SIAM-AG23: 10 – 14 July 2023, Eindhoven, the Netherlands

3 Minisymposia for cryptographers:

- MS Applications of Algebraic Geometry to Post-Quantum Cryptology
- **MS53, 66, 80 Elliptic Curves and Pairings in Cryptography**
- MS92, 105, 118 Applications of Isogenies in Cryptography

Rank-one tensor completion (SIAGA 1, 2017)

`https://www.win.tue.nl/siam-ag23/index.html`
`https://meetings.siam.org/program.cfm?CONFCODE=AG23`

Practical info

Sessions labelled ECC on your A4-program at the back of your nametag

- Session 1: this session, Wednesday 10:30 – 12:30 Room Audi 1
  live stream at https://videocollege.tue.nl/Mediasite/Channel/siam-2023-event/watch/9f5674a21067410294 1f8614f5d2eba91d
- Session 2: this afternoon, Wednesday 14:00 – 16:00 Room Audi 1
  https://videocollege.tue.nl/Mediasite/Channel/siam-2023-event/watch/e7d29808e57f402a825b23107cf071bb1d
- Session 3: tomorrow, Thursday 10:30 – 12:30 Room Audi 1
  https://videocollege.tue.nl/Mediasite/Channel/siam-2023-event/watch/76dd943096194e8c8391bdac686cd8f91d

Elliptic Curves in Cryptography

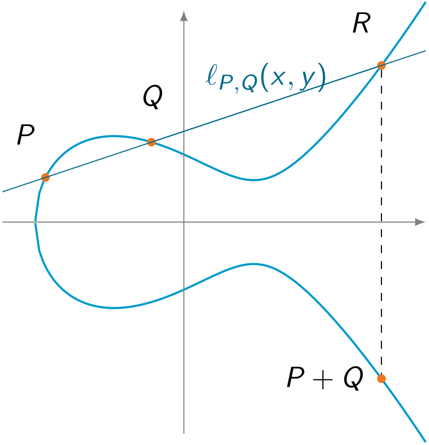
Elliptic curves introduced in 1985 by Miller, Koblitz
Perfect candidates to build a **generic group**
Many curves, usually over prime fields
of sparse binary expansion

- curve 25519 over $GF(2^{255} - 19)$
- NIST-P curves
- Four$\mathbb{Q}$ over $GF(p^2)$, $p = 2^{127} - 1$ Mersenne
- BLS12-381 for pairings ...

Basic crypto operations

Exponentation in a group $\mathbb{G}$ becomes
scalar multiplication

$$m, G \mapsto [m]G = \underbrace{G + G + \ldots + G}_{m \text{ times}}$$



Elliptic curve in Montgomery form and 2-torsion

Curve25519 : $y^2 = x^3 + \underbrace{486662}_{A} x^2 + x$ over GF(p), $p = 2^{255} - 19$

order $\#E(\mathbb{F}_p) = 8r$, 253-bit prime r

2-torsion points $= \{P \in E,\ 2P = \mathcal{O} \iff y_P = 0\}$

- 2-torsion over $\mathbb{F}_p$: $\{\mathcal{O}, (0, 0)\}$
- full 2-torsion over $\mathbb{F}_{p^2}$: $\{\mathcal{O}, (0, 0), (\lambda, 0), (\mu, 0)\}$, $x^2 + Ax + 1 = (x - \lambda)(x - \mu)$

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For an integer ℓ, the ℓ-torsion $E[\ell]$ has order ℓ^2

- $\#E[2] = 4 \subset E(\mathbb{F}_{p^2})$
- $\#E[4] = 16 \subset E(\mathbb{F}_{p^2})$
- $\#E[8] = 64 \subset E(\mathbb{F}_{p^2})$
- $\#E[r] = r^2 \subset E(\mathbb{F}_{p^k})$, $k = (r-1)/6$ of 250 bits for Curve25519

Group operations on curves for crypto

Subgoup membership testing

For curves E over $\mathbb{F}_p$ with a **cofactor** $\#E(\mathbb{F}_p) = h \cdot r$

- Pairings as a new tool by **Dimitri Koshelev** next talk
- Large cofactors for pairings, **Dai Yu**'s talk Session 2

Hashing to a point on the curve

Elligator for curves with $h = 2, 4$,
Wahby–Boneh on the BLS-381 curve and all j-invariant 0 and $p = 1 \bmod 3$ with a $\mathbb{F}_p$-rational small-degree isogeny

- new results by **Jorge Chavez-Saab** just after
- new results by **Dimitri Koshelev** (not in this MS)

Bilinear pairing in cryptography

As a black-box:

$(\mathbb{G}_1, +), (\mathbb{G}_2, +), (\mathbb{G}_T, \cdot)$ three cyclic groups of large prime order r

Bilinear pairing: map $e : \mathbb{G}_1 \times \mathbb{G}_2 \to \mathbb{G}_T$

1. bilinear: $e(P_1 + P_2,\ Q) = e(P_1, Q) \cdot e(P_2, Q)$, $e(P, Q_1 + Q_2) = e(P, Q_1) \cdot e(P, Q_2)$
2. non-degenerate: $e(G_1, G_2) \neq 1$ for $\langle G_1 \rangle = \mathbb{G}_1$, $\langle G_2 \rangle = \mathbb{G}_2$
3. efficiently computable

Mostly used in practice:

$$e([a]P, [b]Q) = e([b]P, [a]Q) = e(P, Q)^{ab}$$

Examples of applications

- 1984: idea of identity-based encryption (IBE) by Shamir
- 1999: first practical identity-based cryptosystem of Sakai-Ohgishi-Kasahara
- 2000: constructive pairings, Joux's tri-partite key-exchange
- 2001: IBE of Boneh-Franklin, short signatures Boneh-Lynn-Shacham

...

- broadcast encryption, re-keying
- aggregate signatures
- attribute-based encryption
  This afternoon, M. Venema, pairing-based ABE
- zero-knowledge (ZK) proofs, non-interactive ZK proofs (NIZK)
  this afternoon, Y. El Housni, pairings in the context of zk-SNARKs
  tomorrow, M. Bellés Muñoz, the quest to finding curves for zk-SNARKs
- tool in isogeny-based post-quantum cryptography, different setting
  (in the other minisymposia, e.g. Giulio's talks MS14)

Bilinear pairings

Security relies on

- Discrete Log Problem (DLP):

$$\text{given } g, h \in \mathbb{G}, \text{ compute } x \text{ s.t. } g^x = h$$

- Diffie-Hellman Problem (DHP):

$$\text{given } g, g^a, g^b \in \mathbb{G}, \text{ compute } g^{ab}$$

- bilinear DLP and DHP
- pairing inversion problem

Pairing-friendly curves should be designed on purpose

In cryptographic setting: $E[r]$ has structure $\mathbb{Z}_r \times \mathbb{Z}_r$ denoted $\mathbb{G}_1 \times \mathbb{G}_2$
(remember the 2-torsion points on Curve25519)

128-, resp. 192-bit security level:

- r large prime $\sim$ 256, resp. 384 bits
- $\#E(\mathbb{F}_p) = h \cdot r$, h small **cofactor**, $\mathbb{G}_1 = E(\mathbb{F}_p)[r]$
- $E[r] \subset E(\mathbb{F}_{p^k})$ and $1 \leq k \leq 54$, $\mathbb{G}_2 \subset E(\mathbb{F}_{p^k})[r]$
  k embedding degree
- $\mathbb{G}_T \subset \mathbb{F}_{p^k}^*$ multiplicative subgroup of order r

Usually $\log k \sim \log r$ (Balasubramanian Koblitz [BK98]).
Plain curves (25519, NIST curves) are never pairing-friendly

Finding pairing-friendly curves

Cocks–Pinch method:
Repeat

1. Start from the subgroup prime order r
2. Choose an embedding degree k and check $r \equiv 1 \bmod k$
3. Set $z \equiv \zeta_k \bmod r$
   (take z at random, repeat $z \mapsto z^{(r-1)/k}$ until $\Phi_k(z) = 1 \bmod r$, $z \neq 1$)
4. Set $t = z + 1$ and lift in $\mathbb{Z}$
5. Set $y = (t-2)/\sqrt{-D} \bmod r$ and lift in $\mathbb{Z}$
6. Set $p = (t^2 + Dy^2)/4$

until $p \in \mathbb{Z}$ and p is prime
Variant: lift $t + h_t \cdot r$, $y + h_y \cdot r$ with small h_t, h_y
Drawback: large cofactor $h \approx r$

Pairing-friendly curves are special

1st ones were *supersingular*, again used in post-quantum crypto.

Ordinary curves:

- 2001: Miyaji–Nakabayashi–Takano curves, $k \in \{3, 4, 6\}$, prime order [MNT01]
- Cocks-Pinch technique
- Barreto–Lynn–Scott curves, $3 \mid k$, $18 \nmid k$ [BLS03]
- Brezing–Weng construction [BW05]
- Freeman $k = 10$ [Fre06], Barreto–Naehrig curves $k = 12$, prime order [BN06]
- Kachisa–Schaefer–Scott curves, $k \in \{8, 16, 18, 32, 36, 40\}$ [KSS08]
- Freeman–Scott–Teske Taxonomy [FST10]
- Scott–Guillevic, $k = 54$ [SG18]
- Gasnier–Guillevic, $k = 20, 22$ (**J. Gasnier, tomorrow**)

Why Barreto–Naehrig'2005 curves were so popular?

$k = 12,\ j = 0,\ D = -3,$

$$E\colon y^2 = x^3 + b$$

$$p(x) = 36x^4 + 36x^3 + 24x^2 + 6x + 1$$
$$r(x) = 36x^4 + 36x^3 + 18x^2 + 6x + 1$$

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$$\left.\begin{array}{l} x_0 = 2^{62} - 2^{54} + 2^{44} \text{ [NAS}^{+}\text{08] (Nogami et al.)} \\ x_0 = -(2^{62} + 2^{55} + 1) \text{ [PSNB11] (Pereira et al.)} \\ x_0 = \texttt{0x44e992b44a6909f1} \text{ in Ethereum, s.t. } 2^{28} \mid r - 1 \end{array}\right\} \begin{array}{l} \#E(\mathbb{F}_p) = r \text{ prime order} \\ r \text{ of 254 bits} \end{array}$$

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$\mathbb{G}_T \subset \mathbb{F}_{p^{12}}$ of $12 \log p \approx 3048$ bits

≈ 3072 bits expected to offer 128 bits of security for RSA and D-Log in the 2000's

- optimal parameter size, optimal $k = 12$
- prime order: no cofactor clearing, no subgroup membership testing
- $\mathbb{F}_{p^{12}}$ towering easier to implement with Karatsuba

$\implies$ BN curves were the perfect match

Choosing pairing-friendly curves

Pairing-based cryptography needs **secure, efficient, compact** pairing-friendly curves

- secure against discrete log in $E(\mathbb{F}_p)$, $E(\mathbb{F}_{p^k})$, $\mathbb{F}_{p^k}$
- efficient for scalar multiplication in E, exponentiation in $\mathbb{F}_{p^k}$, pairing
- compact: key sizes as small as possible

Which curves are the best options?

Pairing-based cryptography

Weil or Tate pairing on an elliptic curve

Discrete logarithm problem with one more dimension

$$e\colon E(\mathbb{F}_p)[r] \times \mathbb{G}_2 \longrightarrow \mathbb{G}_T \subset \mathbb{F}_{p^k}^*, \quad e([a]P,[b]Q) = e(P,Q)^{ab}$$

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Attacks

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- inversion of e : hard problem (exponential)

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- discrete logarithm computation in $E(\mathbb{F}_p)$: hard problem (exponential, in $O(\sqrt{r})$)

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Discrete logarithm problem with one more dimension

$$e\colon E(\mathbb{F}_p)[r] \times \mathbb{G}_2 \longrightarrow \mathbb{G}_T \subset \mathbb{F}^*_{p^k}, \quad e([a]P,[b]Q) = e(P,Q)^{ab}$$

Attacks

- inversion of e : hard problem (exponential)
- discrete logarithm computation in $E(\mathbb{F}_p)$: hard problem (exponential, in $O(\sqrt{r})$)
- discrete logarithm computation in $\mathbb{F}^*_{p^k}$: **easier, subexponential** $\to$ take a large enough field

Discrete Log in $\mathbb{F}_{p^k}$

$\mathbb{F}_{p^k}$ much less investigated than $\mathbb{F}_p$ or integer factorization
Much better results in pairing-related fields

- Special NFS in $\mathbb{F}_{p^k}$: Joux–Pierrot 2013 [JP14]
- Tower NFS (TNFS): Barbulescu–Gaudry–Kleinjung 2015 [BGK15]
- Extended Tower NFS: Kim–Barbulescu [KB16], Kim–Jeong [KJ17], Sarkar–Singh 2016 [SS16]

Use more structure: subfields

Choosing key sizes: Lenstra–Verheul [LV01] extrapolation

Initially for RSA modulus size

For DL in $\mathbb{F}_Q$ of length(Q) bits
n bits of security $\leftrightarrow$ the best (mathematical) attack should take at least 2^n steps

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- fastest Discrete Log computation: with the Number Field Sieve algorithm
- Complexity: $e^{\sqrt[3]{(64/9+o(1))(\ln Q)(\ln\ln Q)^2}}$
- $+o(1)$ not known

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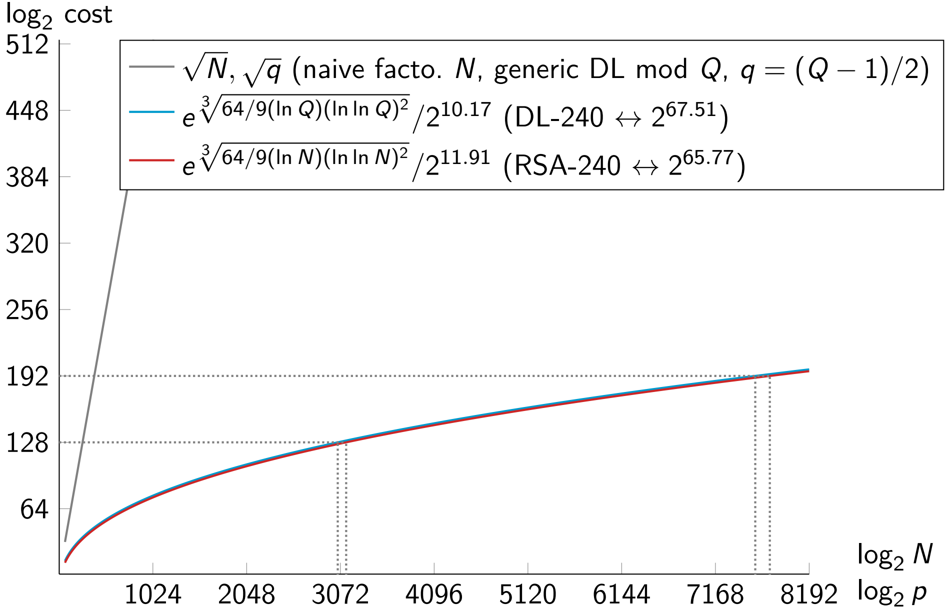
DL in prime field: Replace unknown $+o(1)$ by scaling factor $2^{-10.17}$



RSA-240: 953 core-years, Intel Xeon Gold 6130 CPUs as a reference (2.1GHz) $\approx 953 \cdot 365.25 \cdot 24 \cdot 60 \cdot 60 \cdot 2.1 \cdot 10^9 \approx 2^{65.77}$

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Estimating key sizes for DL in $\mathbb{F}_{p^k}$

- Latest variants of TNFS (Kim–Barbulescu, Kim–Jeong) seem most promising for $\mathbb{F}_{p^k}$ where k is composite
- The asymptotic complexities do not correspond to a fixed k, but to a ratio between k and p
- We need record computations if we want to extrapolate from asymptotic complexities

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Discrete logarithm in GF(p^6) with Tower-NFS [DGP21]

- $Q = p^6$ of 521 bits, total time 24798 core-hours (2.83 core-years) $\leftrightarrow 2^{57.37}$
- Tower-NFS-Conjugation $e^{\sqrt[3]{(48/9+o(1))(\ln Q)(\ln \ln Q)^2}}$
- $e^{\sqrt[3]{(48/9+0)(\ln Q_{\text{DL-521}})(\ln \ln Q_{\text{DL-521}})^2}} = 2^{58.52}$

DL in non-special $\mathbb{F}_{p^6}$ field: too early to apply Lenstra–Verheul extrapolation

Largest record computations in $\mathbb{F}_{p^k}$ with NFS and its variants[1]

Finite field	Size of p^k	Cost: CPU days	Authors	sieving dim
		Tower-NFS		
$\mathbb{F}_{p^6}$	521	1,033	[DGP21] De Micheli et al.'21	6, Tower
$\mathbb{F}_{p^4}$	512	2244	[Rob22] Robinson'22	4, Tower
		NFS and NFS-HD		
$\mathbb{F}_{p^{12}}$	203	11	[HAKT13, HAKT15]	7
$\mathbb{F}_{p^6}$	423	3,400	[MR20]	3
$\mathbb{F}_{p^5}$	324	386	[GGM17]	3
$\mathbb{F}_{p^4}$	392	510	[BGGM15a]	2
$\mathbb{F}_{p^3}$	593	8,400	[GGM16, GMT16]	2
$\mathbb{F}_{p^2}$	595	175	[BGGM15b]	2
$\mathbb{F}_p$	768	1,935,825	[KDLPS17]	2
$\mathbb{F}_p$	795	1,132,275	[BGGHTZ19]	2

[1] Data extracted from DiscreteLogDB by L.Grémy

Complexities $L_{p^k}(\alpha, c) = \exp\Big((c + o(1))(\ln p^k)^{\alpha}(\ln\ln p^k)^{1-\alpha}\Big)$

large characteristic $p = L_{p^k}(\alpha_p),\ \alpha_p > 2/3$: $L_{p^k}(1/3, c)$	
$c = (64/9)^{1/3} \simeq 1.923$	NFS
special p:	
$c = (32/9)^{1/3} \simeq 1.526$	SNFS

medium characteristic $p = L_{p^k}(\alpha_p),\ 1/3 < \alpha_p < 2/3$: $L_{p^k}(1/3, c)$	
$c = (96/9)^{1/3} \simeq 2.201$	prime n NFS-HD (Conjugation)
$c = (48/9)^{1/3} \simeq 1.747$	composite n, best case of TNFS: when parameters fit perfectly
special p:	
$c = (64/9)^{1/3} \simeq 1.923$	NFS-HD+Joux–Pierrot'13
$c = (32/9)^{1/3} \simeq 1.526$	composite n, best case of STNFS

A short-list of pairing-friendly curves at the 128-bit sec level

Webpage at
`https://members.loria.fr/AGuillevic/pairing-friendly-curves/`

k	curve	seed	$\log_2 Q$	$\log_2 r$	ρ	bit sec. $GF(p^k)$
		Curves with fast pairing				
12	BN-382	$-(2^{94}+2^{78}+2^{67}+2^{64}+2^{48}+1)$	382	382	1.0	123
12	BN-446	$2^{110}+2^{36}+1$	446	446	1.0	132
12	BLS12-381	$-(2^{63}+2^{62}+2^{60}+2^{57}+2^{48}+2^{16})$	381	254	1.5	126
12	BLS12	see gitlab	440–448	295–300	1.5	132
		Curves with smallest possible $\mathbb{G}_1$ [CDS20]				
13	`BW13-P310`	`-0x8b0`=-2224	310	267	1.167	140
19	`BW19-P286`	`-0x91`=-145	286	259	1.111	160
		Curves for SNARK $2^L \mid p-1,\ r-1$				
12	BLS12-377	$2^{63}+2^{58}+2^{56}+2^{51}+2^{47}+2^{46}+1$	377	252	1.5	126
24	BLS24-315	$-2^{32}+2^{30}+2^{22}-2^{20}+1$	315	253	1.25	160

Generating new families, choosing curves

- Cycles of curves for SNARKs, **Marta Bellés Muñoz** tomorrow
- New families of paring-friendly curves **Jean Gasnier** tomorrow
- Fastest pairing-friendly curves at the 192-bit security level **Georgios Fotiadis** tomorrow

Pairing computation

$e(P, Q)$: Miller loop + final exponentiation to $(p^k - 1)/r$

Miller loop: evaluate a function $f_{m,P}$ at point Q [Jou04, Ver10]
Contains a scalar multiplication

$$[m]P \text{ where } \log_2 m \approx \frac{\log_2 r}{\varphi(k)} = \frac{log_2 r}{\deg \Phi_k}$$

Φ_k the k-th cyclotomic polynomial
SageMath: `euler_phi(k)`
$\varphi(12) = 4$, $\varphi(16) = 8$, $\varphi(18) = 6$, $\varphi(20) = 8$, $\varphi(24) = 8$

At fixed k, reducing r gives a faster Miller loop

Pairing: Miller loop and final exponentiation

Algorithm 1.1: $\textsc{MillerFunction}(u, P, Q)$

Input: $E, \mathbb{F}_p, \mathbb{F}_{p^k}$, k even, $P \in E(\mathbb{F}_p)[r]$, $Q \in E(\mathbb{F}_{p^k})[r]$ in affine coord., $\pi_p(Q) = [p]Q$, $c \in \mathbb{N}$.

Result: $f = f_{c,Q}(P)$

$f \leftarrow 1$; $R \leftarrow Q$;

for *b from the second most significant bit of c to the least* **do**

  $\ell_0 \leftarrow \ell_{R,R}(P)$; $R \leftarrow [2]R$; `// Dbl step, tangent line`

  $f \leftarrow f^2$; `//` $\mathbf{s}_k$

  if $b = 1$ **then**

    $\ell_1 \leftarrow \ell_{R,Q}(P)$; $R \leftarrow R + Q$; `// Add step, chord line`

    $f \leftarrow f \cdot (\ell_0 \cdot \ell_1)$; `//` $\mathbf{m}_k +$ `sparse-sparse-`$\mathbf{m}_k$

  else

    $f \leftarrow f \cdot \ell_0$; `// full-sparse-`$\mathbf{m}_k$

return f;

Pairing: Miller loop and final exponentiation

Raise to

$$\frac{p^k - 1}{r} = \underbrace{\frac{p^k - 1}{\Phi_k(p)}}_{\text{easy}} \underbrace{\frac{\phi_k(p)}{r}}_{\text{hard}}$$

- More on pairing computation by **Mike Scott** this afternoon
- Pairing computation on BLS curves of odd k by **Laurian Azebaze Guimagang** this session
- Even shorter Miller loop by **Emmanuel Fouotsa** tomorrow
- Pairings inside circuits by **Youssef El Housni** this afternoon

Bibliography I

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