The probability of non-confluent systems

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Motivation

Non-deterministic vs. Probabilistic λ-calculus

Non-determinism	Probabilities
$\mathbf{r} + \mathbf{s}$ non-deterministic superposition (run **r** or **s**, non-deterministically)	$p.\mathbf{r} + q.\mathbf{s}$ probabilistic superposition (run **r** with probability p or **s** with probability q)

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$$\swarrow \qquad \searrow$$

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- Second order propositional logic
- Quantitative characterisation in LL
- Etc.

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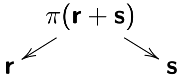

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Goal: To move from ND to Prob. without loosing the connections with logic

Outline

Goal: To move from Non-determinism to Probilities

- General technique
- Application to a particular case

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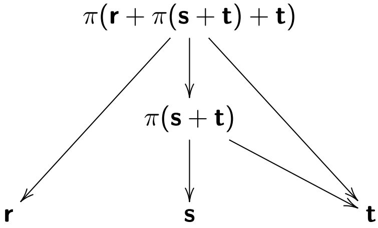

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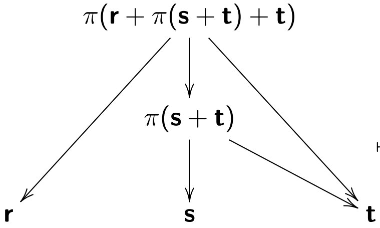
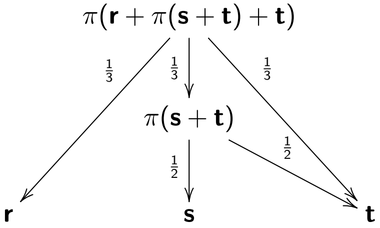


$\mapsto$



$$\sim \quad \frac{1}{3}.\mathbf{r} + \frac{1}{6}.\mathbf{s} + \frac{1}{2}.\mathbf{t}$$

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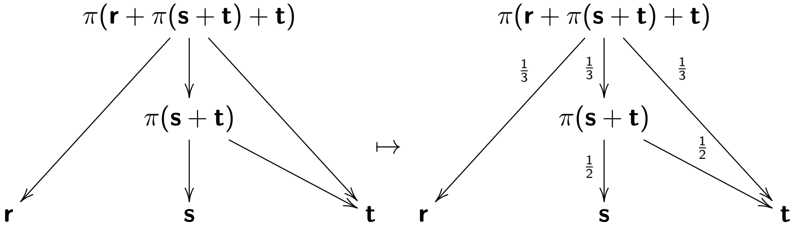

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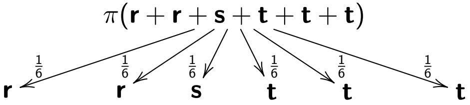
An easier way...



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Generalising the problem to abstract rewrite systems

Idea: to define a variant of a Lebesgue measure for sets of real numbers, on the space of traces

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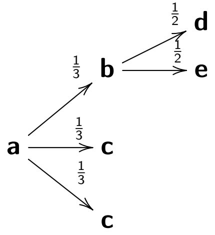
e.g. If 

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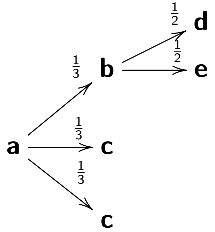

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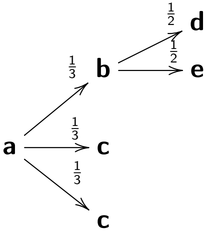

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Formalisation

Strategies

Λ: set of objects $\quad \to : \Lambda \times \Lambda \to \mathbb{N} \quad$ $\mathbf{a} \to \mathbf{b}$ notation for $\to(\mathbf{a}, \mathbf{b}) \neq 0$.

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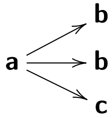
$$\rho(\mathbf{a}) = \sum_{\mathbf{b}} \to(\mathbf{a}, \mathbf{b})$$

e.g. 

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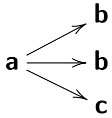

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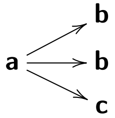

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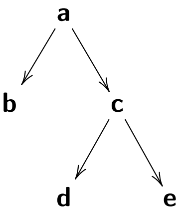

$\Omega = \{f, g, h, i\}$, with

$f(\mathbf{a}) = \mathbf{b}$ $\qquad$ $g(\mathbf{a}) = \mathbf{b}$
$f(\mathbf{c}) = \mathbf{d}$ $\qquad$ $g(\mathbf{c}) = \mathbf{e}$

$h(\mathbf{a}) = \mathbf{c}$ $\qquad$ $i(\mathbf{a}) = \mathbf{c}$
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$B \subseteq \Omega$ of the form

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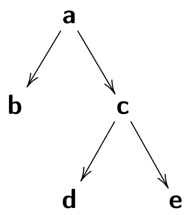

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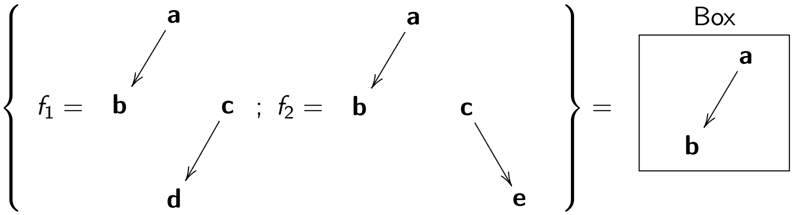

$$\{f_1; f_2\} = \{f \mid f(\mathbf{a}) = \mathbf{b}\}$$

Formalisation

Measure on boxes

Definition (Measure on boxes)

If $B = \{f \mid f(\mathbf{a}_1) = \mathbf{b}_1, \dots, f(\mathbf{a}_n) = \mathbf{b}_n\}$ then

$$\mathrm{p}(B) = \prod_{i=1}^{n} \frac{\to(\mathbf{a}_i, \mathbf{b}_i)}{\rho(\mathbf{a}_i)} \quad \begin{pmatrix} \to(\mathbf{a}_i, \mathbf{b}_i) \\ \text{ways to arrive to } \mathbf{b}_i \text{ from } \mathbf{a}_i \\ \rho(\mathbf{a}_i) \\ \text{nb. of rewrites from } \mathbf{a}_i \end{pmatrix}$$

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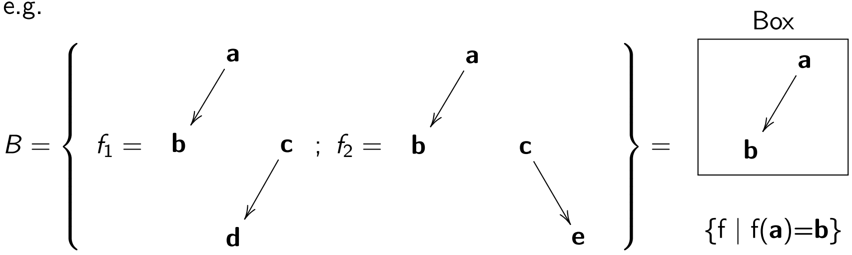

$$\boxed{\mathrm{p}(B) = \frac{\rightarrow(\mathbf{a}, \mathbf{b})}{\rho(\mathbf{a})} = \frac{1}{2}}$$

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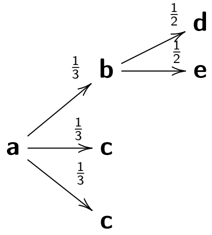
e.g. If 

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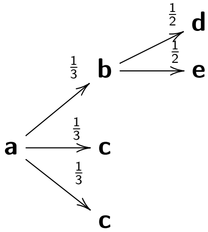
e.g. If 

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$$\mathtt{P}(\emptyset) = 0$$

$$\mathtt{P}(S) = \inf \left\{ \sum_{B \in \mathcal{C}} \mathtt{p}(B) \mid \mathcal{C} \text{ is a countable family of boxes s.t. } S \subseteq \bigcup_{B \in \mathcal{C}} B \right\}$$

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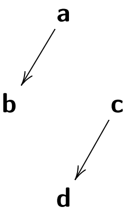
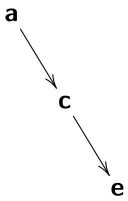
$S = \Big\{ f_1 =$ 

$;\ f_2 =$ 

$\Big\} = \underbrace{\{f_1\}}_{B_1} \cup \underbrace{\{f_2\}}_{B_2}$

$$\boxed{\mathrm{P}(S) = \mathrm{p}(B_1) + \mathrm{p}(B_2) = \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} = \frac{1}{2}}$$

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Lebesgue measure and probability space

Definition (Lebesgue measurable)

A is Lebesgue measurable if $\forall S \in \mathcal{P}(\Omega)$

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Theorem

$(\Omega, \mathcal{A}, \mathrm{P})$ *is a probability space*

- Ω is the set of all possible strategies
- $\mathcal{A}$ is the set of events
- P is the probability function

Proof.

We show that it satisfies the Kolmogorov axioms. □

Outline

Goal: To move from Non-determinism to Probilities

- General technique
- Application to a particular case

From non-determinism to probabilities

The calculus λ_+

$$A, B, C ::= X \mid A \Rightarrow B \mid A \wedge B \mid \forall X.A$$

$$\mathbf{r}, \mathbf{s}, \mathbf{t} ::= x^A \mid \lambda x^A.\mathbf{r} \mid \mathbf{rs} \mid \mathbf{r} + \mathbf{s} \mid \pi_A(\mathbf{r}) \mid \Lambda X.\mathbf{r} \mid \mathbf{r}\{A\}$$

Beta + extra rewrite rules. E.g. $(\mathbf{r} + \mathbf{s})\mathbf{t} \to \mathbf{rt} + \mathbf{st}$

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Non-determinism:

If $\mathbf{r} : A \quad \mathbf{s} : A$

$$\pi_A(\mathbf{r} + \mathbf{s}) \to \mathbf{r} \qquad \pi_A(\mathbf{r} + \mathbf{s}) \to \mathbf{s}$$

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The calculus λ_+^p

Definition (ARS $\lambda_+^\downarrow$)

- Closed normal terms of λ_+ are objects of $\lambda_+^\downarrow$
- If $\mathbf{r}_1, \dots, \mathbf{r}_n$ are objects, then $\mathbf{r}_1 + \cdots + \mathbf{r}_n$ too

The rewrite rules have multiplicities: e.g. $\pi_A(\mathbf{r} + \mathbf{r}) \to \mathbf{r}$ with multiplicity 2

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Theorem

$(\Omega, \mathcal{A}, \mathtt{P})$: *probability space over* $\lambda_+^{\downarrow}$
$B_{\boldsymbol{r}_i} = \{f \mid f(\pi_A(\sum_{j=1}^n m_j.\boldsymbol{r}_j)) = \boldsymbol{r}_i\}$: *a box*

$$\mathtt{P}(B_{\boldsymbol{r}_i}) = \frac{m_i}{\sum_{j=1}^n m_j}$$

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The calculus λ_+^p

Definition (ARS $\lambda_+^\downarrow$)

- Closed normal terms of λ_+ are objects of $\lambda_+^\downarrow$
- If $\mathbf{r}_1, \dots, \mathbf{r}_n$ are objects, then $\mathbf{r}_1 + \cdots + \mathbf{r}_n$ too

The rewrite rules have multiplicities: e.g. $\pi_A(\mathbf{r} + \mathbf{r}) \to \mathbf{r}$ with multiplicity 2

Theorem

$(\Omega, \mathcal{A}, \mathrm{P})$: *probability space over* $\lambda_+^\downarrow$
$B_{\mathbf{r}_i} = \{f \mid f(\pi_A(\sum_{j=1}^n m_j.\mathbf{r}_j)) = \mathbf{r}_i\}$: *a box*

$$\mathrm{P}(B_{\mathbf{r}_i}) = \frac{m_i}{\sum_{j=1}^n m_j}$$

Definition (Probabilistic calculus λ_+^p)

Replace rule "If $\mathbf{r} : A$, then $\pi_A(\mathbf{r} + \mathbf{s}) \to \mathbf{r}$" by
$\pi_A(\sum_{i=1}^n m_i.\mathbf{r}_i + \mathbf{s}) \to \mathbf{r}_i$ with probability $\frac{m_i}{\sum_{i=1}^n m_j}$

From non-determinism to probabilities

$\lambda_+^p \leftarrow$ **Alg**

Algebraic calculi (Probabilistic version)

$$\mathbf{r},\mathbf{s},\mathbf{t} ::= x^A \mid \lambda x^A.\mathbf{r} \mid \mathbf{rs} \mid \Lambda X.\mathbf{r} \mid \mathbf{r}\{A\} \mid \sum_{i=1}^{n} p_i.\mathbf{r}_i \qquad \text{with } \begin{cases} n > 0, \\ p_i \in \mathbb{Q}(0,1] \text{ and} \\ \sum_{i=1}^{n} p_i = 1 \end{cases}$$

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Definition (From Alg to λ_+^p)

$$[\![\sum_{i=1}^{n} \frac{n_i}{d_i}.\mathbf{r}_i]\!] = \pi_A(\sum_{i=1}^{n} m_i.[\![\mathbf{r}_i]\!]) \quad \text{where } \begin{cases} \mathbf{r}_i : A \\ n_i, d_i \in \mathbb{N}^* \\ m_i = n_i(\prod_{\substack{k=1 \\ k\neq i}}^{n} d_k) \end{cases} \text{for } i = 1, \dots, n$$

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Theorem (Alg to λ_+^p)

If $\mathbf{r} \to^* \sum_{i=1}^n p_i.\mathbf{t}_i$ in Alg and $[\![\mathbf{t}_i]\!] \to^* \mathbf{s}_i$, then $[\![\mathbf{r}]\!] \to^* \mathbf{s}_i$ with probability p_i in λ_+^p.

From non-determinism to probabilities

$\lambda_+^p \to$ Alg

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Definition (From λ_+^p to Alg)

If $\pi_A(\mathbf{t}) \to \mathbf{s}_i$ with probability p_i, for $i = 1, \dots, n$, $\quad (\!|\pi_A(\mathbf{t})|\!) = \sum_{i=1}^{n} p_i.(\!|\mathbf{s}_i|\!)$

Remark: if $\mathbf{t}$ normal, no translation

From non-determinism to probabilities

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Theorem (λ_+^p to Alg)

- *If $\mathbf{r} \to \mathbf{s}$, with probability 1, then $(\!|\mathbf{r}|\!) \to (\!|\mathbf{s}|\!)$*
- *If $\mathbf{r} \to \mathbf{s}_i$ with probability p_i, for $i = 1,\dots,n$, then $(\!|\mathbf{r}|\!) = \sum_{i=1}^n p_i.(\!|\mathbf{s}_i|\!)$.* □

Sumarising

- We provide a general technique to transform a non-deterministic calculus into a probabilistic one
- We have a way to transform λ_+ into λ_+^p
- We get a simpler calculus, encoding an algebraic calculus, without losing the connections with logic