Topological Concatenation of 2D Color Codes

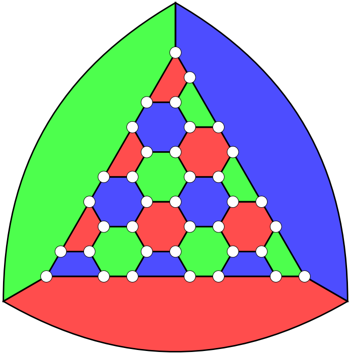
Alexandre Guernut Christophe Vuillot

June 19, 2021

2D triangular color codes

A topological stabilizer code

- Stabilizers are associated to the faces of a tiling of a sphere



(a) Sphere tiling

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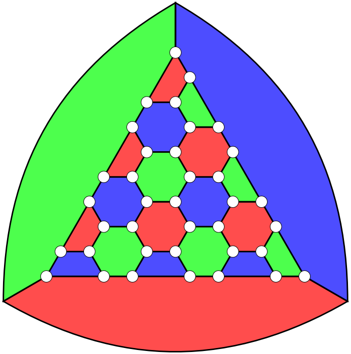

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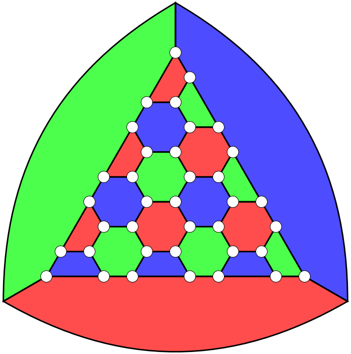
- Each face is associated to an X and a Z stabilizer acting on the vertices



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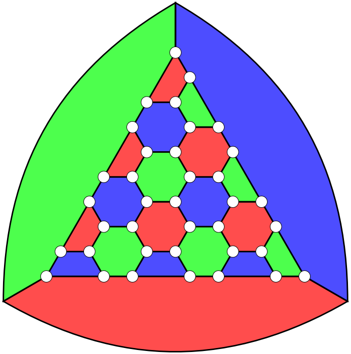

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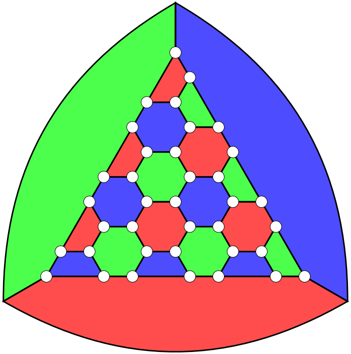

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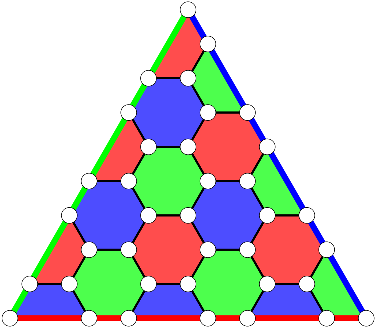
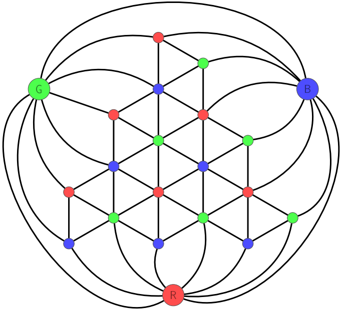


(b) 2D Triangular color code

Dual representation

- The dual graph is often more useful



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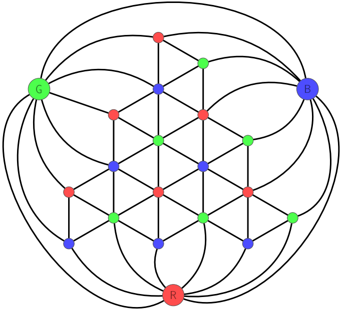

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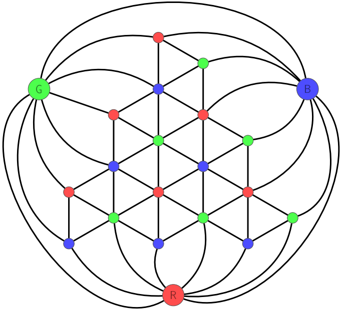
- Colored boundaries are now boundary nodes



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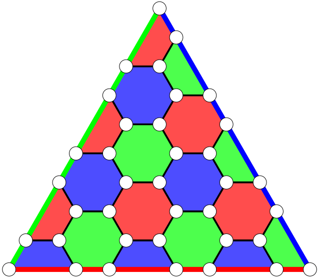

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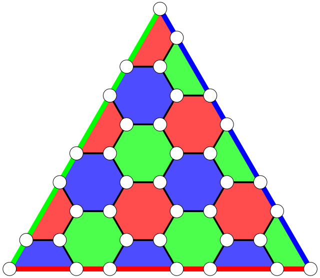

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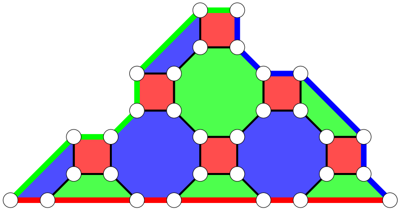

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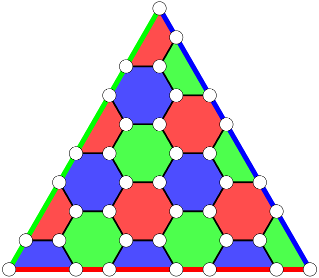
- They are the only ones whose leading coefficient in the number of qubits representation as a function of the distance is less than 1



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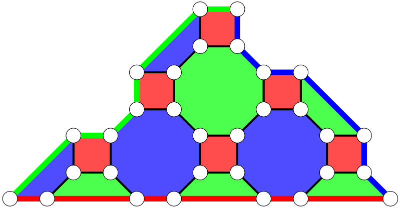

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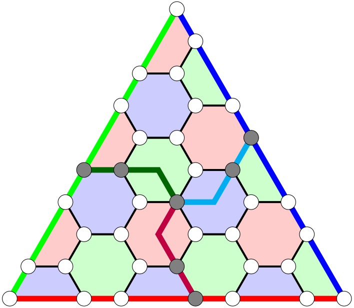

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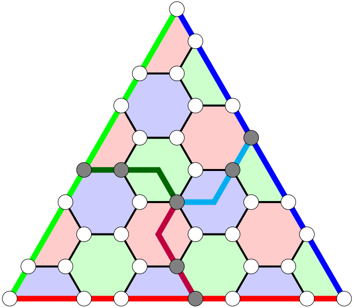

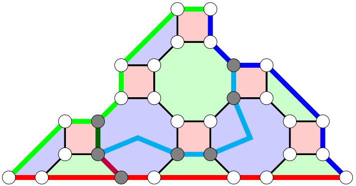

Topological concatenation of color codes

Improving logical error rate

- Once one has both a quantum code and a decoding procedure whose threshold is above the physical error rate

[1] Daniel Gottesman, A Theory of Fault-Tolerant Quantum Computation (1997)

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- In the case of topological codes, we can increase the distance, with a quadratic cost in the number of qubits used (BPT bound)
- Topological concatenation is a hybrid of these two methods

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Plain surgery

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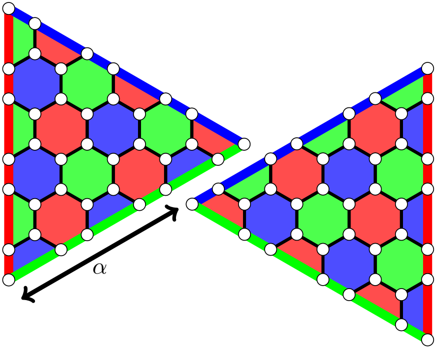

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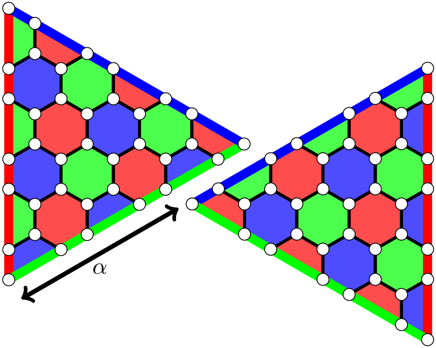

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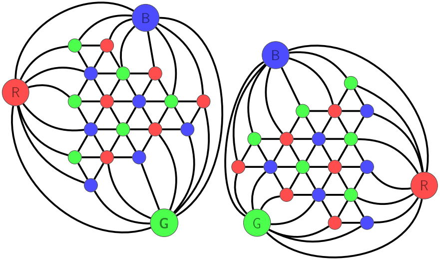

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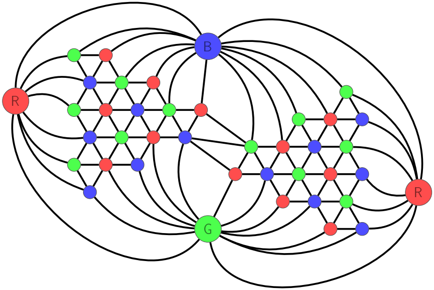

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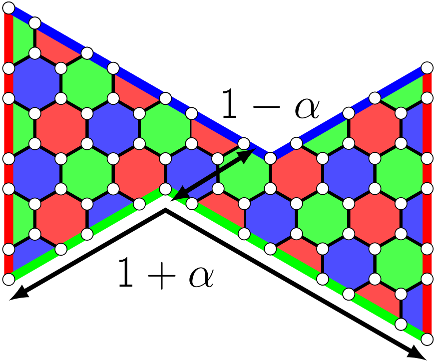

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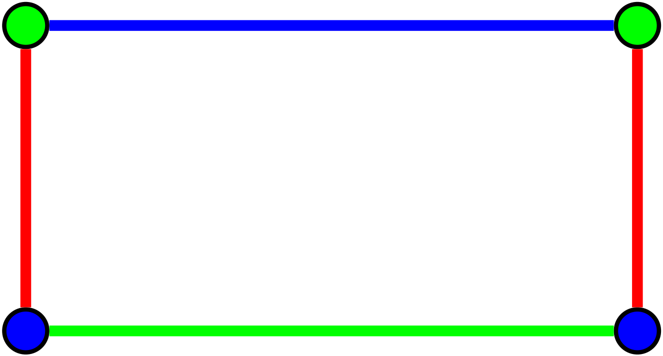

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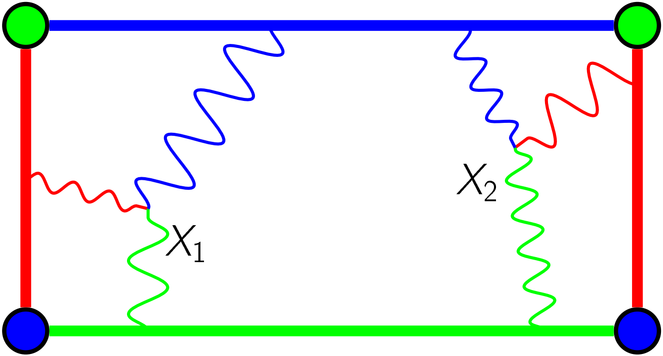

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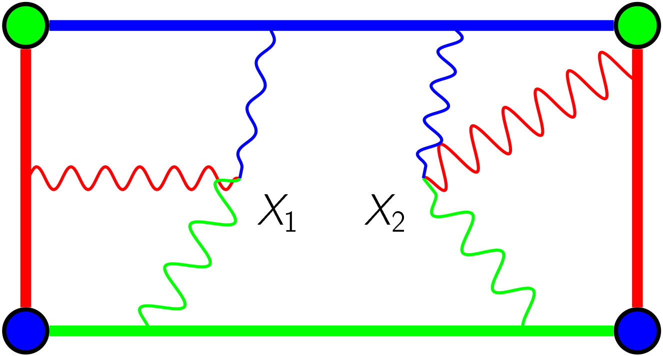

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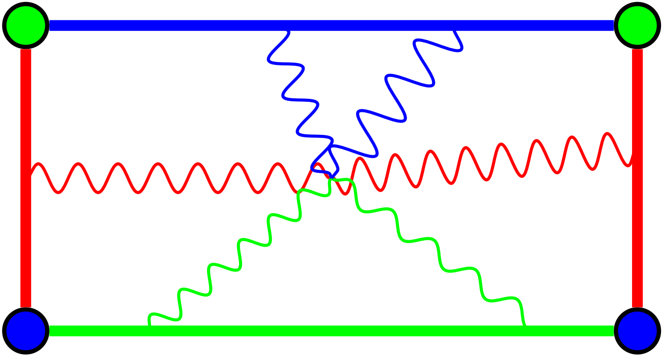

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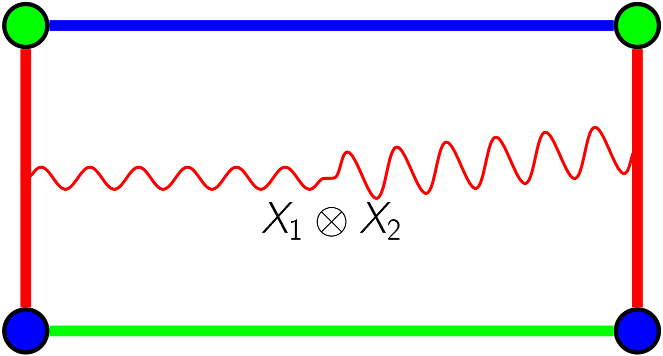
- The product of the two logical operators can be represented by a red-string from left to right:



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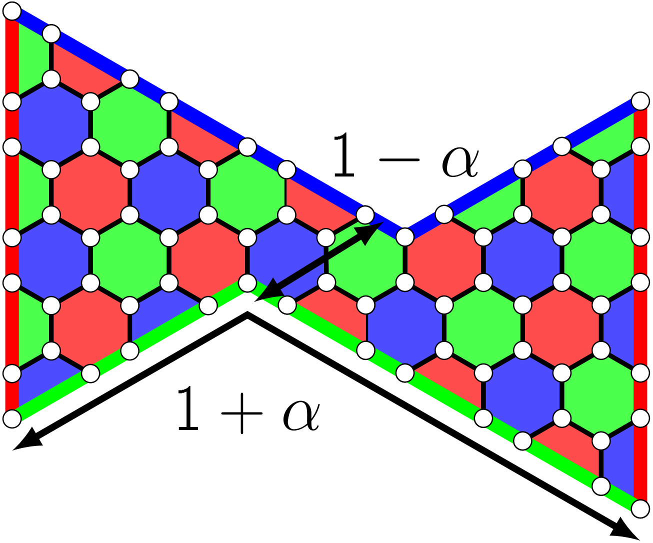
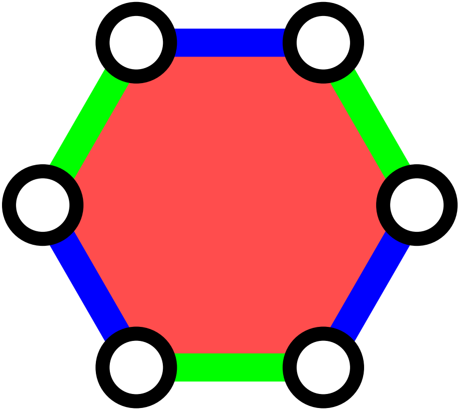


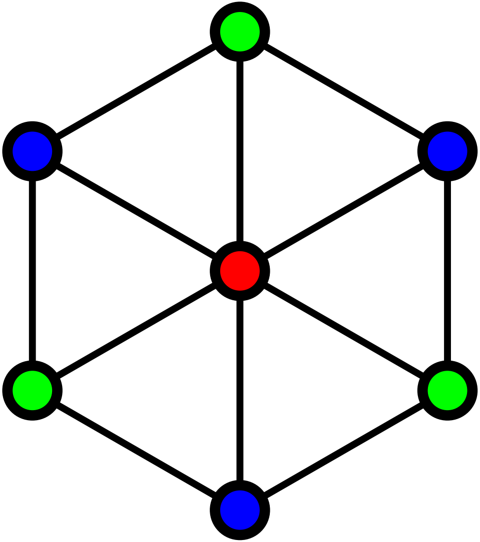
Upper level stabilizers

Stabilizers are product operators on some qubits:



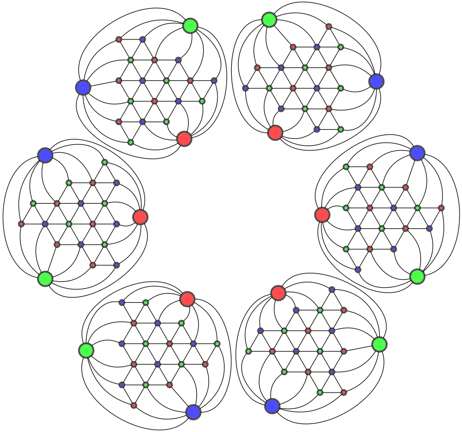
Upper level stabilizers

We can have a look at the dual view:



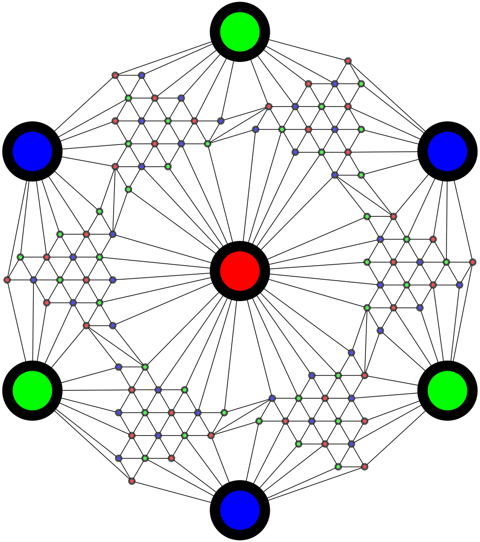
Upper level stabilizers

We can then replace the physical qubits by logical ones:



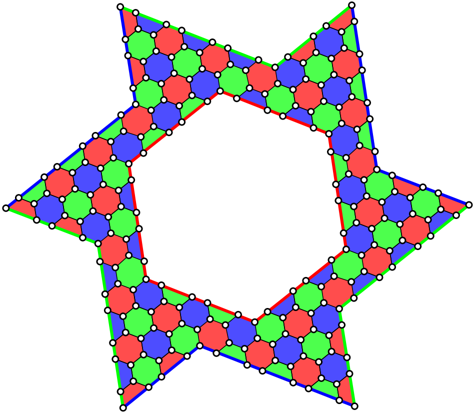
Upper level stabilizers

We can apply the merge procedure to neighboring qubits:



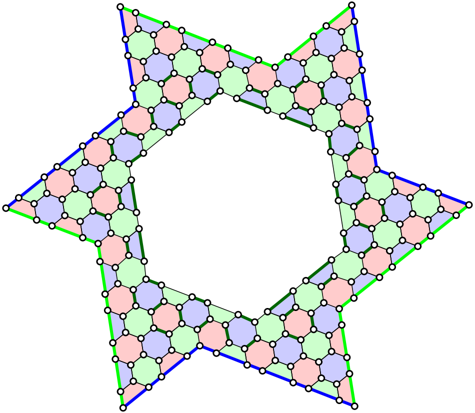
Upper level stabilizers

In the primal view:



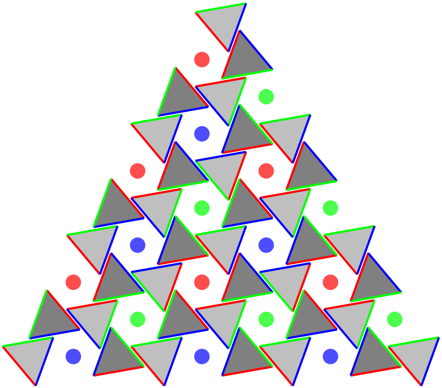
Upper level stabilizers

The large stabilizer can me measured by measuring colored edges in its surroundings:



Topologically concatenated color codes

Repeating the process around all the stabilizers with all qubits considered:



Characteristics of concatenated codes

Theoretical expectations

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$$[\![n_1 n_0, 1, f(\alpha, d_0, d_1)]\!]$$

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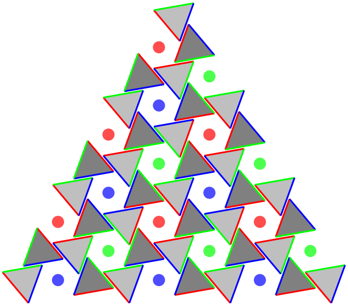

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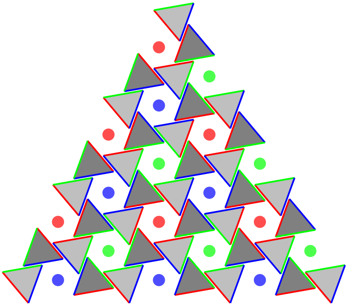


$$d_0 \left((d_1 - 1) \frac{\alpha + 1}{2} + 1 \right)$$

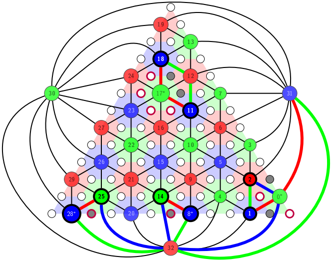
Distance evaluation

Lattice type	$d_0 = d_1$	α	Concatenated distance	n	Qubit gain
6-6-6	11	$\frac{3}{11}$	81	8281	+68%
6-6-6	11	$\frac{7}{11}$	101	8281	+8%
6-6-6	11	$\frac{9}{11}$	111	8281	-11%
6-6-6	111	$\frac{81}{111}$	10671	85.10^6	+0%
6-6-6	111	$\frac{91}{111}$	11221	85.10^6	-10%
4-8-8	11	$\frac{3}{11}$	81	5041	+50%
4-8-8	11	$\frac{7}{11}$	101	5041	-3%
4-8-8	11	$\frac{9}{11}$	111	5041	-20%
4-8-8	111	$\frac{49}{111}$	10671	40.10^6	-1%
4-8-8	111	$\frac{91}{111}$	11221	40.10^6	-37%

Decoding the concatenated color codes

Triangular color code decoder

- A computationally efficient decoder for triangular color codes as been presented by [2]



[2] Chamberland, Triangular color codes on trivalent graphs with flag qubits (2020)
[3] https://github.com/networkx/networkx
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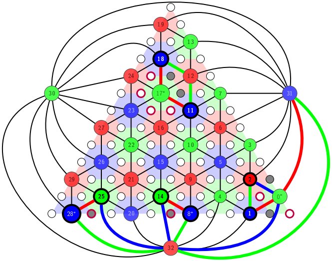
- Our Python implementation uses NetworkX[3] for graphs and PyMatching[4] for syndrome pairings.



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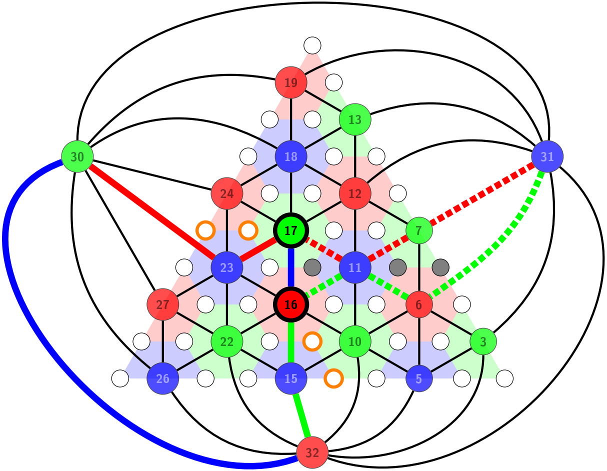
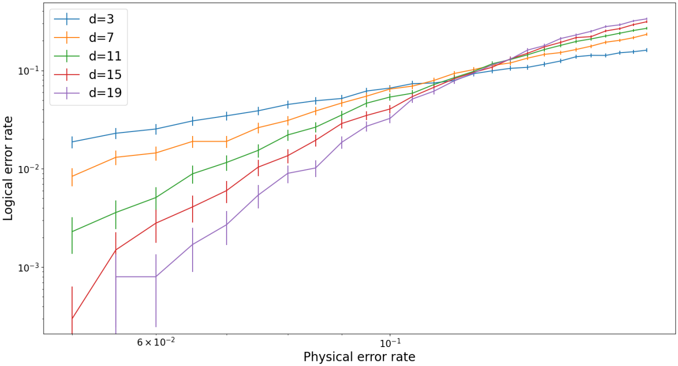
Decoder limitations

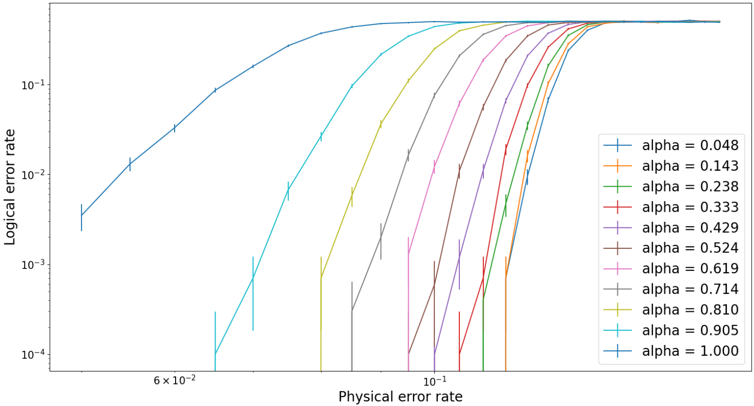


Logical error rate (non concatenated case)



Logical error rate (concatenated case)

Concatenation of a distance 21 code with itself (39601 qubits)



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- We might want to try decoding recursively

Further prospects

- We might want to try using a better decoder
- We might want to try decoding recursively
- We might want to try choosing a different geometry for the upper level code (toric geometry)