Affine computation and affine automaton

Alejandro Díaz-Caro
Universidad Nacional de Quilmes
(Argentina)

Abuzer Yakaryılmaz
Laboratório Nacional de Computação Científica
(Brazil)

Computer Science Symposium in Russia
St. Petersburg, June 9–13, 2016

Outline

Motivation

Affine computation

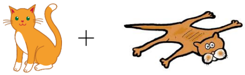
Affine finite Automaton (AfA)

Main results

Probabilistic vs. Quantum

Destructive interference

Probabilistic



Probabilistic vs. Quantum

Destructive interference

Probabilistic

$\frac{1}{2}$  $+ \frac{1}{2}$ 

$$\left(\tfrac{1}{2} + \tfrac{1}{2} = 1\right)$$

Probabilistic vs. Quantum

Destructive interference

Probabilistic

$\frac{1}{2}$  $+ \frac{1}{2}$ 

$$\left(\tfrac{1}{2} + \tfrac{1}{2} = 1\right)$$

Quantum

$\frac{1}{\sqrt{2}}$  $+ \frac{1}{\sqrt{2}}$ 

$$\left(\left|\tfrac{1}{\sqrt{2}}\right|^2 + \left|\tfrac{1}{\sqrt{2}}\right|^2 = 1\right)$$

Probabilistic vs. Quantum

Destructive interference

Probabilistic

$\frac{1}{2}$  $+\ \frac{1}{2}$ 

$$\left(\tfrac{1}{2}+\tfrac{1}{2}=1\right)$$

Quantum

$\frac{1}{\sqrt{2}}$  $-\ \frac{1}{\sqrt{2}}$ 

$$\left(\left|\frac{1}{\sqrt{2}}\right|^2+\left|\frac{-1}{\sqrt{2}}\right|^2=1\right)$$

Probabilistic vs. Quantum

Destructive interference

Probabilistic

$\frac{1}{2}$  $+ \frac{1}{2}$ 

$\left(\frac{1}{2} + \frac{1}{2} = 1\right)$

Quantum

$\frac{1}{\sqrt{2}}$  $- \frac{1}{\sqrt{2}}$ 

$$\left(\left| \frac{1}{\sqrt{2}} \right|^2 + \left| \frac{-1}{\sqrt{2}} \right|^2 = 1 \right)$$

$\frac{1}{2} \Big(\frac{1}{2}$  $+ \frac{1}{2}$  $\Big) + \frac{1}{2} \Big(\qquad \qquad \Big)$

Probabilistic vs. Quantum

Destructive interference

Probabilistic

$\frac{1}{2}$  $+ \frac{1}{2}$ 

$$\left(\tfrac{1}{2} + \tfrac{1}{2} = 1\right)$$

Quantum

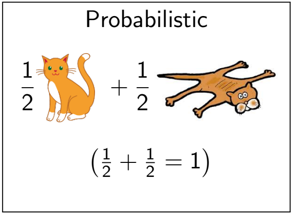
$\frac{1}{\sqrt{2}}$  $- \frac{1}{\sqrt{2}}$ 

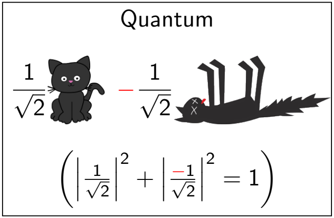
$$\left(\left|\frac{1}{\sqrt{2}}\right|^2 + \left|\frac{-1}{\sqrt{2}}\right|^2 = 1\right)$$

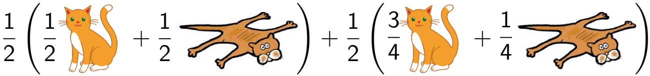
$\frac{1}{2}\Bigg(\frac{1}{2}$  $+ \frac{1}{2}$  $\Bigg) + \frac{1}{2}\Bigg(\alpha$  $+ \beta$  $\Bigg)$

Probabilistic vs. Quantum

Destructive interference







Probabilistic vs. Quantum

Destructive interference

Probabilistic

$\frac{1}{2}$  $+ \frac{1}{2}$ 

$$\left(\tfrac{1}{2} + \tfrac{1}{2} = 1\right)$$

Quantum

$\frac{1}{\sqrt{2}}$  $- \frac{1}{\sqrt{2}}$ 

$$\left(\left|\frac{1}{\sqrt{2}}\right|^2 + \left|\frac{-1}{\sqrt{2}}\right|^2 = 1 \right)$$

$\frac{1}{2}\Big(\frac{1}{2}$  $+ \frac{1}{2}$  $\Big) + \frac{1}{2}\Big(\frac{3}{4}$  $+ \frac{1}{4}$  $\Big)$

$\frac{5}{8}$  $+ \frac{3}{8}$ 

Probabilistic vs. Quantum

Destructive interference

Probabilistic

$\frac{1}{2}$  $+ \frac{1}{2}$ 

$$\left(\frac{1}{2} + \frac{1}{2} = 1\right)$$

Quantum

$\frac{1}{\sqrt{2}}$  $- \frac{1}{\sqrt{2}}$ 

$$\left(\left|\frac{1}{\sqrt{2}}\right|^2 + \left|\frac{-1}{\sqrt{2}}\right|^2 = 1\right)$$

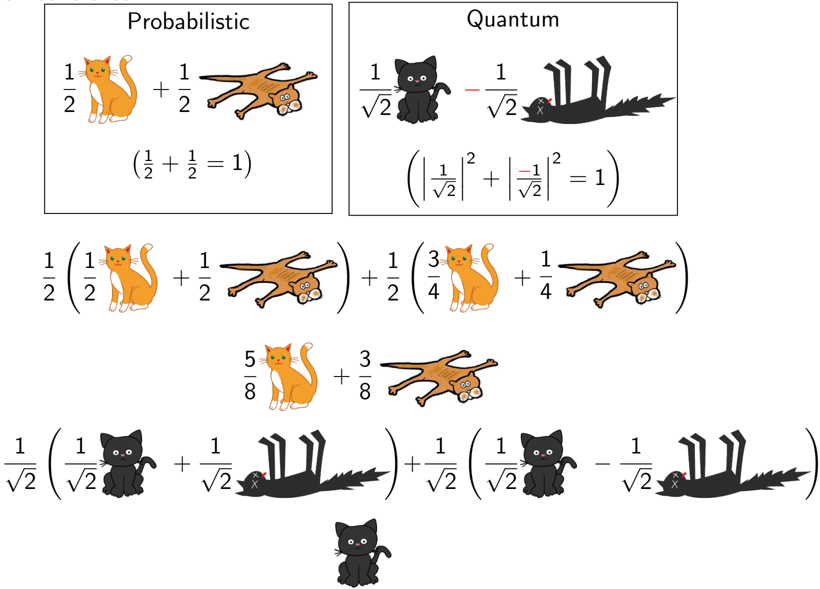
$\frac{1}{2}\Bigg(\frac{1}{2}$  $+ \frac{1}{2}$  $\Bigg) + \frac{1}{2}\Bigg(\frac{3}{4}$  $+ \frac{1}{4}$  $\Bigg)$

$\frac{5}{8}$  $+ \frac{3}{8}$ 

$\frac{1}{\sqrt{2}}\Bigg(\frac{1}{\sqrt{2}}$  $+ \frac{1}{\sqrt{2}}$  $\Bigg) + \frac{1}{\sqrt{2}}\Bigg(\frac{1}{\sqrt{2}}$  $- \frac{1}{\sqrt{2}}$  $\Bigg)$

Probabilistic vs. Quantum

Destructive interference



Is there any computational power in the destructive interference?

Affine systems

Preliminaries

Probabilistic state:	l_1-norm-1 vector (defined on $\mathbb{R}_0^+$)
Probabilistic operator:	Linear operator (stochastic matrix)
Quantum state:	l_2-norm-1 vector (defined on $\mathbb{C}$)
Quantum operator:	Linear operator (unitary matrix)

Affine systems

Preliminaries

Probabilistic state: l_1-norm-1 vector (defined on $\mathbb{R}_0^+$)
Probabilistic operator: Linear operator (stochastic matrix)

Quantum state: l_2-norm-1 vector (defined on $\mathbb{C}$)
Quantum operator: Linear operator (unitary matrix)

Aim

- Generalization of probabilistic system
- Allowing negative values
- Linear operator
- Defined in a simple way

Affine systems

Preliminaries

Probabilistic state: l_1-norm-1 vector (defined on $\mathbb{R}_0^+$)
Probabilistic operator: Linear operator (stochastic matrix)

Quantum state: l_2-norm-1 vector (defined on $\mathbb{C}$)
Quantum operator: Linear operator (unitary matrix)

Aim

- Generalization of probabilistic system
- Allowing negative values
- Linear operator
- Defined in a simple way

Affine state: Barycentric vector (defined on $\mathbb{R}$)
Affine operator: Linear operator (affine transformation)

Affine systems

Getting information

Weighting operator

- Analogous to quantum measurement
- Projects the state into the computational basis
- The weight is the absolut value
- Normalization after measurement (l_1-norm can be > 1)
- Normalized magnitude = probability of observation

10  $-\ 9$ 

Probability of  : $\frac{10}{19}$

Probability of  : $\frac{9}{19}$

Affine systems (AfS)

Formal definition: Affine state

- $E = \{e_1, \ldots, e_n\}$ basis states (deterministic states)
- Affine state: linear combination $a_1 e_1 + \cdots + a_n e_n$ with

$$\sum_{i=1}^{n} a_i = 1 \qquad\qquad a_i \in \mathbb{R}$$

Affine systems (AfS)

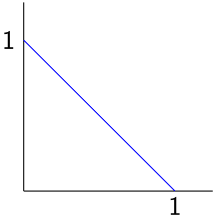
Formal definition: Affine state

- $E = \{e_1, \dots, e_n\}$ basis states (deterministic states)
- Affine state: linear combination $a_1 e_1 + \cdots + a_n e_n$ with

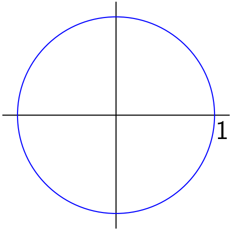
$$\sum_{i=1}^{n} a_i = 1 \qquad\qquad a_i \in \mathbb{R}$$

In $\mathbb{R}^2$

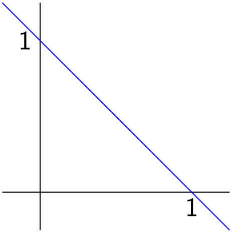
Probabilistic state
Set of states:
A segment



Quantum state
Set of states:
A circle



Affine state
Set of states:
A line



Affine systems (AfS)

Formal definition: Affine transformation and weighting operator

Affine transformation

$A = (a_{ij})_{ij}$ is an **affine transformation** $\iff$ $\forall j, \sum_i a_{ij} = 1$

Affine systems (AfS)

Formal definition: Affine transformation and weighting operator

Affine transformation

$A = (a_{ij})_{ij}$ is an **affine transformation** $\Leftrightarrow$ $\forall j, \sum_i a_{ij} = 1$

Weighting operator

In QC, **sign** of amplitudes does not matter for measurement

We follow the same idea

- **Magnitude** of an affine state:

$$|v| = \sum_i |a_i| + |a_2| + \cdots + |a_n| \geq 1 \qquad (\text{l}_1 \text{ norm})$$

- **Probability** of observing the $j-$th state:

$$\frac{|a_j|}{|v|}$$

Affine finite Automaton (AfA)

Formal definition

An AfA M is a 5-tuple

$$M = (E, \Sigma, \{A_\sigma \mid \sigma \in \Sigma\}, e_s, E_a)$$

where

- E is the **set of deterministic states**
- $e_s \in E$ is the **starting state**
- $E_a \subseteq E$ **set of accepting states**
- Σ is the **input alphabet**
- A_σ is the **affine transformation matrix** for the symbol σ.

Idem PFA except for the transition matrices
(and a PFA with matrices consisting only of 0s and 1s is a DFA)

Affine finite Automaton (AfA)

Language recognition

- Input: $w \in \Sigma^*$
- After reading the whole input, a weighting operator is applied
- Accepting probability of M in w:

$$f_M(w) = \sum_{e_k \in E_a} \frac{|v_f[k]|}{|v_f|} \in [0,1]$$

Affine finite Automaton (AfA)

Language recognition

- Input: $w \in \Sigma^*$
- After reading the whole input, a weighting operator is applied
- Accepting probability of M in w:

$$f_M(w) = \sum_{e_k \in E_a} \frac{|v_f[k]|}{|v_f|} \in [0,1]$$

- A language is recognized by an AfA M with cutpoint $\lambda \in [0,1)$ iff

$$L = \{w \in \Sigma^* \mid f_M(w) > \lambda\}$$

- Nondeterministic *AfA*: cutpoint 0.
- A language is recognized by an AfA M with bound error iff

$$\exists \delta \text{ such that } \begin{cases} \forall w \in L,\ f_M(w) \geq \lambda + \delta \\ \forall w \notin L,\ f_M(w) \leq \lambda - \delta \end{cases}$$

The languages and automata zoo

Cutpoint	Language	Class	Automaton
$CP > 0$	Stochastic lang.	$\mathtt{SL}$	PFA
$CP = 0$	Regular lang.	$\mathtt{REG}$	NFA
Bound error	Regular lang.	$\mathtt{REG}$	BPFA
$CP > 0$	Stochastic lang.	$\mathtt{SL}$	QFA
$CP = 0$	Nondeterministic quantum lang.	$\mathtt{NQAL}$	NQFA
Bound error	Regular lang.	$\mathtt{REG}$	BQFA

$$\mathtt{REG} \subsetneq \mathtt{NQAL} \subsetneq \mathtt{SL}$$

The languages and automata zoo

Cutpoint	Language	Class	Automaton
$CP > 0$	Stochastic lang.	$\mathtt{SL}$	PFA
$CP = 0$	Regular lang.	$\mathtt{REG}$	NFA
Bound error	Regular lang.	$\mathtt{REG}$	BPFA
$CP > 0$	Stochastic lang.	$\mathtt{SL}$	QFA
$CP = 0$	Nondeterministic quantum lang.	$\mathtt{NQAL}$	NQFA
Bound error	Regular lang.	$\mathtt{REG}$	BQFA

$$\mathtt{REG} \subsetneq \mathtt{NQAL} \subsetneq \mathtt{SL}$$

Cutpoint	Language	Class	Automaton
$CP > 0$	Affine lang.	$\mathtt{AfL}$	AfA
$CP = 0$	Nondeterministic affine lang.	$\mathtt{NAfL}$	NAfA
Bound error	Bounded-error affine lang.	$\mathtt{BAfL}$	BAfA

$\mathtt{BAfL}^0$: All non-members are accepted with value 0

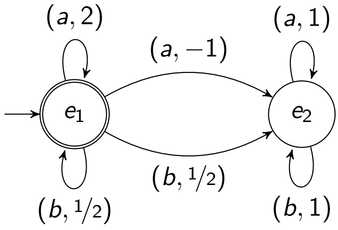
$\mathtt{BAfL}^1$: All members are accepted with value 1.

Bounded-error affine languages (BAfL)

Language $\mathrm{EQ} = \{w \in \{a, b\}^* \mid |w|_a = |w|_b\} \quad \notin \mathtt{REG}$

Bounded-error affine languages (BAfL)

Language $\mathrm{EQ} = \{w \in \{a,b\}^* \mid |w|_a = |w|_b\} \quad \notin \mathtt{REG}$



$$A_a = \begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix}$$

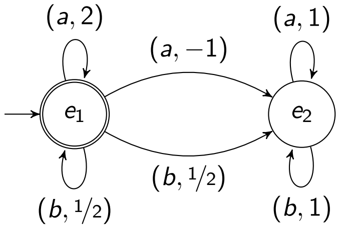
$$A_b = \begin{pmatrix} 1/2 & 0 \\ 1/2 & 1 \end{pmatrix}$$

After reading m as and n bs the state is $\begin{pmatrix} 2^{m-n} \\ 1 - 2^{m-n} \end{pmatrix}$

- $\forall w \in \mathrm{EQ}$, $v_f = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ (accepting value 1)
- $\forall w \notin \mathrm{EQ}$, max accepting value: $v_f = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ (accepting value $2/3$)

Bounded-error affine languages (BAfL)

Language $\mathrm{EQ} = \{w \in \{a,b\}^* \mid |w|_a = |w|_b\} \quad \notin \mathtt{REG}$



$$A_a = \begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix}$$

$$A_b = \begin{pmatrix} 1/2 & 0 \\ 1/2 & 1 \end{pmatrix}$$

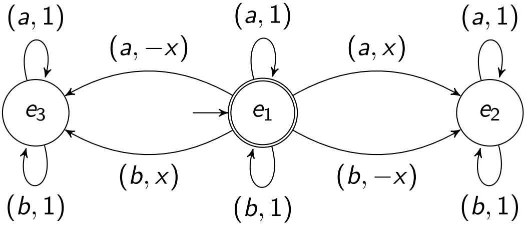
After reading m as and n bs the state is $\begin{pmatrix} 2^{m-n} \\ 1 - 2^{m-n} \end{pmatrix}$

- $\forall w \in \mathrm{EQ}$, $v_f = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ (accepting value 1)
- $\forall w \notin \mathrm{EQ}$, max accepting value: $v_f = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ (accepting value 2/3)

Theorem

$\mathtt{REG} \subsetneq \mathtt{BAfL}$[1]

Bounded-error affine languages (BAfL)



$$A_a = \begin{pmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ -x & 0 & 1 \end{pmatrix}$$

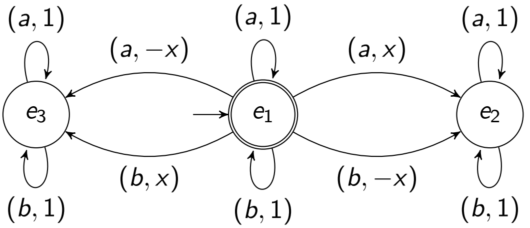
$$A_b = \begin{pmatrix} 1 & 0 & 0 \\ -x & 1 & 0 \\ x & 0 & 1 \end{pmatrix}$$

After reading m as and n bs the state is $\begin{pmatrix} 1 \\ (m-n)x \\ (n-m)x \end{pmatrix}$

Accepting value: $\begin{cases} 1 & \text{if } m = n \\ \dfrac{1}{2x|m-n|+1} & \text{if } m \neq n \end{cases}$

Taking x larger we get smaller error

Bounded-error affine languages (BAfL)



$$A_a = \begin{pmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ -x & 0 & 1 \end{pmatrix}$$

$$A_b = \begin{pmatrix} 1 & 0 & 0 \\ -x & 1 & 0 \\ x & 0 & 1 \end{pmatrix}$$

After reading m as and n bs the state is $\begin{pmatrix} 1 \\ (m-n)x \\ (n-m)x \end{pmatrix}$

$$\text{Accepting value: } \begin{cases} 1 & \text{if } m = n \\ \dfrac{1}{2x|m-n|+1} & \text{if } m \neq n \end{cases}$$

Taking x larger we get smaller error

Theorem

$$\mathtt{REG} \subsetneq \mathtt{BAfL}^0$$

Cutpoint affine languages (AfL)

$$\texttt{LAPINS}' = \{w \in \{a,b,c\}^* \mid |w|_a^2 > |w|_b \text{ and } |w|_b^2 > |w|_c\} \quad \notin \texttt{SL}$$

[Jānis Lapiņš, 1974]

PFAs and QFAs can check one of the conditions, with cutpoint $\frac{1}{2}$, but not both

Cutpoint affine languages ($\texttt{AfL}$)

$$\texttt{LAPINS}' = \{w \in \{a,b,c\}^* \mid |w|_a^2 > |w|_b \text{ and } |w|_b^2 > |w|_c\} \quad \notin \texttt{SL}$$

[Jānis Lapiņš, 1974]

PFAs and QFAs can check one of the conditions, with cutpoint $\frac{1}{2}$, but not both

Theorem

$\texttt{LAPINS}'$ *is recognized by an AfA with cutpoint* $\frac{1}{2}$

Proof (sketch).

Cutpoint affine languages ($\texttt{AfL}$)

$$\texttt{LAPINS}' = \{w \in \{a,b,c\}^* \mid |w|_a^2 > |w|_b \text{ and } |w|_b^2 > |w|_c\} \quad \notin \texttt{SL}$$

[Jānis Lapiņš, 1974]

PFAs and QFAs can check one of the conditions, with cutpoint $\frac{1}{2}$, but not both

Theorem

$\texttt{LAPINS}'$ *is recognized by an AfA with cutpoint* $\frac{1}{2}$

Proof (sketch).

1. Check $|w|_a^2 > |w|_b$ with an AfA simulating the PFA $\begin{pmatrix} |w|_a^2 \\ |w|_b \end{pmatrix}$

Cutpoint affine languages ($\mathtt{AfL}$)

$$\mathtt{LAPINS}' = \{w \in \{a,b,c\}^* \mid |w|_a^2 > |w|_b \text{ and } |w|_b^2 > |w|_c\} \quad \notin \mathtt{SL}$$

[Jānis Lapiņš, 1974]

PFAs and QFAs can check one of the conditions, with cutpoint $\frac{1}{2}$, but not both

Theorem

$\mathtt{LAPINS}'$ is recognized by an AfA with cutpoint $\frac{1}{2}$

Proof (sketch).

1. Check $|w|_a^2 > |w|_b$ with an AfA simulating the PFA $\begin{pmatrix} |w|_a^2 \\ |w|_b \end{pmatrix}$
2. Check $|w|_b^2 > |w|_c$ by producing the state $\begin{pmatrix} |w|_b^2 - |w|_c \\ 1 - (|w|_b^2 - |w|_c) \end{pmatrix}$

Cutpoint affine languages (AfL)

$$\texttt{LAPINS}' = \{w \in \{a,b,c\}^* \mid |w|_a^2 > |w|_b \text{ and } |w|_b^2 > |w|_c\} \quad \notin \texttt{SL}$$

[Jānis Lapiņš, 1974]

PFAs and QFAs can check one of the conditions, with cutpoint $\frac{1}{2}$, but not both

Theorem

$\texttt{LAPINS}'$ *is recognized by an AfA with cutpoint* $\frac{1}{2}$

Proof (sketch).

1. Check $|w|_a^2 > |w|_b$ with an AfA simulating the PFA $\begin{pmatrix}|w|_a^2 \\ |w|_b\end{pmatrix}$
2. Check $|w|_b^2 > |w|_c$ by producing the state $\begin{pmatrix}|w|_b^2 - |w|_c \\ 1-(|w|_b^2 - |w|_c)\end{pmatrix}$
3. Tensor both atomata □

PFA and QFA cannot do steps 1 and 2 at the same time!

Cutpoint affine languages ($\texttt{AfL}$)

$$\texttt{LAPINS}' = \{w \in \{a,b,c\}^* \mid |w|_a^2 > |w|_b \text{ and } |w|_b^2 > |w|_c\} \quad \notin \texttt{SL}$$

[Jānis Lapiņš, 1974]

PFAs and QFAs can check one of the conditions, with cutpoint $\frac{1}{2}$, but not both

Theorem

$\texttt{LAPINS}'$ *is recognized by an AfA with cutpoint* $\frac{1}{2}$

Proof (sketch).

1. Check $|w|_a^2 > |w|_b$ with an AfA simulating the PFA $\begin{pmatrix} |w|_a^2 \\ |w|_b \end{pmatrix}$
2. Check $|w|_b^2 > |w|_c$ by producing the state $\begin{pmatrix} |w|_b^2 - |w|_c \\ 1 - (|w|_b^2 - |w|_c) \end{pmatrix}$
3. Tensor both atomata □

PFA and QFA cannot do steps 1 and 2 at the same time!

Corollary

$\texttt{SL} \subsetneq \texttt{AfL}$

AFAs is more powerful than PFAs and QFAs with cutpoint

Nondeterministic affine languages ($\mathtt{NAfL}$)

($\mathtt{NQAL}$ contains some famous languages like the complement of EQ)

Theorem

$$\mathtt{NAfL} = \mathtt{NQAL}$$

Proof. We prove the double inclusion by showing how to simulate one with the other.

NAfAs have the same power as NQFAs

Summarising

- Bounded and unbounded error:
  AfAs more powerful than QFAs and PFAs
- Nondeterministic computation:
  AfAs equivalent to QFAs
  (and are more powerful than PFAs)

Summarising

- Bounded and unbounded error:
  AfAs more powerful than QFAs and PFAs
- Nondeterministic computation:
  AfAs equivalent to QFAs
  (and are more powerful than PFAs)

Corroboration of the thesis:
The destructive interference
plays a role in the computational power of QC

Summarising

- Bounded and unbounded error:
  AfAs more powerful than QFAs and PFAs
- Nondeterministic computation:
  AfAs equivalent to QFAs
  (and are more powerful than PFAs)

Corroboration of the thesis:
The destructive interference
plays a role in the computational power of QC

Further results

- *Language recognition power and succinctness of affine automata*
  (M. Villagra and A. Yakaryılmaz) *To appear in UCNC'16* `arXiv:1602.05432`
- *Can one quantum bit separate any pair of words with zero-error?*
  (A. Belovs, J. A. Montoya, A. Yakaryılmaz) `arXiv:1602.07967`

Backup slides

Weighting operator

Can we use the weighting operator as a projective measurement?

Answer: No

$$v = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

Weighting based on separation $\{e_1\}$ and $\{e_2, e_3\}$:

Probability $1/3$ of $\{e_1\}$
Probability $2/3$ of $\{e_2, e_3\}$

But

$$v' = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \qquad \text{Not affine! (not even after normalization)}$$

Conclusion: After weighting, the system must collapse to a single state