A lambda calculus for density matrices with classical and probabilistic controls

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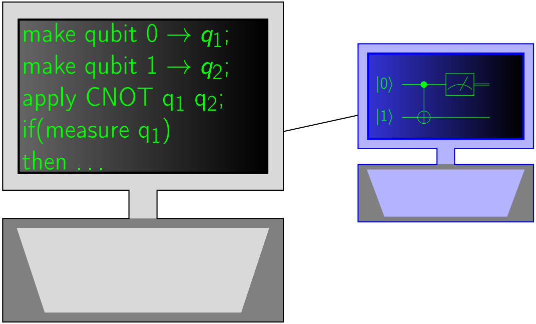
APLAS 2017

November 27-29, 2017, Suzhou, China

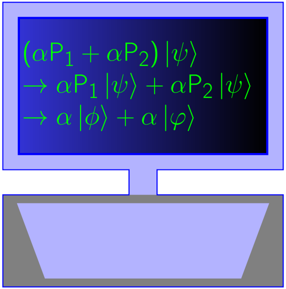
Motivation

Two paradigms

Classical control / quantum data



Quantum control



In this work we propose a paradigm in between:
"Probabilistic control" **or** ***"Weak quantum control"***

Outline

Density matrices and quantum mechanics
- Postulates of quantum mechanics
- Density matrices
- Postulates of quantum mechanics with density matrices

λ_ρ
- Untyped
- Typed language
- Denotational semantics

λ_ρ°
- Taking advantage of density matrices

Conclusions

Postulates of quantum mechanics

Postulate 1: State space

The state of an isolated quantum system can be fully described by a *state vector*, which is a unit vector in a complex Hilbert space*.

* Hilbert space: Vector space with inner product, complete in its norm

Examples

Space	Vectors
$\mathbb{C}^2$	$\|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\quad \|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $\quad \frac{1}{\sqrt{2}}\|0\rangle + \frac{1}{\sqrt{2}}\|1\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$
$\mathbb{C}^4 = \mathbb{C}^2 \otimes \mathbb{C}^2$	$\|00\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ $\qquad \frac{1}{\sqrt{3}}\|00\rangle + \frac{\sqrt{2}}{\sqrt{3}}\|11\rangle = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ 0 \\ 0 \\ \frac{\sqrt{2}}{\sqrt{3}} \end{pmatrix}$

Postulates of quantum mechanics

Postulate 2: Evolution

The evolution of an isolated quantum system can be described by a *unitary matrix**.

$$|\psi'\rangle = U\,|\psi\rangle$$

* U unitary if $U^\dagger = U^{-1}$.

Examples

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad H\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = |+\rangle \qquad H\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix} = |-\rangle$$

$$Not = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad Not\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad Not\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad Z\,|+\rangle = |-\rangle \qquad Z\,|-\rangle = |+\rangle$$

Postulates of quantum mechanics

Postulate 3: Measurement

The quantum measurement is described by a collection of *measurement matrices** $\{M_i\}_i$, where i is the output of the measurement.

Condition over $\{M_i\}_i$: $\sum_i M_i^\dagger M_i = I$

The probability of measuring i is: $p_i = \langle\psi| M_i^\dagger M_i |\psi\rangle$

The state after measuring i is: $|\psi'\rangle = \frac{M_i|\psi\rangle}{\sqrt{p_i}}$

* square matrices with complex coefficients $\langle\psi| = |\psi\rangle^\dagger$

Example

$$\{M_0, M_1\} \text{ with } M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\ M_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

$$p_0 = \begin{pmatrix} \frac{\sqrt{2}}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}^2 \begin{pmatrix} \frac{\sqrt{2}}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix} = \frac{2}{3} \qquad \frac{1}{\sqrt{p_0}} M_0 \begin{pmatrix} \frac{\sqrt{2}}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix} = \frac{1}{\sqrt{p_0}} \begin{pmatrix} \frac{\sqrt{2}}{\sqrt{3}} \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$p_1 = \begin{pmatrix} \frac{\sqrt{2}}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}^2 \begin{pmatrix} \frac{\sqrt{2}}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix} = \frac{1}{3} \qquad \frac{1}{\sqrt{p_0}} M_1 \begin{pmatrix} \frac{\sqrt{2}}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix} = \frac{1}{\sqrt{p_1}} \begin{pmatrix} 0 \\ \frac{1}{\sqrt{3}} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

In general, with those $\{M_0, M_1\}$, the vector $\begin{pmatrix} a \\ b \end{pmatrix}$ measures 0 with probability $|a|^2$ and 1 with probability $|b|^2$, and the sates after measuring are $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ y $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ respectively.

Postulates of quantum mechanics

Postulate 4: Composed system

The sate space of a composed system is the tensor product of the state space of its components.
Given n systems in states $|\psi_1\rangle, \dots, |\psi_n\rangle$, the composed system is

$$|\psi_1\rangle \otimes |\psi_2\rangle \otimes \dots \otimes |\psi_n\rangle$$

Example

System 1: $|\psi\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$ System 2: $|\phi\rangle = \begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix}$

Composed system $|\psi\rangle \otimes |\phi\rangle$:

$$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \otimes \begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{10}} \\ \frac{2}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} \\ \frac{2}{\sqrt{10}} \end{pmatrix}$$

Density matrices

A representation of our ignorance about the system

Definition (Density matrix)

Mixed state: A distribution set of pure states: $\{(p_i, |\psi_i\rangle)\}_i$
Density matrix: $\rho = \sum_i p_i |\psi_i\rangle \langle\psi_i|$

Characterisation: ρ density matrix $\Leftrightarrow \mathrm{tr}(\rho) = 1 \wedge \rho$ positive

Let $M = \{M_0, M_1\}$, with M_0 and M_1 projecting to the canonical base
After measuring $\left(\begin{smallmatrix}\alpha\\\beta\end{smallmatrix}\right)$: $\begin{cases}\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) \text{ with probability } |\alpha|^2\\ \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right) \text{ with probability } |\beta|^2\end{cases}$

Example: Pre and post measure

$\{(|\alpha|^2, \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)), (|\beta|^2, \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right))\} \Rightarrow \rho = |\alpha|^2 \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\left(\begin{smallmatrix}1 & 0\end{smallmatrix}\right) + |\beta|^2 \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)\left(\begin{smallmatrix}0 & 1\end{smallmatrix}\right) = \begin{pmatrix}|\alpha|^2 & 0\\ 0 & |\beta|^2\end{pmatrix}$

$\{(1, \left(\begin{smallmatrix}\alpha\\\beta\end{smallmatrix}\right))\} \Rightarrow \rho = \left(\begin{smallmatrix}\alpha\\\beta\end{smallmatrix}\right)\left(\begin{smallmatrix}\alpha^* & \beta^*\end{smallmatrix}\right) = \begin{pmatrix}|\alpha|^2 & \alpha\beta^*\\ \alpha^*\beta & |\beta|^2\end{pmatrix}$

Postulates of quantum mechanics

with density matrices

Postulate 1 (with vectors): State space

The state of an isolated quantum system can be fully described by a *state vector*, which is a unit vector in a complex Hilbert space.

Postulate 1 (with matrices): State space

The state of an isolated quantum system can be fully described by a *density matrix*, which is a square matrix ρ with trace 1 acting on a complex Hilbert space.

If a quantum system is in state ρ_i with probability p_i, the density matrix of the system is

$$\sum_i p_i \rho_i$$

Postulates of quantum mechanics

with density matrices

Postulate 2 (with vectors): Evolution

The evolution of an isolated quantum system can be described by a *unitary matrix*.

$$|\psi'\rangle = U|\psi\rangle$$

Postulate 2 (with matrices): Evolution

The evolution of an isolated quantum system can be described by a *unitary matrix*.

$$\rho' = U\rho U^\dagger$$

Postulates of quantum mechanics

with density matrices

Postulate 3 (with vectors): Measurement

The quantum measurement is described by a collection of *measurement matrices* $\{M_i\}_i$, where i is the output of the measurement.

Condition over $\{M_i\}_i$: $\sum_i M_i^\dagger M_i = I$

The probability of measuring i is: $p_i = \langle\psi| M_i^\dagger M_i |\psi\rangle$

The state after measuring i is: $|\psi'\rangle = \frac{M_i|\psi\rangle}{\sqrt{p_i}}$

Postulate 3 (with matrices): Measurement

The quantum measurement is described by a collection of *measurement matrices* $\{M_i\}_i$, where i is the output of the measurement.

Condition over $\{M_i\}_i$: $\sum_i M_i^\dagger M_i = I$

The probability of measuring i is: $p_i = \mathrm{tr}(M_i^\dagger M_i \rho)$

The state after measuring i is: $\rho' = \frac{M_i \rho M_i^\dagger}{p_i}$ $(|\psi'\rangle\langle\psi'|)$

Postulates of quantum mechanics

with density matrices

Postulate 4 (with vectors): Composed system

The sate space of a composed system is the tensor product of the state space of its components.
Given n systems in states $|\psi_1\rangle, \dots, |\psi_n\rangle$, the composed system is

$$|\psi_1\rangle \otimes |\psi_2\rangle \otimes \cdots \otimes |\psi_n\rangle$$

Postulate 4 (with matrices): Composed system

The sate space of a composed system is the tensor product of the state space of its components.
Given n systems in states $\rho_1, \dots, \rho_n$, the composed system is

$$\rho_1 \otimes \rho_2 \otimes \cdots \otimes \rho_n$$

Example

[Nielsen-Chuang p371]

Experiment 1: Toss a coin

Experiment 2: Toss a coin to decide whether or not to apply Z to $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$

Experiment 1

$$\{(1/2, \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)), (1/2, \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right))\}$$

$$\rho_1 = 1/2 \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & 0 \end{smallmatrix}\right) + 1/2 \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) \left(\begin{smallmatrix} 0 & 1 \end{smallmatrix}\right) = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$$

Experiment 2

$$\{(1/2, \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}), (1/2, \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix})\}$$

$$\rho_2 = 1/2 \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \left(\begin{smallmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{smallmatrix}\right) + 1/2 \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix} \left(\begin{smallmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{smallmatrix}\right) = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$$

Same density matrix does not imply same mixed state

But mixed states with same density matrices are **indistinguishable**

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- Postulates of quantum mechanics
- Density matrices
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λ_ρ

- Untyped
- Typed language
- Denotational semantics

λ_ρ°

- Taking advantage of density matrices

Conclusions

Untyped λ_ρ

$$
\begin{aligned}
t := {} & x \mid \lambda x.t \mid tt && \text{(lambda calculus)}\\
& \mid \rho^n \mid U^n t \mid \pi^n t \mid t \otimes t && \text{(the 4 postulates)}\\
& \mid (b^m, \rho^n) \mid \mathsf{letcase}\ x = r\ \mathsf{in}\ \{t \dots t\} && \text{(classical control over meas.)}
\end{aligned}
$$

where

- $\pi^n = \{\pi_0, \dots, \pi_{2^n-1}\}$ is a measurement in the computational base
- b^m is a m-bits number

$$
\begin{aligned}
(\lambda x.t)r &\longrightarrow_1 t[r/x]\\
U^m \rho^n &\longrightarrow_1 \rho'^n\\
\pi^m \rho^n &\longrightarrow_{p_i} (i^m, \rho_i^n)\\
\rho_1 \otimes \rho_2 &\longrightarrow_1 \rho'\\
\mathsf{letcase}\ x = (b^m, \rho^n)\ \mathsf{in}\ \{t_0, \dots, t_{2^m-1}\} &\longrightarrow_1 t_{b^m}[\rho^n/x]
\end{aligned}
$$

Types

$$A := n \mid (m,n) \mid A \multimap A$$

$$\frac{}{\Gamma, x : A \vdash x : A}\ \mathrm{ax} \qquad \frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x.t : A \multimap B}\ \multimap_i$$

$$\frac{\Gamma \vdash t : A \multimap B \quad \Delta \vdash r : A}{\Gamma, \Delta \vdash tr : B}\ \multimap_e \qquad \frac{}{\Gamma \vdash \rho^n : n}\ \mathrm{ax}_\rho$$

$$\frac{\Gamma \vdash t : n}{\Gamma \vdash U^m t : n}\ \mathrm{u} \qquad \frac{\Gamma \vdash t : n}{\Gamma \vdash \pi^m t : (m,n)}\ \mathrm{m}$$

$$\frac{\Gamma \vdash t : n \quad \Delta \vdash r : m}{\Gamma, \Delta \vdash t \otimes r : n+m}\ \otimes \qquad \frac{}{\Gamma \vdash (b^m, \rho^n) : (m,n)}\ \mathrm{ax}_{\mathrm{am}}$$

$$\frac{\Delta, x : n \vdash t_0 : A \quad \dots \quad \Delta, x : n \vdash t_{2^m-1} : A \quad \Gamma \vdash r : (m,n)}{\Gamma, \Delta \vdash \mathsf{letcase}\ x = r\ \mathsf{in}\ \{t_0, \dots, t_{2^m-1}\} : A}\ \mathrm{lc}$$

with $m \leq n$ and $0 \leq b^m < 2^m$.

Denotational semantics

Intuition

$$[\![\pi^n\rho^n]\!] = \{(p_0,\rho_0),\dots,(p_{2^n-1},\rho_{2^n-1})\}$$

where, with probability p_i the final state is ρ_i

$$(\!|\pi^n\rho^n|\!) = \sum_i p_i\rho_i$$

In general:

$$[\![t]\!] = \{(p_i, e_i)\}_i$$

with e_i density matrix or function from density matrices to density matrices

$$(\!|t|\!) = \sum_i p_i e_i$$

where $(a.f + b.g)(x) = a.f(x) + b.g(x)$

$$(\!|n|\!) = (\!|(m,n)|\!) = \mathcal{D}_n \qquad (\!|A \multimap B|\!) = \mathcal{D}_{\mathcal{D}_A \multimap \mathcal{D}_B} = \mathcal{D}_A \multimap \mathcal{D}_B$$

Example 1

Experiment 1: Toss a coin

Experiment 2: Toss a coin to decide whether or not to apply Z to $\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$

Example 1

Experiment 1: Toss a coin

Experiment 2: Toss a coin to decide whether or not to apply Z to $\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$

Experiment 1: $\pi^1 \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$

Experiment 2: $\mathsf{letcase}\ x = \pi^1 \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}\ \mathsf{in}\ \left\{ \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \mathsf{Z} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \right\}$

Example 1

Experiment 1: Toss a coin

Experiment 2: Toss a coin to decide whether or not to apply Z to $\begin{pmatrix} \frac12 & \frac12 \\ \frac12 & \frac12 \end{pmatrix}$

Experiment 1: $\pi^1 \begin{pmatrix} \frac12 & \frac12 \\ \frac12 & \frac12 \end{pmatrix}$

$$[\![\pi^1 \begin{pmatrix} \frac12 & \frac12 \\ \frac12 & \frac12 \end{pmatrix}]\!] = \{(\frac12, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}), (\frac12, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix})\}$$

Experiment 2: $\text{letcase } x = \pi^1 \begin{pmatrix} \frac12 & \frac12 \\ \frac12 & \frac12 \end{pmatrix} \text{ in } \{\begin{pmatrix} \frac12 & \frac12 \\ \frac12 & \frac12 \end{pmatrix}, \mathsf{Z} \begin{pmatrix} \frac12 & \frac12 \\ \frac12 & \frac12 \end{pmatrix}\}$

$$[\![\text{letcase } x = \pi^1 \begin{pmatrix} \frac12 & \frac12 \\ \frac12 & \frac12 \end{pmatrix} \text{ in } \{\begin{pmatrix} \frac12 & \frac12 \\ \frac12 & \frac12 \end{pmatrix}, \mathsf{Z} \begin{pmatrix} \frac12 & \frac12 \\ \frac12 & \frac12 \end{pmatrix}\}]\!]$$
$$= \{(\frac12, \begin{pmatrix} \frac12 & \frac12 \\ \frac12 & \frac12 \end{pmatrix}), (\frac12, \begin{pmatrix} \frac12 & -\frac12 \\ -\frac12 & \frac12 \end{pmatrix})\}$$

Example 1

Experiment 1: Toss a coin

Experiment 2: Toss a coin to decide whether or not to apply Z to $\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$

Experiment 1: $\pi^1 \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$

$$[\![\pi^1 \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}]\!] = \{(\frac{1}{2}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}), (\frac{1}{2}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix})\}$$

Experiment 2: $\text{letcase } x = \pi^1 \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \text{ in } \{\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \mathsf{Z} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}\}$

$$[\![\text{letcase } x = \pi^1 \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \text{ in } \{\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \mathsf{Z} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}\}]\!]$$
$$= \{(\frac{1}{2}, \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}), (\frac{1}{2}, \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix})\}$$

$$(\!|\text{letcase } x = \pi^1 \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \text{ in } \{\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \mathsf{Z} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}\}|\!) = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} = (\!|\pi^1 \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}|\!)$$

Example 2

Measure a given ρ and then toss a coin to decide whether to return the resulting state of the measurement, or the output of a tossing a new coin.

$$
\begin{aligned}
t = (&\text{letcase } y = \pi^1 \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \\
&\quad \text{in } \{\lambda x.\text{letcase } z = \pi^1 \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \text{ in } \{z, z\}, \lambda x.x\} \\
&) \\
&(\text{letcase } z = \pi^1 \rho \text{ in } \{z, z\})
\end{aligned}
$$

Example 2

A possible trace (confluence of trees to be proven following [DC-Martínez LSFA'17])

$$\underbrace{(\text{letcase } y = \pi^1 \begin{pmatrix} \frac12 & \frac12 \\ \frac12 & \frac12 \end{pmatrix} \text{ in } \{\overbrace{\lambda x.\text{letcase } z = \underbrace{\pi^1 \begin{pmatrix} \frac12 & \frac12 \\ \frac12 & \frac12 \end{pmatrix}}_{s} \text{ in } \{z,z\}}^{t_o}, \overbrace{\lambda x.x}^{t_1}\})}_{r_1} \underbrace{(\text{letcase } z = \pi^1 \rho \text{ in } \{z,z\})}_{r_2}$$

Let $\rho = \begin{pmatrix} 3/4 & \sqrt{3}/4 \\ \sqrt{3}/4 & 1/4 \end{pmatrix}$

- $r_1 r_2$
  - $\xrightarrow{3/4}$ $r_1 |0\rangle$ $\xrightarrow{1}$ $r_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$
    - $\xrightarrow{1/2}$ $r_1^0 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $\xrightarrow{1}$ $t_0 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $\xrightarrow{1}$ letcase $y = s$ in $\{y, y\}$
      - $\xrightarrow{1/2}$ $|0\rangle$ $\xrightarrow{1}$ $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$
      - $\xrightarrow{1/2}$ $|1\rangle$ $\xrightarrow{1}$ $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$
    - $\xrightarrow{1/2}$ $r_1^1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $\xrightarrow{1}$ $t_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $\xrightarrow{1}$ $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$
  - $\xrightarrow{1/4}$ $r_1 |1\rangle$ $\xrightarrow{1}$ $r_1 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$
    - $\xrightarrow{1/2}$ $r_1^0 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ $\xrightarrow{1}$ $t_0 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ $\xrightarrow{1}$ letcase $y = s$ in $\{y, y\}$
      - $\xrightarrow{1/2}$ $|0\rangle$ $\xrightarrow{1}$ $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$
      - $\xrightarrow{1/2}$ $|1\rangle$ $\xrightarrow{1}$ $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$
    - $\xrightarrow{1/2}$ $r_1^1 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ $\xrightarrow{1}$ $t_1 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ $\xrightarrow{1}$ $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$$[\![r_1 r_2]\!] = \{(\tfrac58, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}), (\tfrac38, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix})\}$$

$$(\!| r_1 r_2 |\!) = \begin{pmatrix} \frac58 & 0 \\ 0 & \frac38 \end{pmatrix}$$

Outline

λ_ρ°: taking advantage of density matrices

$$
\begin{aligned}
t := {} & x \mid \lambda x.t \mid tt && \text{(lambda calculus)}\\
& \mid \rho^n \mid U^n t \mid \pi^n t \mid t \otimes t && \text{(the 4 postulates)}\\
& \mid (b^m, \rho^n) \mid \text{letcase } x = r \text{ in } \{t \dots t\} && \text{(classical control over meas.)}\\
& \mid \sum_{i=1}^{n} p_i t_i \mid \text{letcase}^\circ\ x = r \text{ in } \{t \dots t\} && \text{(probabilistic control)}
\end{aligned}
$$

$$
\begin{aligned}
(\lambda x.t)r &\to t[r/x]\\
U^m \rho^n &\to \rho'^n\\
\pi^m \rho^n &\to_{p_i} (i^m, \rho_i^n)\\
\rho_1 \otimes \rho_2 &\to \rho'\\
\text{letcase } x = (b^m, \rho^n) \text{ in } \{t_0, \dots, t_{2^m-1}\} &\to t_{b^m}[\rho^n/x]\\
\text{letcase}^\circ\ x = \pi^m \rho^n \text{ in } \{t_0, \dots, t_{2^m-1}\} &\to \sum_i p_i t_i[\rho_i^n/x]
\end{aligned}
$$

Example 2 again

Measure a given ρ and then toss a coin to decide whether to return the resulting state of the measurement, or the output of a tossing a new coin.

$$t = (\text{letcase}^\circ\ y = \pi^1 \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$\text{in } \{\lambda x.\text{letcase}^\circ\ z = \pi^1 \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \text{ in } \{z, z\}, \lambda x.x\}$$

$$)$$

$$(\text{letcase}^\circ\ z = \pi^1 \rho \text{ in } \{z, z\})$$

$$t \to^* \begin{pmatrix} \frac{5}{8} & 0 \\ 0 & \frac{3}{8} \end{pmatrix}$$

Summarising

- λ_ρ: classical control/quantum data (data = density matrices)
- λ_ρ°: probabilistic control/quantum data
- **Same denotational semantics**

Future works

- Comparison between $\lambda_\rho/\lambda_\rho^\circ$, and Selinger-Valiron's λ_q

  (with Agustín Borgna (UBA))
- Implementation of a simulator in Haskell

  (with Alan Rodas and Pablo E. Martínez López (UNQ))
- Polymorphic extension and proofs of SN and confluence

  (with Lucas Romero (UBA))
- Studding a fixed point operator

  (with Malena Ivnisky and Hernán Melgratti (UBA))