# A concrete categorical semantics of Lambda-$\mathcal{S}$

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## Motivation

We are interested in the most natural way of **forbidding dupplication** in **quantum programming languages** and **formal logics**

# Outline

Quantum mechanics, in two slides

Motivation, better explained

Lambda-S

Concrete categorical semantics

# Quantum mechanics, in two slides

## (I) The postulates 1 and 2

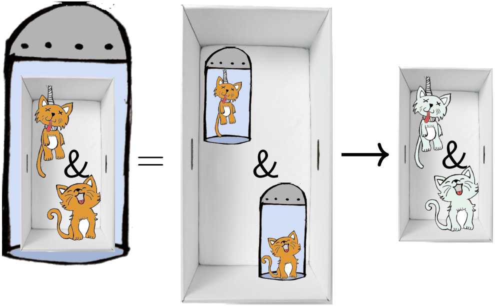
### Postulate 1: Quantum states


(A bit) more precisely:
**Normalized vectors** $\in \mathbb{C}^{2^n}$

$$\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \alpha\,|0\rangle + \beta\,|1\rangle \in \mathbb{C}^2$$

$$\begin{pmatrix}\alpha\\ \beta\\ \gamma\\ \delta\end{pmatrix} = \begin{array}{l}\alpha(|0\rangle\otimes|0\rangle) + \beta(|0\rangle\otimes|1\rangle)\\ +\gamma(|1\rangle\otimes|0\rangle) + \delta(|1\rangle\otimes|1\rangle)\\ \in \mathbb{C}^4\end{array}$$

### Postulate 2: Evolution




(A bit) more precisely:
**Unitary transformation** (matrix)

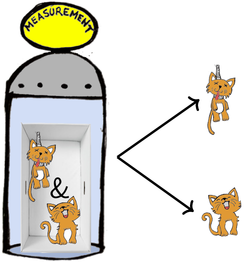
$$U\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = U(\alpha\,|0\rangle + \beta\,|1\rangle)$$

$$= \delta\,|0\rangle + \gamma\,|1\rangle = \begin{pmatrix}\delta\\ \gamma\end{pmatrix}$$

# Quantum computing, in two slides

## (II) The postulates 3 and 4

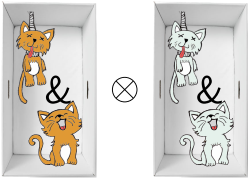
### Postulate 3: Measurement




(A bit) more precisely:

$\sum_{i=0}^{2^n} \alpha_i \left|i\right\rangle$ **collapses** to $\left|k\right\rangle$ with probability $|\alpha_k|^2$

### Postulate 4: Composition



More precisely: Tensor product

$$\begin{pmatrix}\alpha\\ \beta\end{pmatrix} \otimes \begin{pmatrix}\gamma\\ \delta\end{pmatrix} = \begin{pmatrix}\alpha\gamma\\ \alpha\delta\\ \beta\gamma\\ \beta\delta\end{pmatrix} \in \mathbb{C}^2 \otimes \mathbb{C}^2 = \mathbb{C}^4$$

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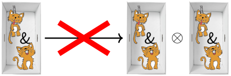
### Consequence: No cloning

A superposed state cannot be cloned



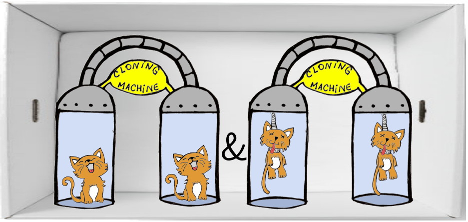
# Motivation

## Two approaches in the literature to deal with no cloning

**Linear-logic approach**


e.g. $\lambda x.(x \otimes x)$ is forbidden

**Linear-algebra approach**




e.g. $f(\alpha\,|0\rangle + \beta\,|1\rangle) \to \alpha f(|0\rangle) + \beta f(|1\rangle)$

# Motivation

## Measurement


The linear-algebra approach does not make sense here...


...but the linear-logic one, does

---

e.g.

$$(\lambda x.\pi x)\ (\alpha.\,|0\rangle + \beta.\,|1\rangle) \longrightarrow \alpha.(\lambda x.\pi x)\,|0\rangle + \beta.(\lambda x.\pi x)\,|1\rangle$$ **Wrong!**

(Measurement operator)

# Key point

# We need to distinguish superposed states from basis states using types

Basis states can be cloned
Superposed states cannot

Functions receiving superposed states, cannot clone its argument

# Lambda-$\mathcal{S}$

$$\Psi := \mathbb{B} \mid S(\Psi) \mid \Psi \times \Psi \qquad \text{Qubit types}$$
$$A := \Psi \mid \Psi \Rightarrow A \mid S(A) \qquad \text{Types}$$

## Examples

$$\mathbb{B} = \{\left|0\right\rangle, \left|1\right\rangle\} \subseteq \mathbb{C}^2 \qquad S(A) = \mathsf{span}(A)$$

$$\underbrace{\left|0\right\rangle}_{\mathbb{B}} \times \underbrace{(1/\sqrt{2}.\left|0\right\rangle + 1/\sqrt{2}.\left|1\right\rangle)}_{S(\mathbb{B})} \in \{\left|0\right\rangle, \left|1\right\rangle\} \times \mathbb{C}^2$$

$$\underbrace{1/\sqrt{2}.(\left|0\right\rangle \times \left|0\right\rangle) + 1/\sqrt{2}.(\left|0\right\rangle \times \left|1\right\rangle)}_{S(\mathbb{B}\times\mathbb{B})} \in \mathbb{C}^4 \simeq \mathsf{span}(\mathbb{B} \times \mathbb{B})$$

# Two kinds of linearity

**Call-by-name**

$$\underbrace{(\lambda x^{S(\Psi)}.t)}_{\text{linear abstraction}}\ \underbrace{u}_{S(\Psi)} \quad \to \quad t[u/x]$$

**Call-by-base**

$$(\lambda x^{\mathbb{B}}.t)\underbrace{(b_1 + b_2)}_{S(\mathbb{B})} \quad \to \quad (\lambda x^{\mathbb{B}}.t)\underbrace{b_1}_{\mathbb{B}} + (\lambda x^{\mathbb{B}}.t)\underbrace{b_2}_{\mathbb{B}} \quad \to \quad t[b_1/x] + t[b_2/x]$$

# Some information is lost on reduction

## Subtyping

$$\{|0\rangle, |1\rangle\} \subset \mathbb{C}^2 \qquad \text{then} \qquad \mathbb{B} \leq S(\mathbb{B})$$

$$\mathsf{span}(\mathsf{span}(A)) = \mathsf{span}(A) \qquad \text{then} \qquad S(S(\mathbb{B})) = S(\mathbb{B})$$

$$\{|0\rangle, |1\rangle\} \times \mathbb{C}^2 \subset \mathbb{C}^2 \times \mathbb{C}^2 \qquad \text{then} \qquad \mathbb{B} \times S(\mathbb{B}) \leq S(\mathbb{B} \times \mathbb{B})$$

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$$|0\rangle \times (|0\rangle + |1\rangle) \quad : \quad \mathbb{B} \times S(\mathbb{B})$$

$$|0\rangle \times |0\rangle + |0\rangle \times |1\rangle \quad : \quad S(\mathbb{B} \times \mathbb{B})$$

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**Same happens in math!**

$$(X-1)(X-2) \longrightarrow X^2 - 3X + 2$$

we lost the information that it was a product

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**Solution: casting**

$$|0\rangle \times (|0\rangle + |1\rangle) \not\rightarrow |0\rangle \times |0\rangle + |0\rangle \times |1\rangle$$

$$\Uparrow |0\rangle \times (|0\rangle + |1\rangle) \rightarrow |0\rangle \times |0\rangle + |0\rangle \times |1\rangle$$

# Measurement of the first $j$ qubits

$$\frac{\Gamma \vdash t : S(\mathbb{B}^n)}{\Gamma \vdash \pi_j t : \mathbb{B}^j \times S(\mathbb{B}^{n-j})}$$

## Example: Measurement

$$\begin{tikzcd}
& \pi_2(\ 2\,|\mathbf{01}1\rangle\ +\ |\mathbf{01}0\rangle\ +\ 3\,|\mathbf{11}1\rangle\ ) \arrow[dl, "\frac{5}{14}"'] \arrow[dr, "\frac{9}{14}"] & \\
|\mathbf{01}\rangle \times (\frac{2}{\sqrt{5}}\,|1\rangle + \frac{1}{\sqrt{5}}\,|0\rangle) & & |\mathbf{11}1\rangle
\end{tikzcd}$$

i.e. $\pi_j(|\psi\rangle) = \pi_j(\frac{|\psi\rangle}{\||\psi\rangle\|})$

## Example: Error

$$\pi_1(|0\rangle - |0\rangle) \to \pi_1(\vec{0}) \to \text{error}$$

where $|xyz\rangle = |x\rangle \times |y\rangle \times |z\rangle$

# A concrete categorical semantics

$$\begin{tikzcd}(\mathbf{Set},\times,\mathbf{1}) \arrow[rr, bend left, "(S{,}m)"] \arrow[rr, phantom, "\perp"] & & (\mathbf{Vec},\otimes,I) \arrow[ll, bend left, "(U{,}n)"]\end{tikzcd}$$

$$\boxed{\begin{aligned}[\![\mathbb{B}]\!] &= \mathbb{B}\\ [\![\Psi \Rightarrow A]\!] &= [\![\Psi]\!] \Rightarrow [\![A]\!]\\ [\![S(A)]\!] &= US[\![A]\!]\\ [\![A \times B]\!] &= [\![A]\!] \times [\![B]\!]\end{aligned}}$$

- $S$ functor s.t. $S(A) = \{\sum_i \alpha_i a_i \mid a_i \in A, \alpha_i \in \mathbb{C}\}$ vector space
- $U$ forgetful functor s.t.
  - for each vector space $V$, $U(V)$ is the underlying set of vectors in $V$
  - for each linear map $f$, $U(f)$ forgets of its linear property
- $m$: natural transformation

$$m_{AB} : S(A) \otimes S(B) \to S(A \times B)$$
$$(\sum_{a\in A} \alpha_a a) \otimes (\sum_{b\in B} \beta_b b) \mapsto \sum_{(a,b)\in A\times B} \alpha_a\beta_b(a,b)$$

- $n$: natural transformation

$$n_{AB} : U(V) \times U(W) \to U(V \otimes W)$$
$$(v,w) \mapsto v \otimes w$$

# Soundness example

$$\dfrac{\dfrac{\vdash f : \mathbb{B} \Rightarrow \mathbb{B}}{\vdash f : S(\mathbb{B} \Rightarrow \mathbb{B})} \qquad \dfrac{\vdash |0\rangle : \mathbb{B} \quad \vdash |1\rangle : \mathbb{B}}{\vdash |0\rangle + |1\rangle : S(\mathbb{B})}}{\vdash f(|0\rangle + |1\rangle) : S(\mathbb{B})} \qquad\qquad \dfrac{\dfrac{\vdash f : \mathbb{B} \Rightarrow \mathbb{B} \quad \vdash |0\rangle : \mathbb{B}}{\vdash f\,|0\rangle : \mathbb{B}} \qquad \dfrac{\vdash f : \mathbb{B} \Rightarrow A \quad \vdash |1\rangle : \mathbb{B}}{\vdash f\,|1\rangle : \mathbb{B}}}{\vdash f\,|0\rangle + f\,|1\rangle : S(\mathbb{B})}$$

$$\begin{tikzcd}
 & US(\mathbb{B}) \times US([\mathbb{B},\mathbb{B}]) \arrow[r, "n"] & U(S(\mathbb{B}) \otimes S([\mathbb{B},\mathbb{B}])) \arrow[dr, bend left, "U(m)"] & \\
\mathbb{B}^2 \times [\mathbb{B},\mathbb{B}] \arrow[ur, bend left, "+\times\preceq"] & & & US(\mathbb{B} \times [\mathbb{B},\mathbb{B}]) \arrow[d, "US(\varepsilon^{\mathbb{B}})"] \\
\mathbf{1} \arrow[u, "(|0\rangle\times|1\rangle)\times f"] \arrow[dr, bend right, "(|0\rangle\times f)\times(|1\rangle\times f)"] & & & US(\mathbb{B}) \\
 & (\mathbb{B} \times [\mathbb{B},\mathbb{B}])^2 \arrow[r, "(\varepsilon^{\mathbb{B}})^2"] & \mathbb{B}^2 \arrow[ur, bend right, "+"'] &
\end{tikzcd}$$

# Projection

## Two more ingredients

- A distribution monad $(D, \hat{\eta}, \hat{\mu})$ $\qquad D : \mathbf{Set} \to \mathbf{Set}$

$$D(A) = \{\sum_{i=1}^{n} p_i \chi_{a_i} \mid \sum_{i=1}^{n} p_i = 1, \quad a_i \in A\}$$

$\chi_a$: characteristic function of $a$

$$\begin{aligned} \hat{\eta} : A &\to D(A) \\ a &\mapsto 1\chi_a \end{aligned} \qquad \begin{aligned} \hat{\mu} : D(D(A)) &\to D(A) \\ \sum_{i=1}^{n} p_i \chi_{(\sum_{j=1}^{m_i} q_{ij}\chi_{a_{ij}})} &\mapsto \sum_{i=1}^{n}\sum_{j=1}^{m_i} p_i q_{ij} \chi_{a_{ij}} \end{aligned}$$

- An error monad $(E, \tilde{\eta}, \tilde{\mu})$ $\qquad E : \mathbf{Set} \to \mathbf{Set}$

$$E(A) = A + \{\mathrm{e}\}$$

e: error

$$\begin{aligned} \tilde{\eta} : A &\to E(A) \\ a &\mapsto \mathrm{inl}(a) \end{aligned} \qquad \begin{aligned} \tilde{\mu} : E(E(A)) &\to E(A) \\ \mathrm{inl}(a) &\mapsto a \\ \mathrm{inr}(\mathrm{e}) &\mapsto \mathrm{inr}(\mathrm{e}) \end{aligned}$$

# Projection

## Factorization arrow in Set

$$\begin{tikzcd}
\mathbb{B}^j \times US(\mathbb{B}^{n-j}) \arrow[r, "\eta\times\mathsf{Id}"] \arrow[dd, "\mathsf{Id}"] & US(\mathbb{B}^j) \times US(\mathbb{B}^{n-j}) \arrow[r, "n"] & U(S(\mathbb{B}^j) \otimes S(\mathbb{B}^{n-j})) \arrow[d, "U(m)"] \\
 & & US(\mathbb{B}^j \times \mathbb{B}^{n-j}) \arrow[d, equal] \\
\mathbb{B}^j \times US(\mathbb{B}^{n-j}) & & US(\mathbb{B}^n) \arrow[ll, "\varphi_j"']
\end{tikzcd}$$

**Example**

$$\varphi_1 : US(\mathbb{B}^2) \to \mathbb{B} \times US(\mathbb{B})$$
$$a \mapsto \begin{cases} |x\rangle \times (\alpha_1 |y\rangle + \alpha_2 |z\rangle) & \text{if } a = \alpha_1 |xy\rangle + \alpha_2 |xz\rangle \\ |00\rangle & \text{otherwise} \end{cases}$$

# Projection

## Projection arrow in Set

$$\begin{aligned}\mathrm{Norm} : US(\mathbb{B}^n) &\to EUS(\mathbb{B}^n)\\ a &\mapsto \begin{cases} \frac{q}{\sqrt{(\overline{q}^\dagger \circ \overline{q})(1)}} & \text{if } q \neq \vec{0} \\ \mathrm{e} & \text{if } q = \vec{0} \end{cases}\end{aligned}$$

where $\overline{q} : I \to S(\mathbb{B}^j)$ such that $1 \mapsto q$
For each $k = 0, \dots, 2^j - 1$

$$\begin{tikzcd}
US(\mathbb{B}^n) \arrow[r, "U((\overline{|k\rangle} \circ \overline{|k\rangle}^\dagger) \otimes I)"] \arrow[d, "\pi_{jk}"] & US(\mathbb{B}^n) \arrow[d, "\mathrm{Norm}"] \\
E(\mathbb{B}^j \times US(\mathbb{B}^{n-j})) & EUS(\mathbb{B}^n) \arrow[l, "E(\varphi_j)"]
\end{tikzcd}$$

$$\begin{aligned}\pi_j : US(\mathbb{B}^n) &\to D(E(\mathbb{B}^j \times US(\mathbb{B}^{n-j})))\\ |\psi\rangle &\mapsto \sum_{k=0}^{2^j-1} p_k \chi_{\pi_{jk}|\psi\rangle}\end{aligned}$$

where
$p_k = \overline{\mathrm{Norm}(|\psi\rangle)}^\dagger \circ P_k \circ \overline{\mathrm{Norm}(|\psi\rangle)}$
with $P_k = (\overline{|k\rangle} \circ \overline{|k\rangle}^\dagger) \otimes I$.

# Summarising

- Extension of first-order lambda-calculus for quantum computing
- Logical-linearity and algebraic-linearity both used for no-cloning
- Categorical semantics: dual of linear logic

Other publications

- A. Díaz-Caro & G. Dowek. "Typing quantum superpositions and measurement". LNCS 10687:281–293 (**TPNC 2017**).
- J. P. Rinaldi. "Demostrando normalización fuerte sobre una extensión cuántica del lambda cálculo". **Master's thesis**. Universidad Nacional de Rosario. June 2018.

Works in progress

- Abstract categorical model (A. Díaz-Caro & O. Malherbe)
- Implementation in Haskell (I. Grimma, A. Díaz-Caro & P. E. Martínez López)

# Backup slides

# Why first order

$$\mathsf{CM} = \lambda y^{S(\mathbb{B})}.((\lambda x^{\mathbb{B}\Rightarrow S(\mathbb{B})}.(x\,|0\rangle) \times (x\,|0\rangle))\ (\lambda z^{\mathbb{B}}.y))$$

$$\begin{aligned}
&\mathsf{CM}\ (\alpha.\,|0\rangle + \beta.\,|1\rangle)\\
&\to (\lambda x^{\mathbb{B}\Rightarrow S(\mathbb{B})}.(x\,|0\rangle) \times (x\,|0\rangle))\ (\lambda z^{\mathbb{B}}.(\alpha.\,|0\rangle + \beta.\,|1\rangle))\\
&\to ((\lambda z^{\mathbb{B}}.(\alpha.\,|0\rangle + \beta.\,|1\rangle))\,|0\rangle) \times ((\lambda z^{\mathbb{B}}.(\alpha.\,|0\rangle + \beta.\,|1\rangle))\,|0\rangle)\\
&\to^2 (\alpha.\,|0\rangle + \beta.\,|1\rangle) \times (\alpha.\,|0\rangle + \beta.\,|1\rangle)
\end{aligned}$$

# Deutsch algorithm

## Preliminaries

### Hadamard

$$H\,|0\rangle = \frac{1}{\sqrt{2}}\,|0\rangle + \frac{1}{\sqrt{2}}\,|1\rangle \qquad H\,|1\rangle = \frac{1}{\sqrt{2}}\,|0\rangle - \frac{1}{\sqrt{2}}\,|1\rangle$$

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$$H = \lambda x^{\mathbb{B}}.{}^{1}\!/\!\sqrt{2}.(|0\rangle + x?{-|1\rangle}\cdot|1\rangle)$$

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### Oracle

A "black box" implementing a function $f : \{0,1\} \to \{0,1\}$

$$U_f(|x\rangle \times |y\rangle) = |x\rangle \times |y \oplus f(x)\rangle$$

# Deutsch algorithm

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### Oracle

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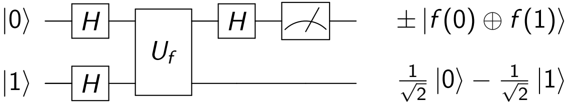
$$U_f(|x\rangle \times |y\rangle) = |x\rangle \times |y \oplus f(x)\rangle$$

$$\begin{aligned} not &= \lambda x^{\mathbb{B}}.x?|0\rangle{\cdot}|1\rangle \\ U_f &= \lambda x^{\mathbb{B}\times\mathbb{B}}.(head\ x) \times ((tail\ x)?not(f(head\ x)){\cdot}f(head\ x)) \end{aligned}$$

# Deutsch in $\lambda$



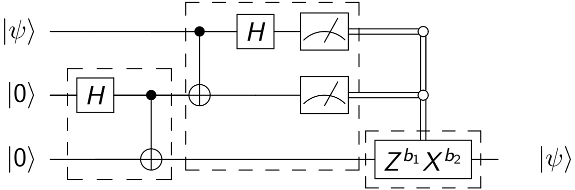

$$\begin{aligned}
not &= \lambda x^{\mathbb{B}}.x?|0\rangle{\cdot}|1\rangle \\
H &= \lambda x^{\mathbb{B}}.{}^{1}\!/\!\sqrt{2}.(|0\rangle + x?{-}|1\rangle{\cdot}|1\rangle) \\
H^{\times 2} &= \lambda x^{\mathbb{B}\times\mathbb{B}}.(H(head\ x)) \times (H(tail\ x)) \\
U_f &= \lambda x^{\mathbb{B}\times\mathbb{B}}.(head\ x) \times ((tail\ x)?not(f(head\ x)){\cdot}f(head\ x)) \\
H_1 &= \lambda x^{\mathbb{B}\times\mathbb{B}}.(H(head\ x)) \times (tail\ x)
\end{aligned}$$

$$Deutsch_f = \pi_1(\Uparrow H_1(U_f \Uparrow\Uparrow H^{\times 2}(|0\rangle \times |1\rangle)$$

$$\vdash Deutsch_f : \mathbb{B} \times S(\mathbb{B})$$

$$\begin{aligned}
Deutsch_{id} &\longrightarrow^{*}_{(1)} \pi_1({}^{1}\!/\!\sqrt{2}.\,|1\rangle \times |0\rangle - {}^{1}\!/\!\sqrt{2}.\,|1\rangle \times |1\rangle) \\
&\longrightarrow_{(1)} |1\rangle \times ({}^{1}\!/\!\sqrt{2}.\,|0\rangle - {}^{1}\!/\!\sqrt{2}.\,|1\rangle)
\end{aligned}$$

# Teleportation in $\lambda$




$$\text{epr} = \lambda x^{\mathbb{B}\times\mathbb{B}}.\text{cnot}(H_1\ x)$$

$$\text{alice} = \lambda x^{S(\mathbb{B})\times S(\mathbb{B}\times\mathbb{B})}.\pi_2(\Uparrow H_1^3(cnot_{12}^3 \Uparrow\Uparrow x))$$

$$U^b = (\lambda b^{\mathbb{B}}.\lambda x^{\mathbb{B}}.b?Ux{\cdot}x)\ b$$

$$\text{bob} = \lambda x^{\mathbb{B}\times\mathbb{B}\times\mathbb{B}}.Z^{head\ x}not^{head\ (tail\ x)}.(tail\ (tail\ x))$$

$$Teleportation = \lambda q^{S(\mathbb{B})}.\text{bob}\ (\Uparrow \text{alice}\ (q \times (\text{epr}\ \ |0\rangle \times |0\rangle)))$$

$$\vdash Teleportation : S(\mathbb{B}) \Rightarrow S(\mathbb{B})$$

$$Teleportation\ q \longrightarrow_{(1)} q$$