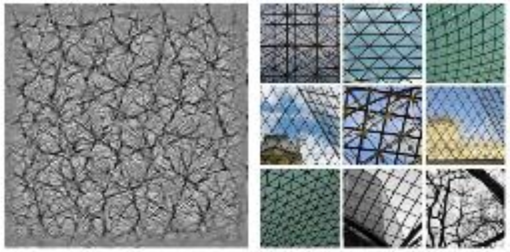


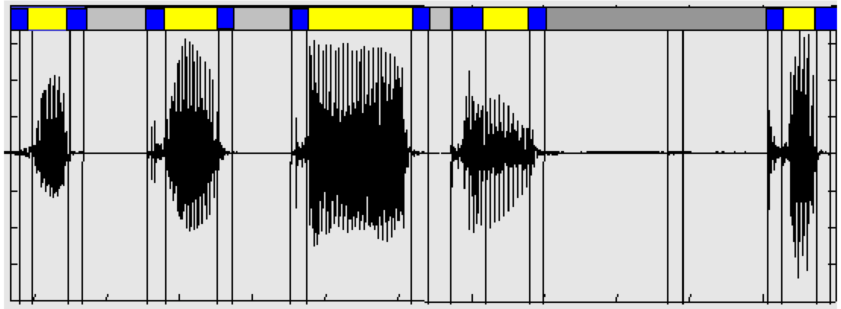
- Application to feature learning from images:

Mathilde Caron, Piotr Bojanowski, Armand Joulin, and Matthijs Douze. **"Deep clustering for unsupervised learning of visual features."** In *Proceedings of the European Conference on Computer Vision (ECCV)*, pp. 132-149. 2018.



Typical applications of clustering

Image segmentation ($\{x_n\}_{n=1}^N$ are local descriptors)



Audio segmentation ($\{x_n\}_{n=1}^N$ are sound segment)

Anti-spam filters

Data Quantification

DNA sequence analysis

Medical imaging

Speech diarization

Species classificaton (biology)

Social network analysis

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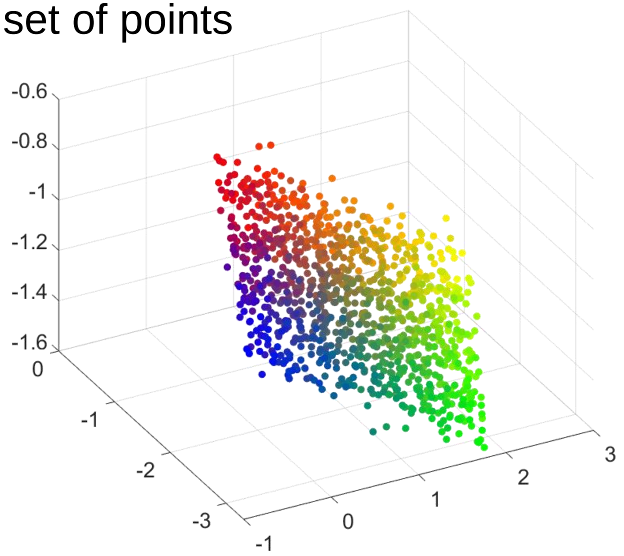

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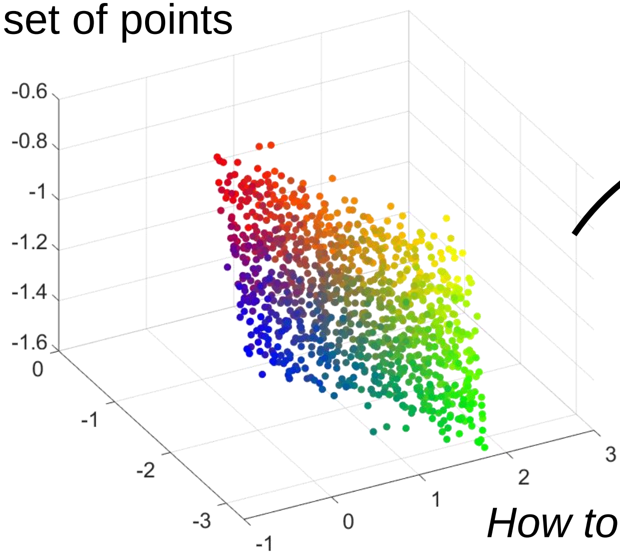

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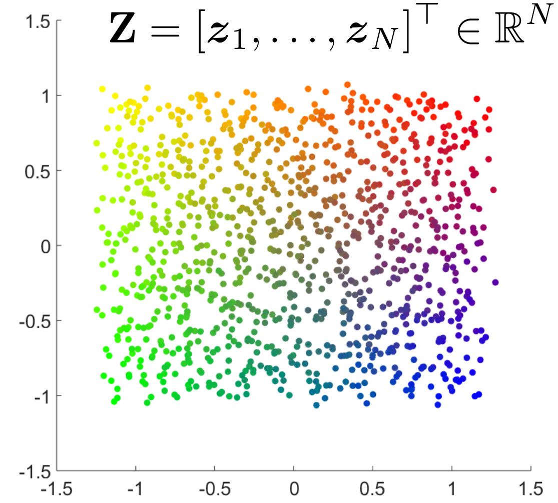

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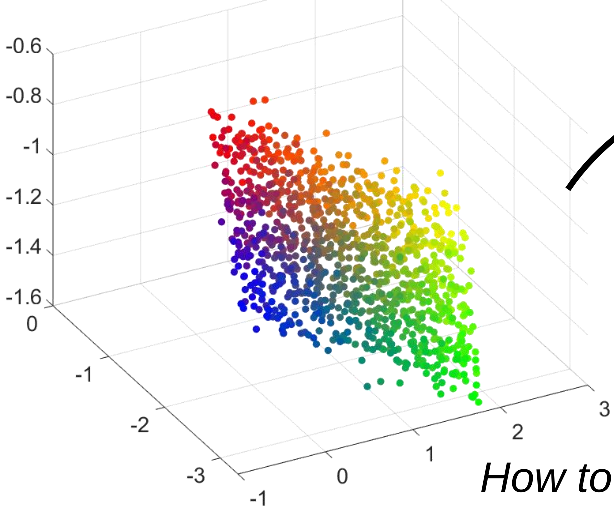

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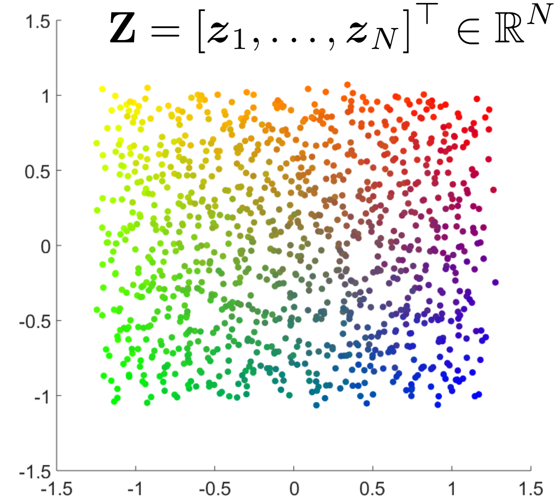

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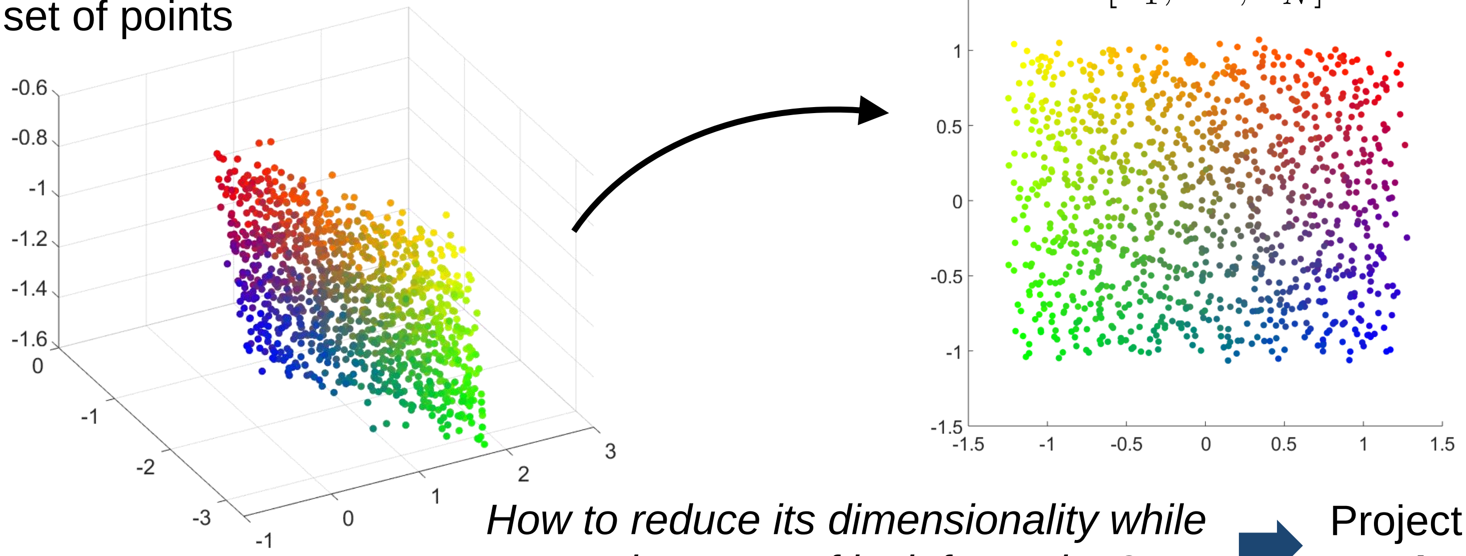

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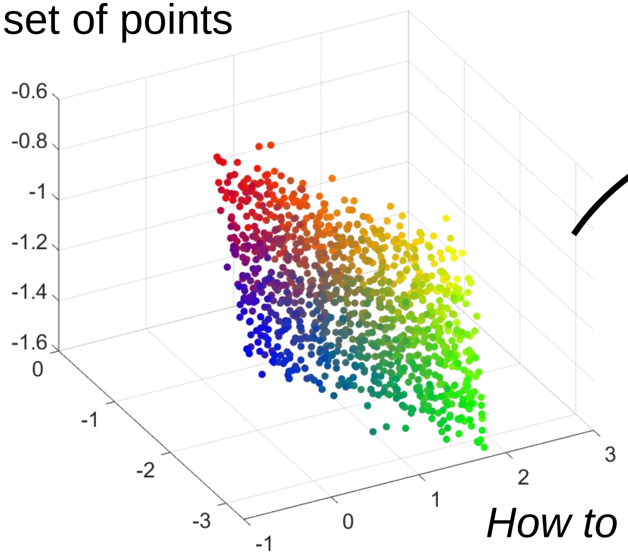

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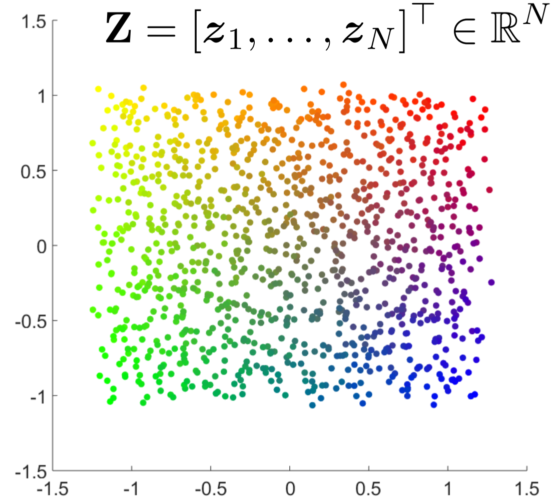

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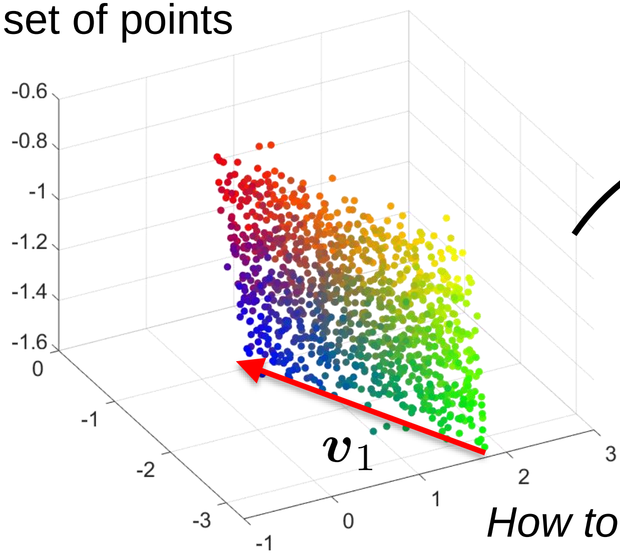

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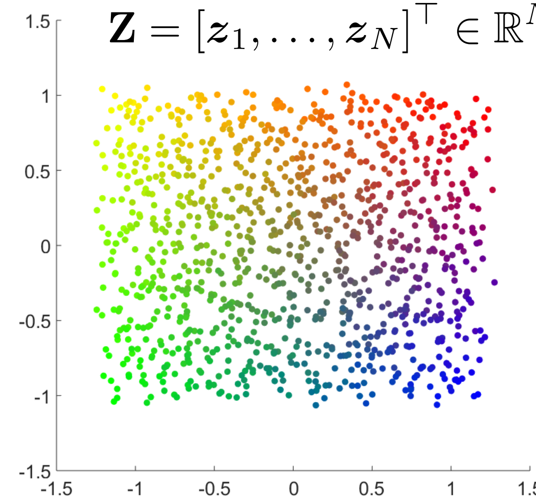

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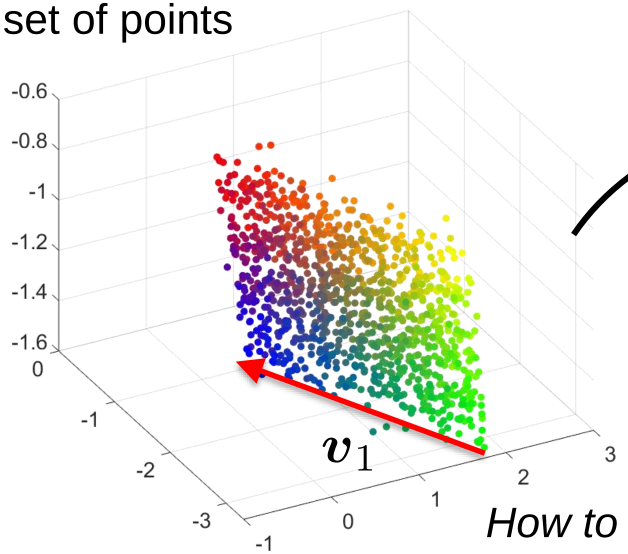

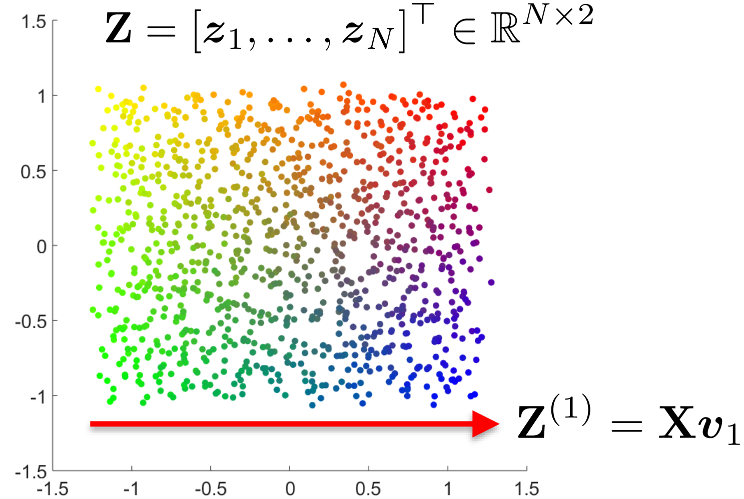

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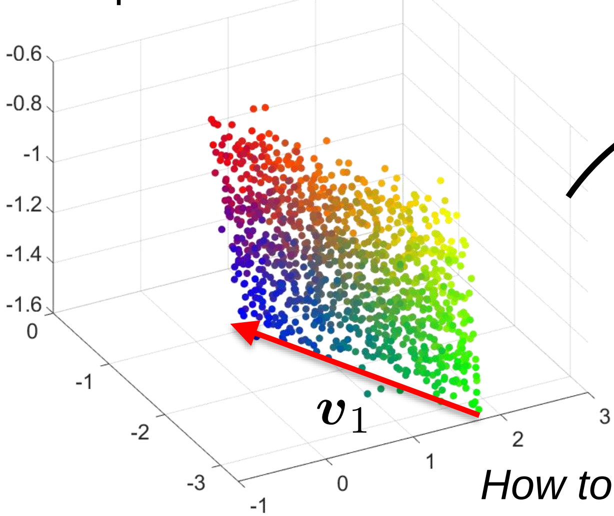

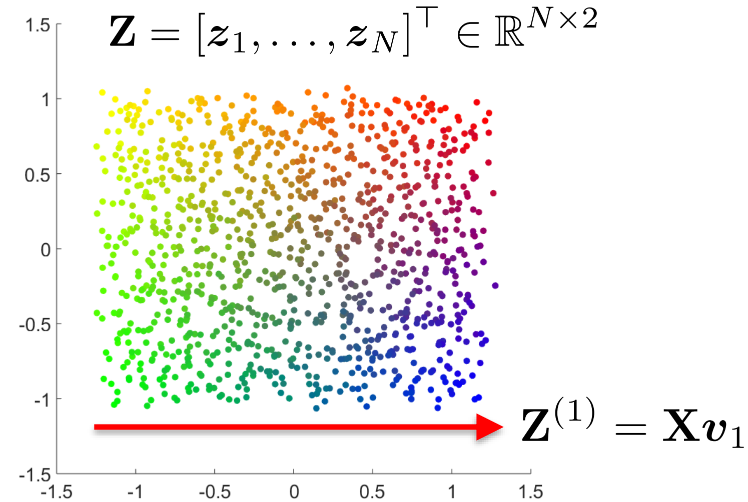

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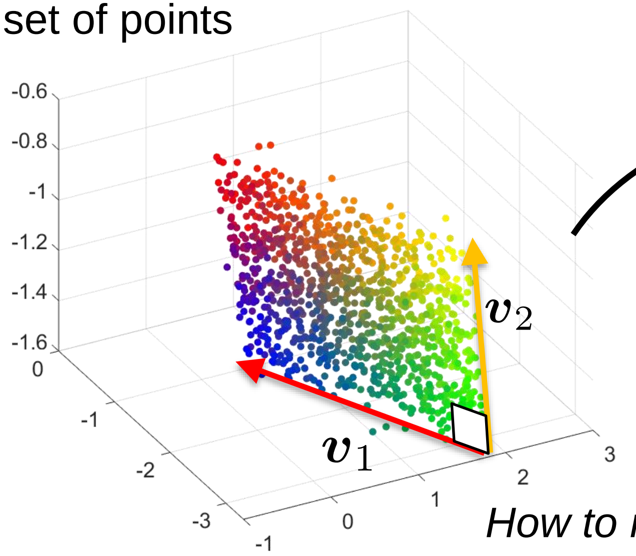

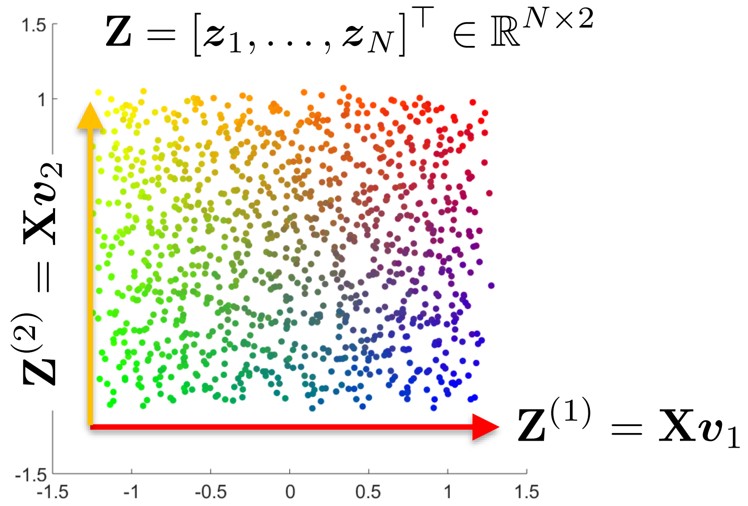

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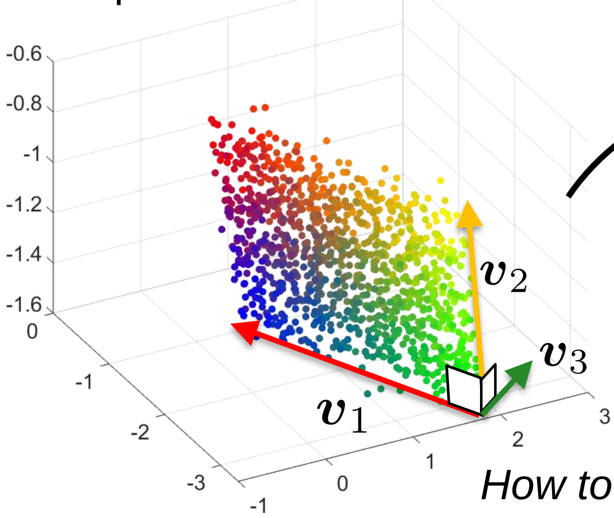

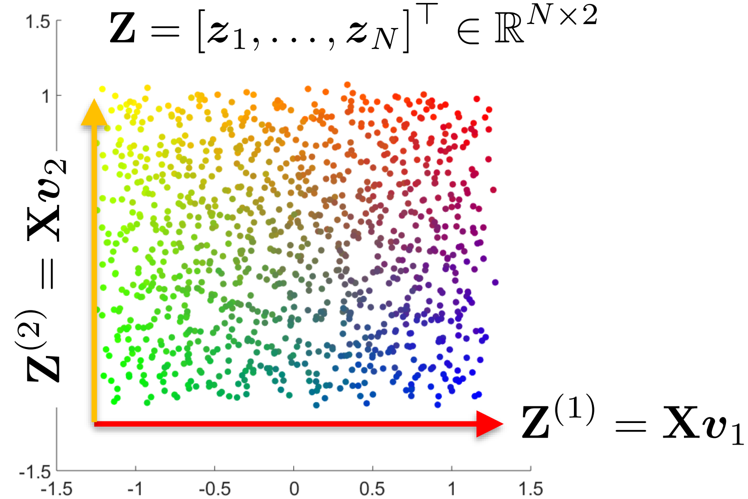

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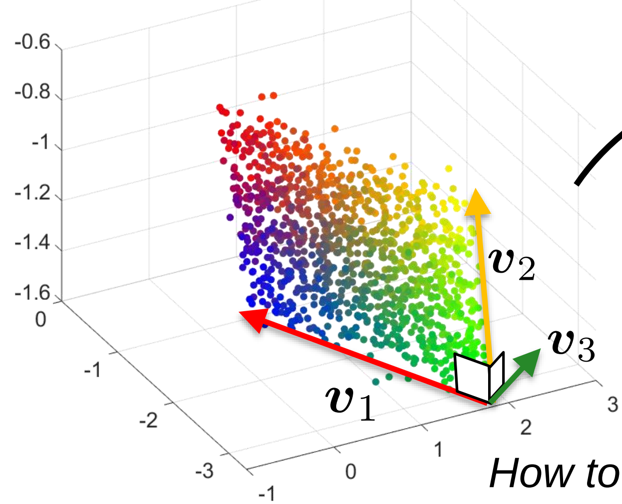

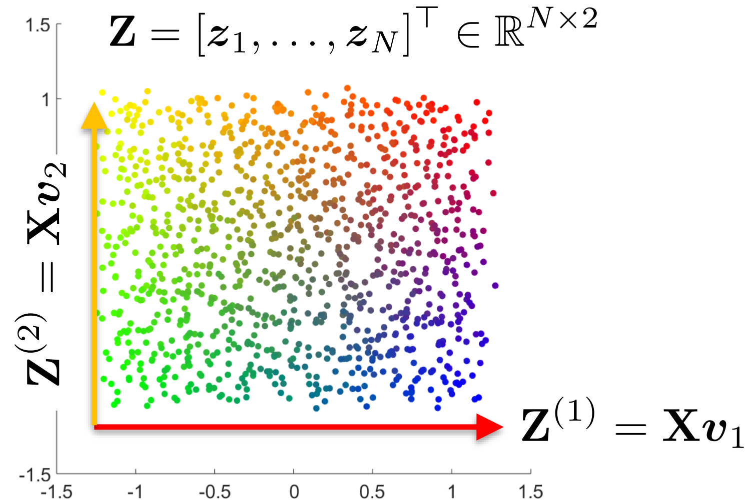

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Principal
Component
Anaysis

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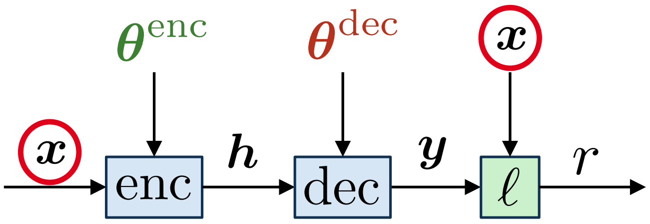

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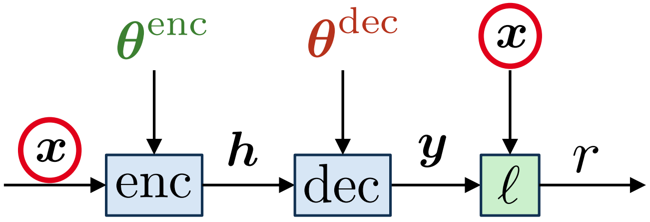

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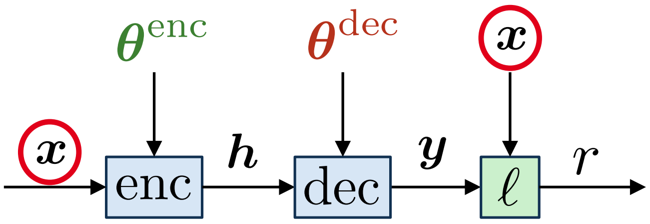

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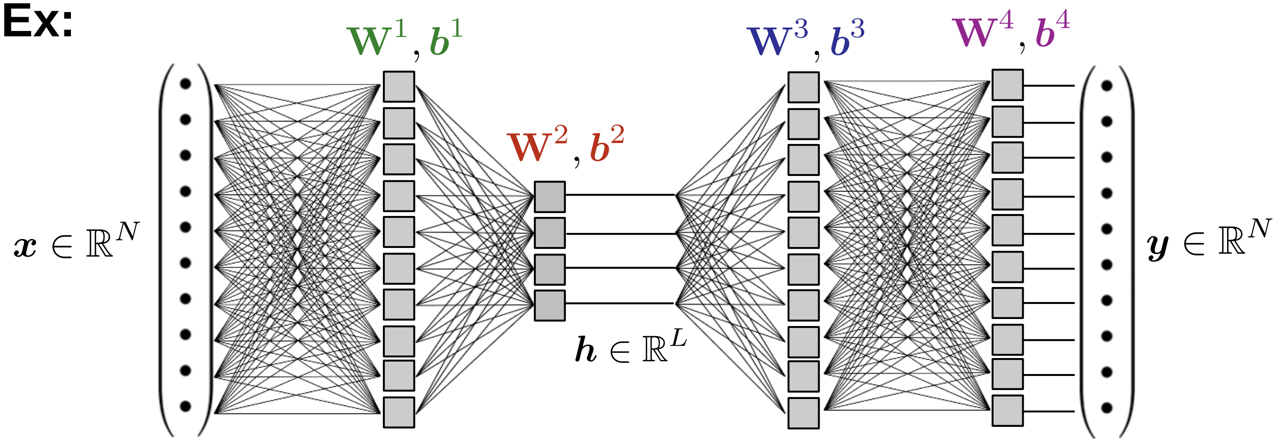

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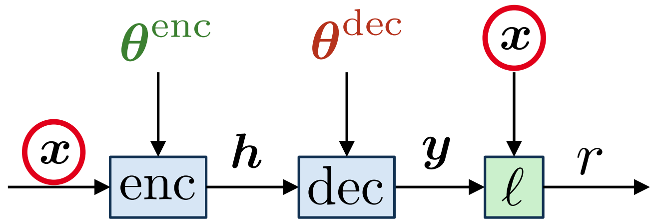

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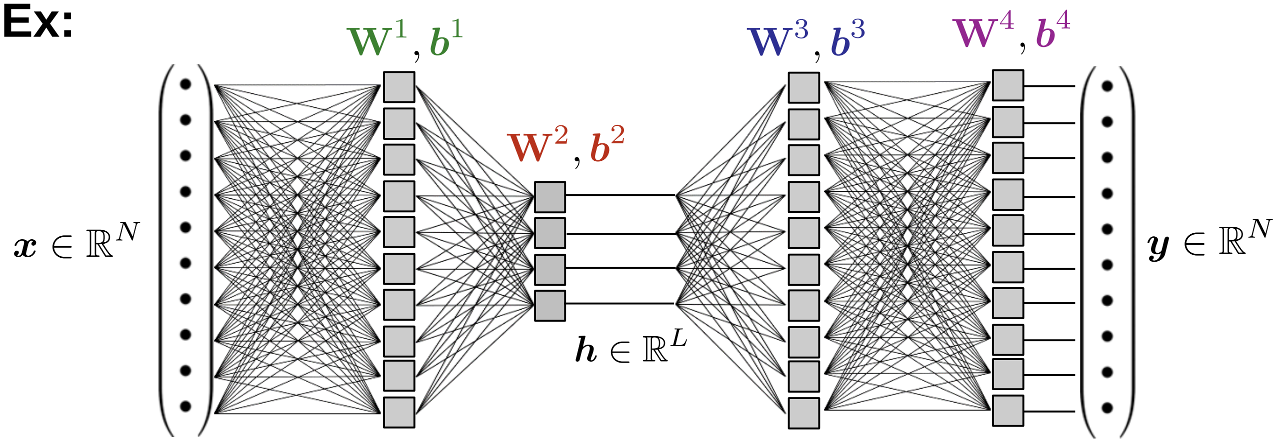

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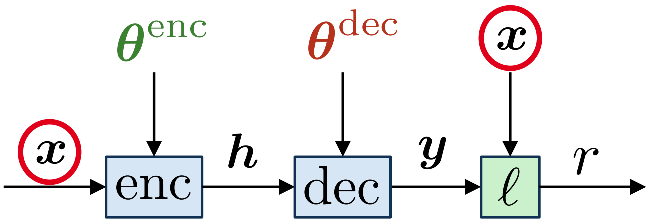

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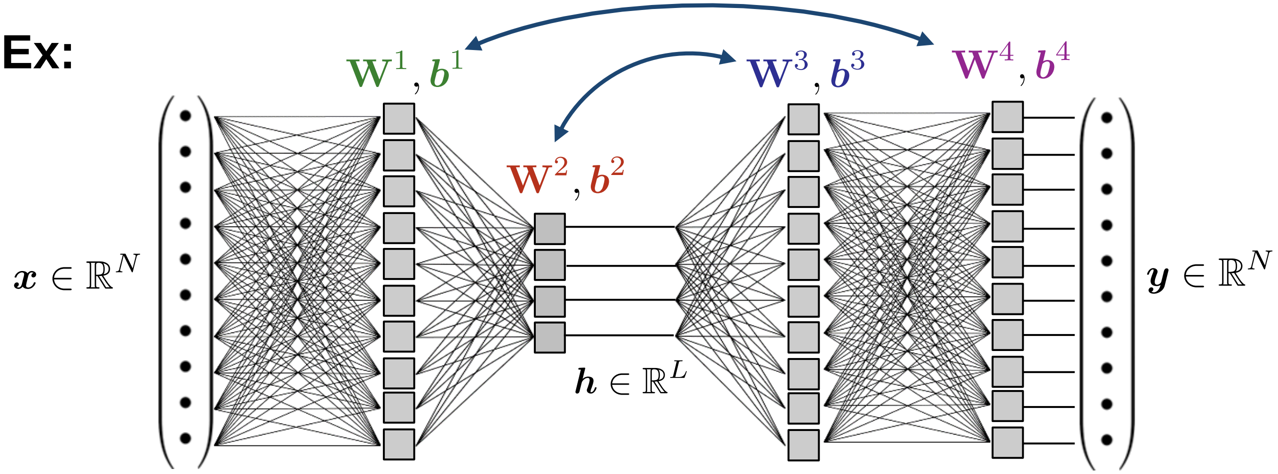

The encoder and decoder parameters can be ***tied*** together

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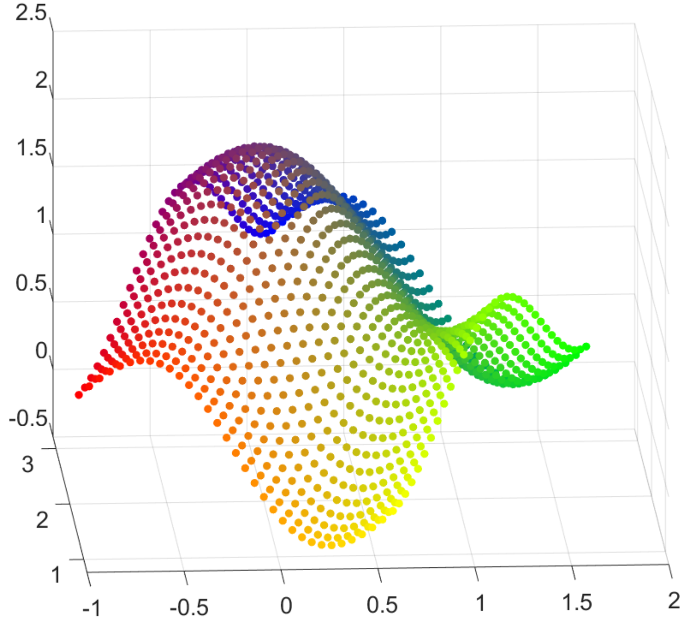
Manifold Learning

- Local Tangent Space Alignment (LTSA)



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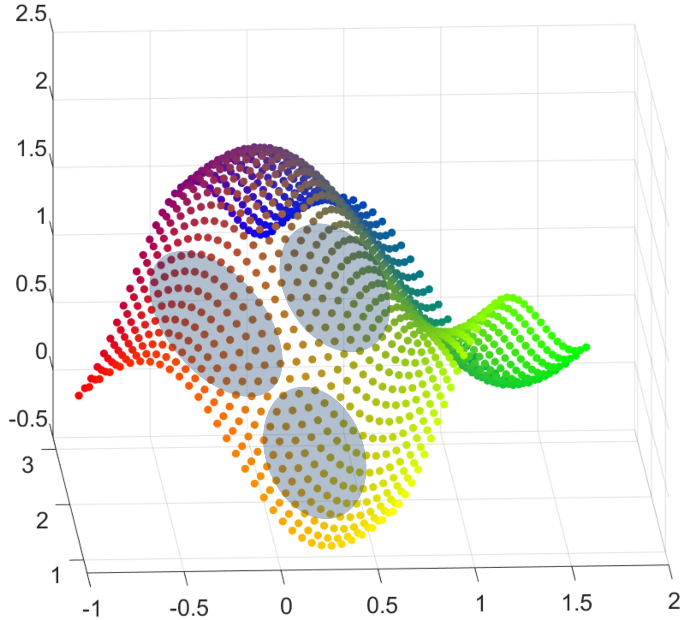

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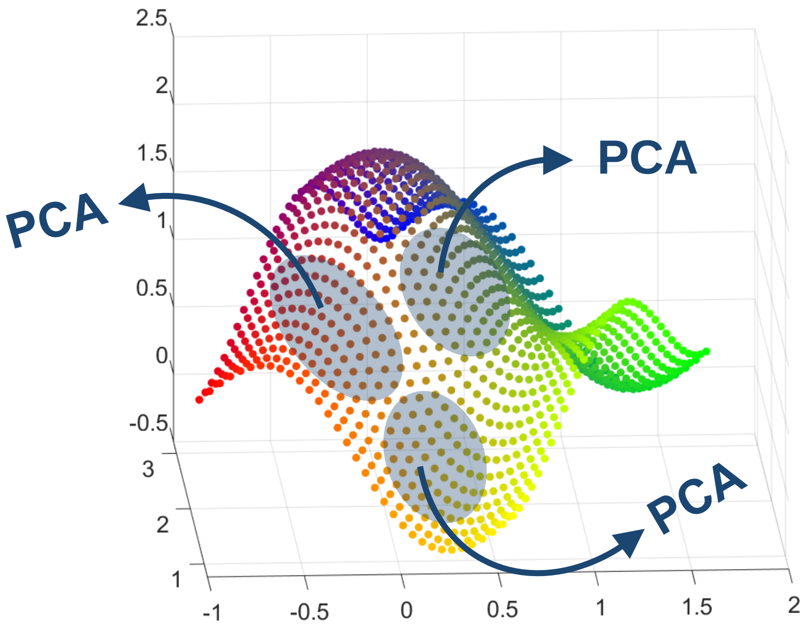

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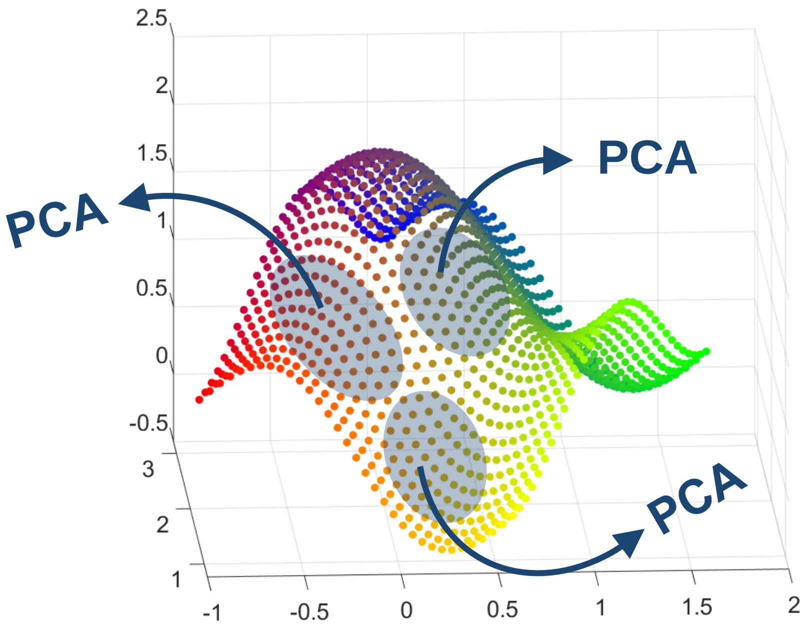
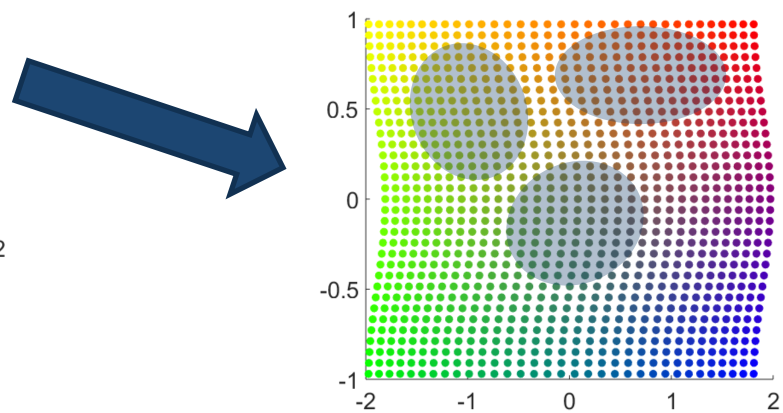


1. Builds local k-nearest neighborhoods on the data

2. Applies PCA to each neighborhood

3. Patch the local PCAs together



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Manifold Learning

- Graph-Based Methods: *Isomap, LLE, Laplacian Eigenmap,…*

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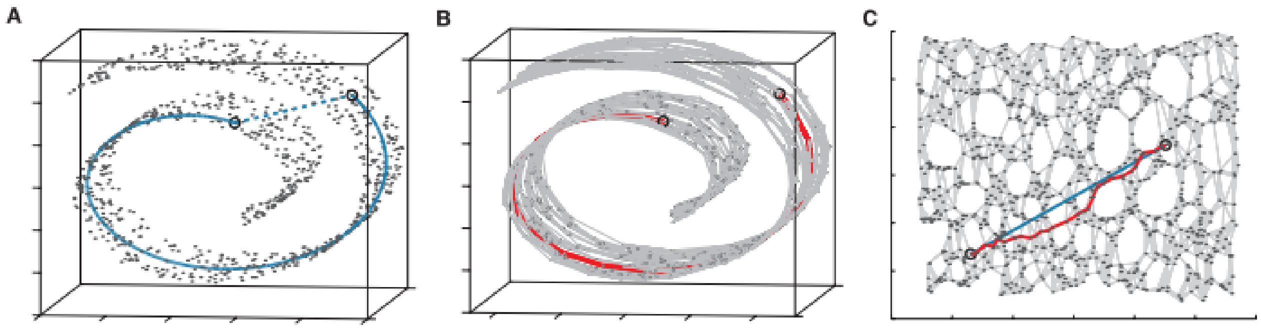

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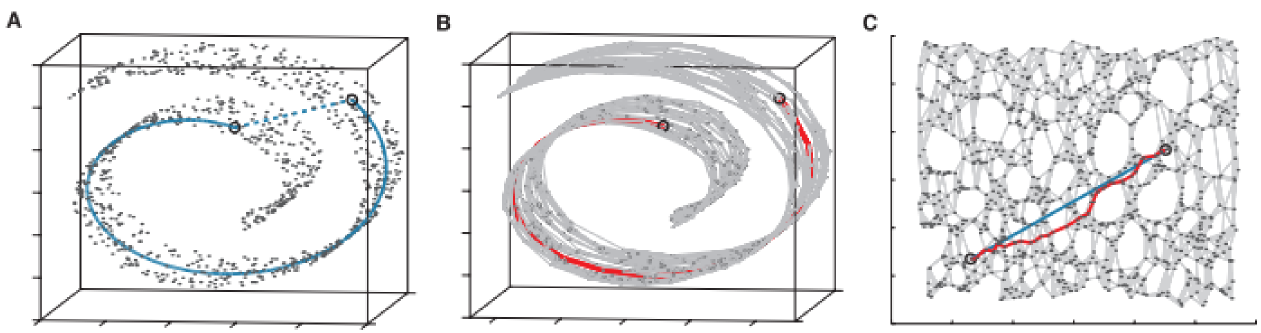

- ***Stochastic neighbor embedding*** (**SNE**): match *neighborhood probabilities* in the high- and low-dim. spaces

Hinton, Geoffrey E., and Sam Roweis. "Stochastic neighbor embedding." *Advances in neural information processing systems* 15 (2002).

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Manifold Learning

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- The variances of the Gaussians are fixed to attain a given ***entropy*** value.
- The latent representation is found by minimizing the ***Kullback-Leibler divergence*** wrt. these distributions using **gradient descent**.
- Easy to use thanks to the Scikit Learn implementation

sklearn.manifold.TSNE

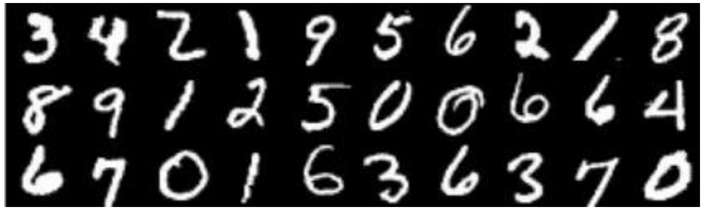
class `sklearn.manifold.TSNE`(*n_components=2, *, perplexity=30.0, early_exaggeration=12.0, learning_rate='auto', n_iter=1000, n_iter_without_progress=300, min_grad_norm=1e-07, metric='euclidean', metric_params=None, init='pca', verbose=0, random_state=None, method='barnes_hut', angle=0.5, n_jobs=None, square_distances='deprecated'*) [source]

Documentation:
https://scikit-learn.org/stable/modules/generated/sklearn.manifold.TSNE.html

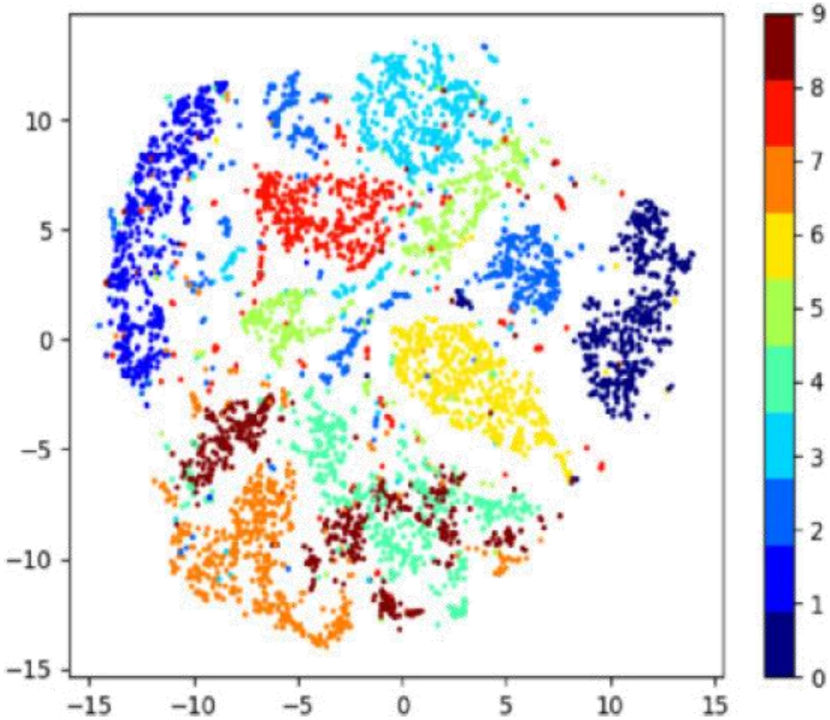
Manifold Learning

Examples of t-SNE visualizations

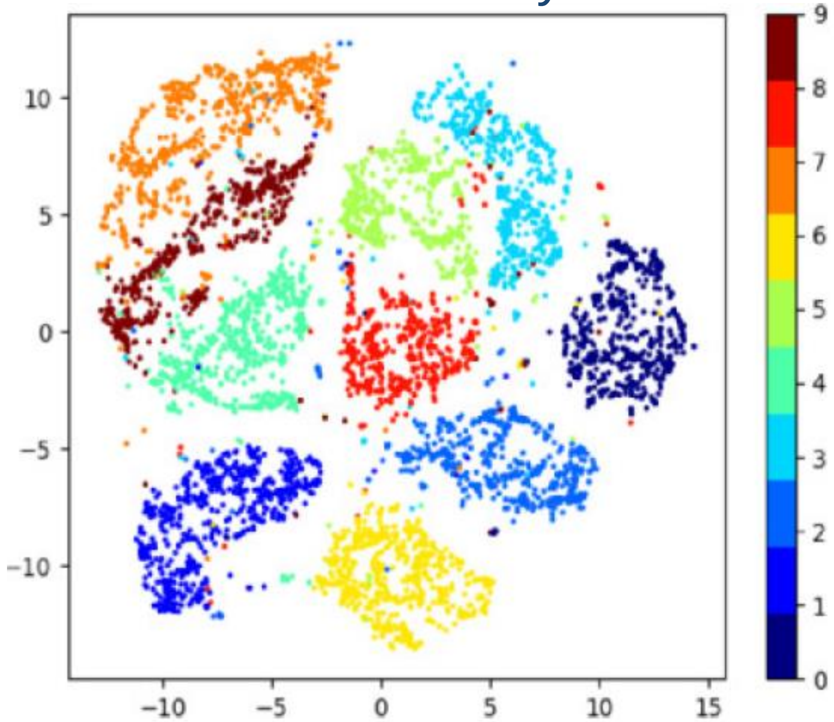
- MNIST dataset



T-SNE on image pixels



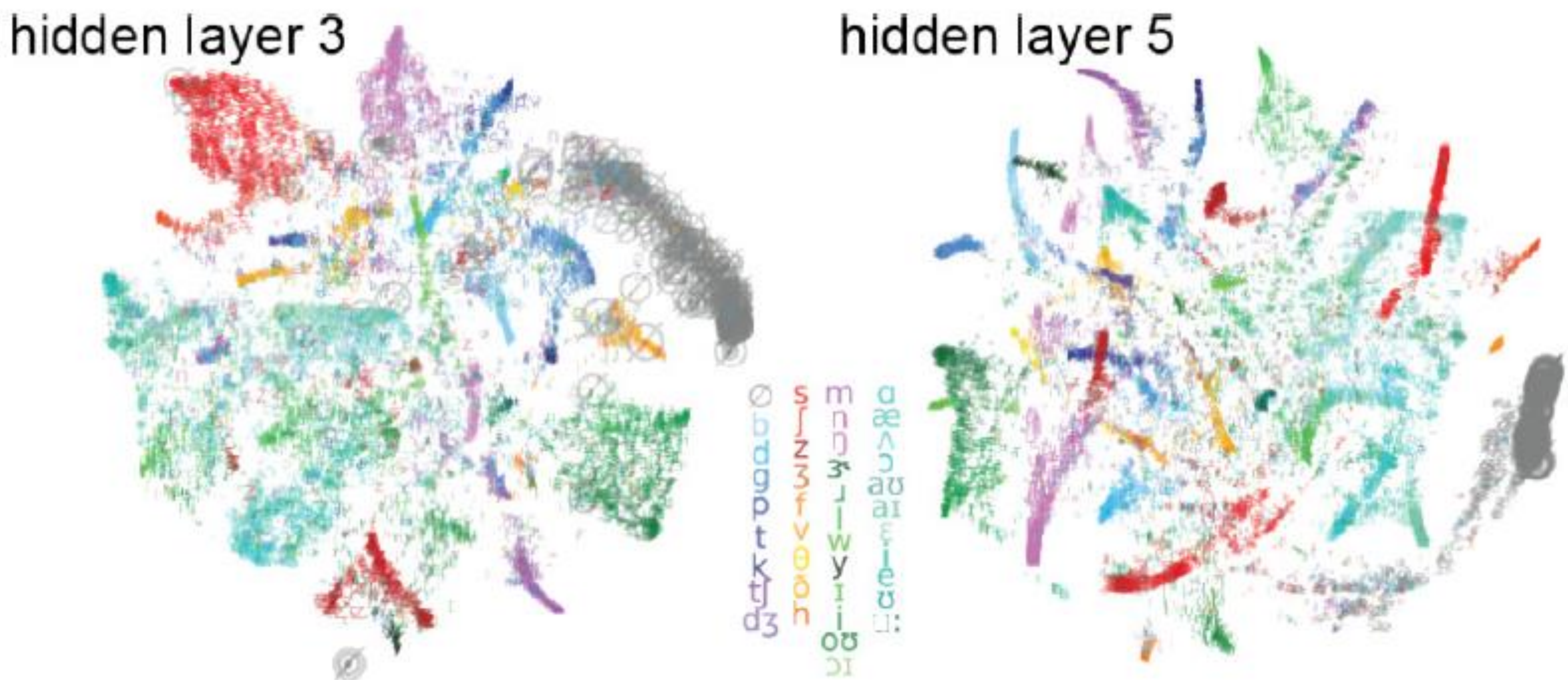
T-SNE on auto-encoder bottleneck layer



Manifold Learning

Examples of t-SNE visualizations

- Speech recognition (Wall Street Journal Dataset)



And in the Deep Learning era?

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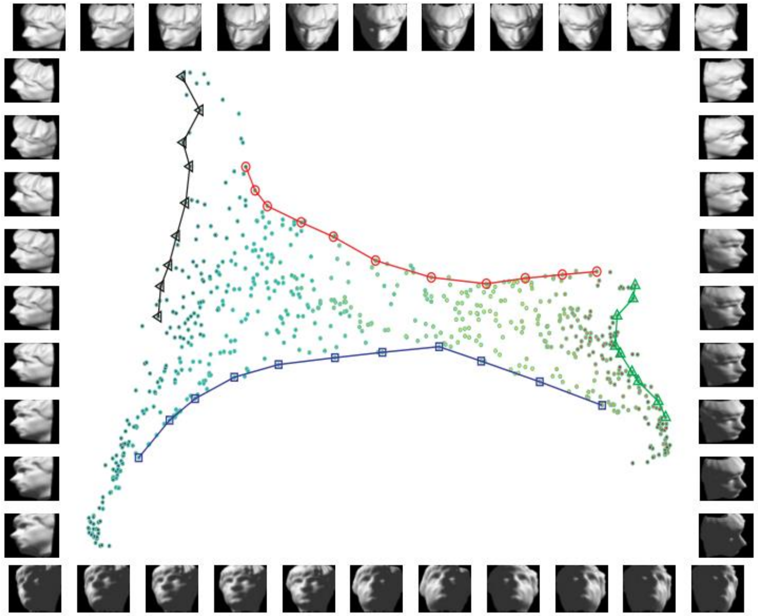
Typical applications:

Dataset Visualization



Data generation (Glow)

Pre-processing to speed up learning

Compression

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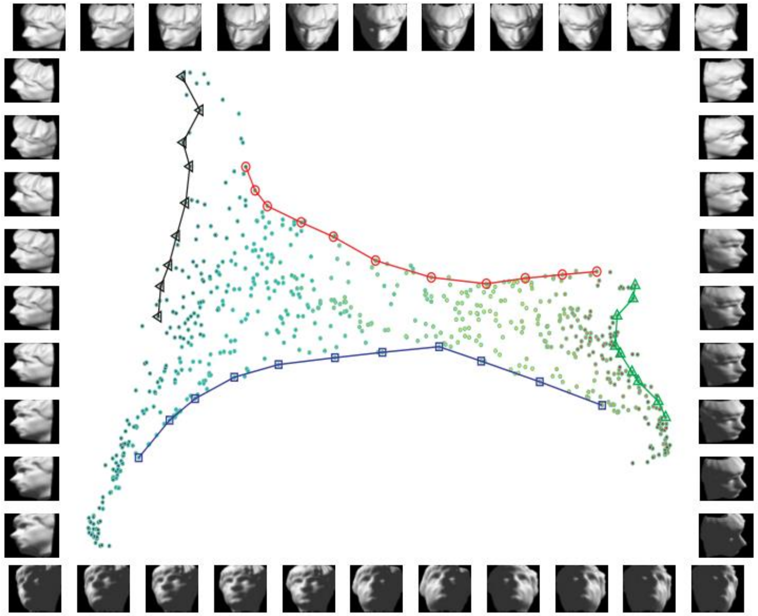

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+ extensions (Normalizing Flows, Glow…)

Typical applications:

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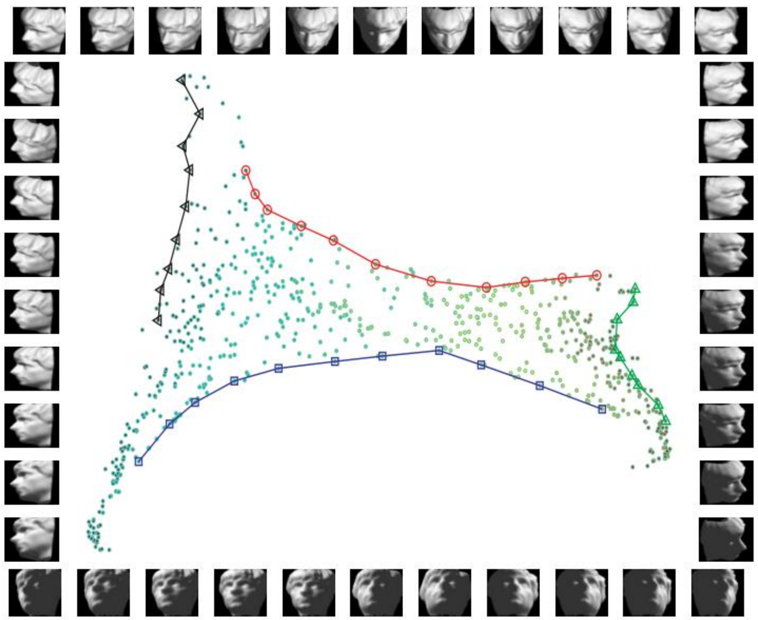

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Supervised Learning

Labeled training data

$$\mathcal{T} = \{(\boldsymbol{x}_n, \boldsymbol{y}_n)\}_{n=1}^{N}$$

$[\boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_N]$ Input ⟷ $[\boldsymbol{y}_1, \boldsymbol{y}_2, \dots, \boldsymbol{y}_N]$ Labels/Targets

Training

Test $\tilde{\boldsymbol{x}}$ → Learned Model f_θ → **Estimate** $\tilde{\boldsymbol{y}}$

1. Discrete case: («one-hot»)
- Classification

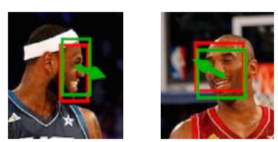
Ex. application: *dog breed*

$\boldsymbol{y} =$ 1

2. Continuous case:
- Regression

Ex. application: *head pose*

$\boldsymbol{y} =$



3. Sparse case:
- Multi-classification

Ex. application: *image labelling*

$\boldsymbol{y} =$ 0.1 0.6 0.3

Unsupervised Learning

Unlabeled training data

$$\mathcal{T} = \{\boldsymbol{x}_n\}_{n=1}^{N}$$

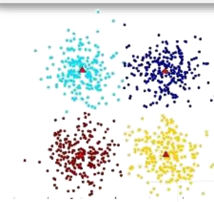
$[\boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_N]$ Input

Training

Learned Model f_θ → **Representation** $[\boldsymbol{z}_1, \boldsymbol{z}_2, \dots, \boldsymbol{z}_N]$

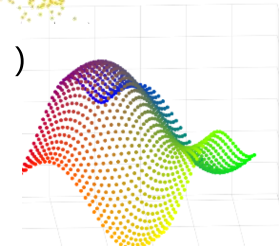
1. Discrete case:
- Clustering

$\boldsymbol{z} =$



2. Continuous case: ($\dim(\boldsymbol{z}) \ll \dim(\boldsymbol{x})$)
- Dimensionality Reduction

$\boldsymbol{z} =$



3. Sparse case:
- Dictionary Learning

$\boldsymbol{z} =$