

Towards a Computational Quantum Logic: An Overview of an Ongoing Research Program

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Abstract

This extended abstract provides an overview of a long-term collaboration aimed at extending the Curry-Howard-Lambek correspondence to the realm of quantum computation. I will introduce the Lambda-S calculus, a (partial) dual to intuitionistic linear logic whose proof terms serve as a foundation for a quantum programming language. Additionally, I will discuss the \mathcal{L}^S calculus, a proof language for intuitionistic linear logic, which also enables the construction of quantum programming languages. These frameworks offer a logical and computational foundation for reasoning about quantum programs and provide a glimpse into the structure of a potential quantum logic as the dual of linear logic.

Introduction. Quantum logic is a formal system inspired by the structure of quantum theory, originally developed by Garrett Birkhoff and John von Neumann [BvN36]. Unlike classical logic, which is based on Boolean algebra, quantum logic weakens the distributive law, leading to an orthocomplemented lattice structure. This formulation aligns with the mathematical properties of quantum mechanics, where propositions correspond to projections on a Hilbert space. However, while quantum logic has been explored as a foundational system for reasoning about quantum mechanics, its connection to computation has been less explored. Indeed, the connection between intuitionistic logic, typed lambda calculus, and Cartesian closed categories has been a fruitful area of research, with the Curry-Howard-Lambek correspondence [SU06, Cro93] providing a deep connection between these areas. If we are to extend this correspondence to quantum computation, we need to start from a logical foundation that captures the structure of quantum computation. In this extended abstract, I will provide an overview of a research program aimed at developing a *computational* quantum logic, which will serve as a foundation for quantum programming languages. This program is based on two main frameworks: the Lambda-S calculus, which is an extension of the lambda calculus to quantum computing, and the \mathcal{L}^S calculus, which is a proof language for intuitionistic linear logic whose proof terms can be used to construct quantum programs. The idea was to start from computing to logic (the Lambda-S calculus), and then from logic to computing (the \mathcal{L}^S calculus), in order to meet in the middle.

The Lambda-S Calculus. From Computing to Logic. Lambda-S [DCDR19] and Lambda-S₁ [DCGMV19] are quantum lambda calculi designed to handle quantum superpositions and control while preserving computational properties such as strong normalization and subject reduction. Both calculi extend the lambda calculus with algebraic linearity and type-based constraints to enforce quantum mechanics principles, particularly the no-cloning theorem.

Lambda-S is an extension of simply typed lambda calculus with linear combinations of terms, it incorporates a type constructor $S(A)$ where a simply type A denotes a set of basis vectors (terms) and $S(A)$ its span. This system ensures that terms in A are duplicable, whereas terms in $S(A)$ are not, reflecting the impossibility of cloning unknown quantum states. Lambda-S has been given a categorical semantics through an adjunction between Cartesian and additive symmetric monoidal categories, where S is a functor transforming sets into vector spaces, and its adjoint forgets the vectorial structure [DCM23, DCM20].

Lambda-S₁ extends Lambda-S by enforcing norm-preserving constraints on superpositions, making it a stricter model suitable for representing unitary transformations explicitly. It has been obtained via a realizability model in [DCGMV19]. The categorical semantics of Lambda-S₁ [DCM22] is structured around adjunctions but diverges from Lambda-S by ensuring that all terms maintain a unitary norm, addressing the long-standing issue of preserving quantum unitarity in quantum control lambda calculi.

Both calculi serve as foundations for quantum programming languages and categorical models of quantum computation, bridging classical and quantum computational paradigms through rigorous mathematical structures.

One of the most notable results of this *side* of the research program, is the fact that we proved the model of Lambda-S to be a (partial) dual of known models for intuitionistic linear logic (ILL). Partial, because it favoritizes a computational basis, but some preliminary results suggest that it can be extended to a proper dual [Mon25]. Usually, ILL is interpreted in a monoidal closed category, using an adjunction with a Cartesian closed category to interpret duplicable data. Lambda-S, on the other hand, is interpreted in a Cartesian closed category, using the same adjunction, but the other way around, to interpret non-duplicable data. This duality is a strong indication that the structure of quantum computation is the dual of linear logic, and that the Curry-Howard-Lambek correspondence can be extended to quantum computation, with this linear logic dual as the logical side and Lambda-S as the computational side.

The \mathcal{L}^S Calculus. From Logic to Computing. The \mathcal{L}^S calculus extends intuitionistic multiplicative additive linear logic (IMALL) with algebraic structures, incorporating sum and scalar multiplication within proof terms. It builds on the \odot -calculus [DCD23], which introduced an algebraic connective for quantum superpositions, and refines this idea within a fully linear framework [DCD24]. In a recent draft, we replace the \odot connective with an alternative rule for disjunction introduction, enabling a structured representation of the quantum measurement [DCD25]. Its categorical semantics formalizes this approach, aligning it with monoidal structures that preserve linearity [DCM24]. Moreover, the calculus has been extended [DCDIM24] with $!$, that is, from IMALL to ILL, and polymorphism, ensuring expressive power suitable for quantum programming languages.

The \mathcal{L}^S calculus provides a logical foundation for quantum programming languages, enabling the construction of quantum circuits and algorithms through proof terms. Its algebraic structure aligns with quantum mechanics principles, allowing for a direct representation of quantum superpositions and measurements. The calculus serves as a bridge between linear logic and quantum computation, providing a formal framework for reasoning about quantum programs.

The fact that both languages, Lambda-S and \mathcal{L}^S , use the same adjunction

$$\begin{array}{ccc}
 \text{Lambda-S} \text{ ---} & & \text{---} \text{ } \mathcal{L}^S \\
 \text{---} & \xrightarrow{(F,m)} & \text{---} \\
 (\mathcal{S}, \times, I) & \perp & (\mathcal{V}, \otimes, 1) \\
 & \xleftarrow{(G,n)} & \\
 & & \text{---}
 \end{array}$$

where \mathcal{S} is a Cartesian closed category, \mathcal{V} is a monoidal closed category, but with Lambda-S being interpreted in \mathcal{S} , using the monad GF to interpret the non-duplicable terms, and \mathcal{L}^S being interpreted in \mathcal{V} , using the comonad FG to interpret the duplicable terms, is a strong indication that the structure of quantum computation can be seen as the dual of linear logic, and that the Curry-Howard-Lambek correspondence can be extended to quantum computation following this path.

This ongoing research program presents many open challenges; however, the promising results so far suggest that computational quantum logic is a viable and interesting direction.

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