Understanding the special tower number field sieve algorithm and applications to pairing-based cryptography

Aurore Guillevic

Inria Nancy, France

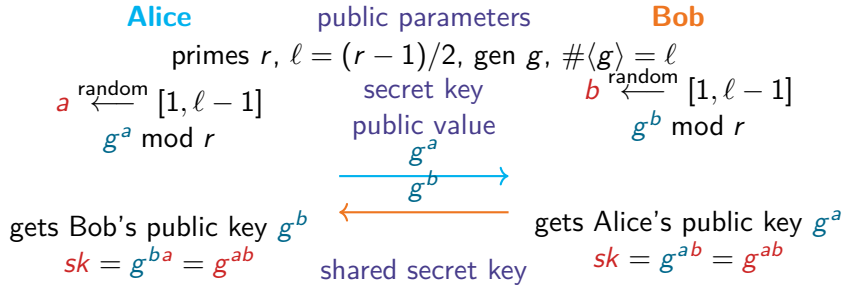
Université de Toulon, April 20, 2021

Joint work with Shashank Singh, IISER Bhopal, India

Outline

Outline

Diffie–Hellman key exchange, discrete logarithm problem



it works because $(g^a)^b = (g^b)^a = g^{ab}$

Diffie-Hellman Problem

Given $\mathbf{G} = \langle g\rangle$, g, g^a, g^b, computes g^{ab}.

Discrete Logarithm Problem

Given $\mathbf{G} = \langle g\rangle$, g, g^a, computes a.

Choosing key sizes

Symmetric ciphers (AES): key sizes are 128, 192 or 256 bits.
Perfect symmetric cipher: trying all keys of size n bits takes 2^n tests
$\rightarrow$ **brute-force search**

perfect symmetric cipher with secret key in $[0, 2^n - 1]$, of n bits $\leftrightarrow$ n bits of security

For DL-based key exchange with p, ℓ of length(p), length(ℓ) bits:
n bits of security $\leftrightarrow$ the best (mathematical) attack should take at least 2^n steps

- what is the fastest attack?
- how much time does it take with respect to length(p), length(ℓ)?

RSA and Diffie–Hellman keys are much larger.
Cipher suite: a pair of symmetric and asymmetric ciphers offering the same level of security.

Outline

Discrete logarithm problem

G multiplicative group of order ℓ
g generator, $\mathbf{G} = \{1, g, g^2, g^3, \ldots, g^{\ell-2}, g^{\ell-1}\}$

Given $h \in \mathbf{G}$, find integer $x \in \{0, 1, \ldots, \ell - 1\}$ such that $h = g^x$.
Exponentiation easy: $(g, x) \mapsto g^x$
Discrete logarithm hard in well-chosen groups **G**
Common choices of **G**:

- prime finite field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ (1976)
- characteristic 2 field $\mathbb{F}_{2^n}$ ($\approx$ 1979)
- elliptic curve $E(\mathbb{F}_p)$ (1985)

Discrete log problem

How fast can we invert the exponentiation function $(g, x) \mapsto g^x$?

- $g \in G$ generator, exists always a preimage $x \in \{1, \ldots, \#G\}$
- naive search, try them all: $\#G$ tests
- $O(\sqrt{\#G})$ generic algorithms
- independent search in each distinct subgroup
  + Chinese remainder theorem (Pohlig-Hellman)

$\rightarrow$ choose G of large prime order (no subgroup)

$\rightarrow$ complexity of inverting exponentiation in $O(\sqrt{\#G})$

$\rightarrow$ security level 128 bits means $\sqrt{\#G} \geq 2^{128}$
take $\#G = 2^{256}$
analogy with symmetric crypto, keylength 128 bits (16 bytes)

Discrete log problem

How fast can we invert the exponentiation function $(g, x) \mapsto g^x$?

- $g \in G$ generator, exists always a preimage $x \in \{1, \ldots, \#G\}$
- naive search, try them all: $\#G$ tests
- $O(\sqrt{\#G})$ generic algorithms
- independent search in each distinct subgroup
  + Chinese remainder theorem (Pohlig-Hellman)

$\rightarrow$ choose G of large prime order (no subgroup)

$\rightarrow$ complexity of inverting exponentiation in $O(\sqrt{\#G})$

$\rightarrow$ security level 128 bits means $\sqrt{\#G} \geq 2^{128}$
take $\#G = 2^{256}$
analogy with symmetric crypto, keylength 128 bits (16 bytes)

Use additional structure of G if any.

Discrete log problem when $\mathbf{G} = (\mathbb{Z}/p\mathbb{Z})^*$

Index calculus algorithm [Western–Miller 68, Adleman 79],
prequel of the Number Field Sieve algorithm (NFS)

- p prime, $(p-1)/2$ prime, $\mathbf{G} = (\mathbb{Z}/p\mathbb{Z})^*$, gen. g, target h
- get many multiplicative relations in $\mathbf{G}$
- get one multiplicative relation involving the target h
- take logarithms: linear relations *in the exponents*
- solve a linear system to get discrete logarithms
- get $x = \log h$

Index calculus in $(\mathbb{Z}/p\mathbb{Z})^*$: example

$p = 2039$, $\ell = (p-1)/2 = 1019$ prime

Smoothness bound $B = 17$

$\mathcal{F}_{17} = \{2, 3, 5, 7, 11, 13, 17\}$ small primes up to B, $\#\mathcal{F} = 7$

B-smooth integer: $n = 2^{e_1} 3^{e_2} \cdots 17^{e_7}$, $e_i \geq 0$

is g^s mod p smooth? $1 \leq s \leq 59$ is enough

$$
\begin{array}{lll}
g^{1} & \equiv & 7 = 7 \\
g^{5} & \equiv & 495 = 3^2 \cdot 5 \cdot 11 \\
g^{30} & \equiv & 204 = 2^2 \cdot 3 \cdot 17 \\
g^{44} & \equiv & 255 = 3 \cdot 5 \cdot 17 \\
g^{52} & \equiv & 1088 = 2^6 \cdot 17 \\
g^{57} & \equiv & 264 = 2^3 \cdot 3 \cdot 11 \\
g^{59} & \equiv & 702 = 2 \cdot 3^3 \cdot 13
\end{array}
\rightarrow
\begin{array}{c}
\begin{array}{ccccccc} 2 & 3 & 5 & 7 & 11 & 13 & 17 \end{array} \\
\begin{bmatrix}
 & & & 1 & & & \\
 & 2 & 1 & & 1 & & \\
2 & 1 & & & & & 1 \\
 & 1 & 1 & & & & 1 \\
6 & & & & & & 1 \\
3 & 1 & & & 1 & & \\
1 & 3 & & & & 1 &
\end{bmatrix}
\end{array}
\cdot \boldsymbol{x} =
\begin{bmatrix} 1 \\ 5 \\ 30 \\ 44 \\ 52 \\ 57 \\ 59 \end{bmatrix}
$$

$\boldsymbol{x} = [325, 259, 664, 1, 861, 995, 140]$ mod 1019

$7^{325} = -2, 7^{259} = -3, 7^{664} = 5, 7^{1} = 7, 7^{861} = 11, 7^{995} = 13, 7^{140} = 17$

Index calculus in $(\mathbb{Z}/p\mathbb{Z})^*$: example

$\boldsymbol{x} = [325, 259, 664, 1, 861, 995, 140] \bmod 1019$
$\boldsymbol{v} = [1344, 1278, 664, 1, 861, 995, 140] \bmod p - 1 = 2038$
$7^{1344} = 2, 7^{1278} = 3, 7^{664} = 5, 7^1 = 7, 7^{861} = 11, 7^{995} = 13, 7^{140} = 17$
Target $h = 314$, $g^7 \cdot 314 = 405 = 3^4 \cdot 5 \bmod p$
$\log_7 314 = -7 + 4 \log_7 3 + \log_7 5 = 1693 \bmod p - 1$
$7^{1693} = 314 \bmod p$

Index calculus in $(\mathbb{Z}/p\mathbb{Z})^*$

Multiplicative relations over the **integers**

Smooth integers $n = p_1^{e_1} p_2^{e_2} \cdots p_i^{e_i}$, $p_i \leq B$ are quite common $\rightarrow$ it works

Complexity $e^{\sqrt{(2+o(1))(\log p)(\log\log p)}}$ (Pomerance 87)

Index calculus in $(\mathbb{Z}/p\mathbb{Z})^*$

Multiplicative relations over the **integers**

Smooth integers $n = p_1^{e_1} p_2^{e_2} \cdots p_i^{e_i}$, $p_i \leq B$ are quite common $\rightarrow$ it works

Complexity $e^{\sqrt{(2+o(1))(\log p)(\log\log p)}}$ (Pomerance 87)

Improvements in the 80's, 90's:

- Sieve (faster relation collection)
- Smaller integers to factor
- Multiplicative relations in **number fields**
- Better **sparse linear algebra**
- Independent targets h

Where is the Number Field?

1985: ElGamal, DL in GF(p^2) with two quadratic number fields
1986: Coppersmith–Odlyzko–Schroeppel, DL algorithm in GF(p)
1995: Weber–Denny, record computation 85 dd with $\mathbb{Q}[\sqrt{-2}]$

Where is the Number Field?

1985: ElGamal, DL in GF(p^2) with two quadratic number fields
1986: Coppersmith–Odlyzko–Schroeppel, DL algorithm in GF(p)
1995: Weber–Denny, record computation 85 dd with $\mathbb{Q}[\sqrt{-2}]$

- If $p = 1 \bmod 4$, exists u, v s.t. $p = u^2 + v^2$, $\theta = \sqrt{-1}$
- If $p = 3 \bmod 8$, exists u, v s.t. $p = u^2 + 2v^2$, $\theta = \sqrt{-2}$
- If $p = 7 \bmod 8$, exists u, v s.t. $p = u^2 - 2v^2$, $\theta = \sqrt{2}$

and $|u|, |v| < \sqrt{p}$
$u/v \equiv m \mod p$ and $m^2 + s = 0 \mod p$

Where is the Number Field?

1985: ElGamal, DL in GF(p^2) with two quadratic number fields
1986: Coppersmith–Odlyzko–Schroeppel, DL algorithm in GF(p)
1995: Weber–Denny, record computation 85 dd with $\mathbb{Q}[\sqrt{-2}]$

- If $p = 1 \bmod 4$, exists u, v s.t. $p = u^2 + v^2$, $\theta = \sqrt{-1}$
- If $p = 3 \bmod 8$, exists u, v s.t. $p = u^2 + 2v^2$, $\theta = \sqrt{-2}$
- If $p = 7 \bmod 8$, exists u, v s.t. $p = u^2 - 2v^2$, $\theta = \sqrt{2}$

and $|u|, |v| < \sqrt{p}$
$u/v \equiv m \mod p$ and $m^2 + s = 0 \mod p$

Define a map from $\mathbb{Z}[\theta]$ to $\mathbb{Z}/p\mathbb{Z}$

$$\begin{aligned} \phi\colon \mathbb{Z}[\theta] &\to \mathbb{Z}/p\mathbb{Z} \\ \theta &\mapsto m \bmod p \text{ where } m = u/v,\ m^2 + s = 0 \bmod p \end{aligned}$$

ring homomorphism $\phi(a + b\theta) = a + bm$

$$\phi(\underbrace{a + b\theta}_{\substack{\text{factor in} \\ \mathbb{Z}[\theta]}}) = a + bm = (a + b\ \underbrace{u/v}_{=m}) = (\underbrace{av + bu}_{\text{factor in } \mathbb{Z}})v^{-1} \mod p$$

Example in $\mathbb{Z}[\sqrt{2}]$

$p = 2039 = 7 \bmod 8$, $\ell = (p-1)/2 = 1019$ prime

$p = 3^2 - 2 \cdot 32^2$

$\max(|a|, |b|) = A = 13$, $B = 13$ smoothness bound

Example in $\mathbb{Z}[\sqrt{2}]$

$p = 2039 = 7 \bmod 8$, $\ell = (p-1)/2 = 1019$ prime

$p = 3^2 - 2 \cdot 32^2$

$\max(|a|, |b|) = A = 13$, $B = 13$ smoothness bound

Rational side

$\mathcal{F}_0 = \{2, 3, 5, 7, 11, 13\}$ small primes up to 13, $\#\mathcal{F}_0 = 6$

$g(x) = 32x - 3$

Example in $\mathbb{Z}[\sqrt{2}]$

$p = 2039 = 7 \bmod 8$, $\ell = (p-1)/2 = 1019$ prime
$p = 3^2 - 2 \cdot 32^2$
$\max(|a|, |b|) = A = 13$, $B = 13$ smoothness bound

Rational side

$\mathcal{F}_0 = \{2, 3, 5, 7, 11, 13\}$ small primes up to 13, $\#\mathcal{F}_0 = 6$
$g(x) = 32x - 3$

Algebraic side: think about the complex numbers in $\mathbb{C}$

$-i(1+i)^2 = 2$, $(2+i)(2-i) = 5$, $(2+3i)(2-3i) = 13$
$2 = (\theta)^2$, $7 = (\theta+3)(\theta-3)$
$\mathcal{F}_1 = \{(\theta), (\theta+3), (\theta-3)\}$ small prime ideals above 2,7, $\#\mathcal{F}_1 = 3$
"primes" of norm up to B
$f(x) = x^2 - 2$

Example in $\mathbb{Z}[\sqrt{2}]$

$p = 2039 = 7 \bmod 8$, $\ell = (p-1)/2 = 1019$ prime
$p = 3^2 - 2 \cdot 32^2$
$\max(|a|, |b|) = A = 13$, $B = 13$ smoothness bound

Rational side

$\mathcal{F}_0 = \{2, 3, 5, 7, 11, 13\}$ small primes up to 13, $\#\mathcal{F}_0 = 6$
$g(x) = 32x - 3$

Algebraic side: think about the complex numbers in $\mathbb{C}$

$-i(1+i)^2 = 2$, $(2+i)(2-i) = 5$, $(2+3i)(2-3i) = 13$
$2 = (\theta)^2$, $7 = (\theta+3)(\theta-3)$
$\mathcal{F}_1 = \{(\theta), (\theta+3), (\theta-3)\}$ small prime ideals above 2,7, $\#\mathcal{F}_1 = 3$
"primes" of norm up to B
$f(x) = x^2 - 2$

Units

$\mathcal{U}_1 = \{(\theta+1)\}$ fundamental unit

Example in $\mathbb{Z}[\sqrt{-2}]$

a, b	$av + bu =$ factor in $\mathbb{Z}$	$a^2 + 2b^2$	factor in $\mathbb{Z}[\sqrt{-2}]$	units
$-9, 4$	$-300 = -2^2 \cdot 3 \cdot 5^2$	$49 = 7^2$	$(\theta - 3)^2$	$-(\theta + 1)^2$
$-5, 3$	$-169 = -13^2$	$7 = 7$	$(\theta - 3)$	$(\theta + 1)^2$
$-3, 1$	$-99 = -3^2 \cdot 11$	$7 = 7$	$(\theta + 3)$	-1
$-1, 0$	$-32 = -2^5$	$1 =$		-1
$-1, 1$	$-35 = -5 \cdot 7$	$-1 =$		$-(\theta + 1)$
$0, 1$	$-3 = -3$	$-2 = -2$	(θ)	-1
$1, 0$	$32 = 2^5$	$1 =$		1
$1, 2$	$26 = 2 \cdot 13$	$-7 = -7$	$(\theta + 3)$	$-(\theta + 1)^{-1}$
$2, 3$	$55 = 5 \cdot 11$	$-14 = -2 \cdot 7$	$(\theta)(\theta - 3)$	1
$3, 2$	$90 = 2 \cdot 3^2 \cdot 5$	$1 =$		$(\theta + 1)^{-2}$
$4, 1$	$125 = 5^3$	$14 = 2 \cdot 7$	$(\theta)(\theta + 3)$	$(\theta + 1)^{-1}$
$12, 11$	$351 = 3^3 \cdot 13$	$-98 = -2 \cdot 7^2$	$(\theta)(\theta - 3)^2$	-1

Example in $\mathbb{Z}[i]$: Matrix

Build the matrix of relations:

- one row per (a, b) pair s.t. both norms are smooth
- one column per prime of $\mathcal{F}_{\text{rat}}$
- one column for $1/V$
- one column per prime ideal of $\mathcal{F}_{\text{alg}}$
- one column per unit $(-1, i)$
- store the exponents

Example in $\mathbb{Z}[\sqrt{-2}]$

$$M = \begin{array}{c} \begin{array}{ccccccccccc} 2 & 3 & 5 & 7 & 11 & 13 & 1/v & \theta & \theta+3 & \theta-3 & \theta+1 \end{array} \\ \left[\begin{array}{ccccccccccc} 2 & 1 & 2 & & & & -1 & & & -2 & -2 \\ & & & & & 2 & -1 & & & -1 & -2 \\ & 2 & & & 1 & & -1 & & -1 & & \\ 5 & & & & & & -1 & & & & \\ & & 1 & 1 & & & -1 & & & & -1 \\ & 1 & & & & & -1 & -1 & & & \\ 5 & & & & & & -1 & & & & \\ 1 & & & & & 1 & -1 & & -1 & & 1 \\ & & 1 & & 1 & & -1 & -1 & & -1 & \\ 1 & 2 & 1 & & & & -1 & & & & 2 \\ & & 3 & & & & -1 & -1 & -1 & & 1 \\ & 3 & & & & 1 & -1 & -1 & & -2 & \end{array}\right] \end{array}$$

$\boldsymbol{x} = [1, 515, 140, 301, 335, 928, 5, 510, 341, 979, 436]$

$\boldsymbol{x}/301 = [325, 259, 664, 1, 861, 995, 606, 672, 773, 247, 59]$

We had before: $\boldsymbol{x} = [325, 259, 664, 1, 861, 995, 140] \bmod 1019$

Number Field Sieve

Since 1993 (Gordon, Schirokauer):

$$L_p(1/3, c) = e^{(c+o(1))(\log p)^{1/3}(\log\log p)^{2/3}}$$

- polynomial selection
- **relation collection** $L_p(1/3, 1.923)$
  sieve to enumerate efficiently (a, b) pairs
- **sparse linear algebra** $L_p(1/3, 1.923)$
  compute right kernel mod prime ℓ, block-Wiedemann alg.
- individual discrete logarithm

Latest record computation: 795-bit (240dd) $p = \text{RSA-240} + 49204$ prime,
$\ell = (p-1)/2$ prime
Boudot, Gaudry, Guillevic, Heninger, Thomé, Zimmermann, Crypto'2020

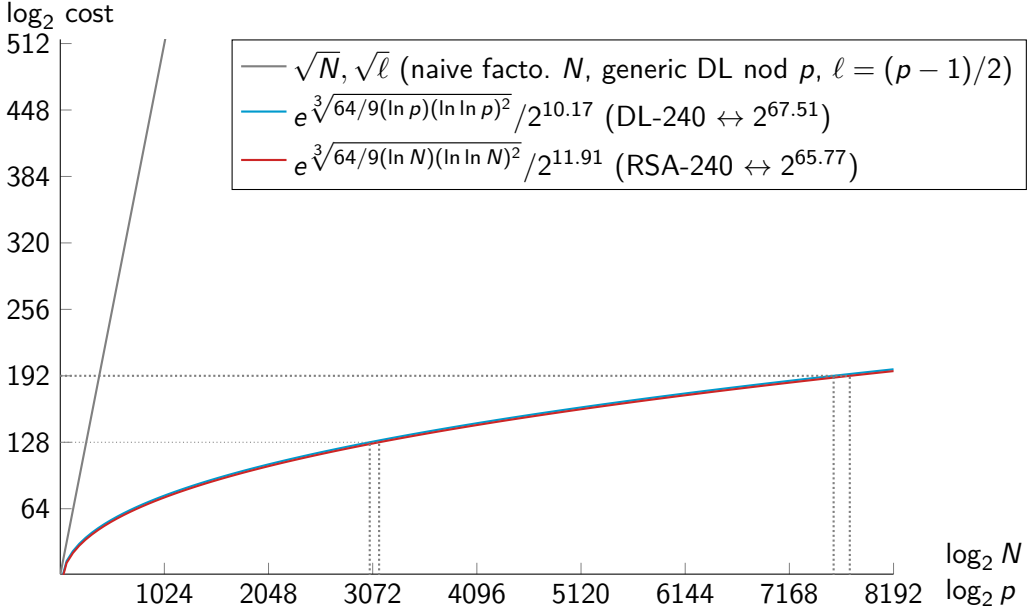
Variants of NFS: Complexities

large characteristic $p = L_{p^n}(\alpha),\ \alpha > 2/3$:

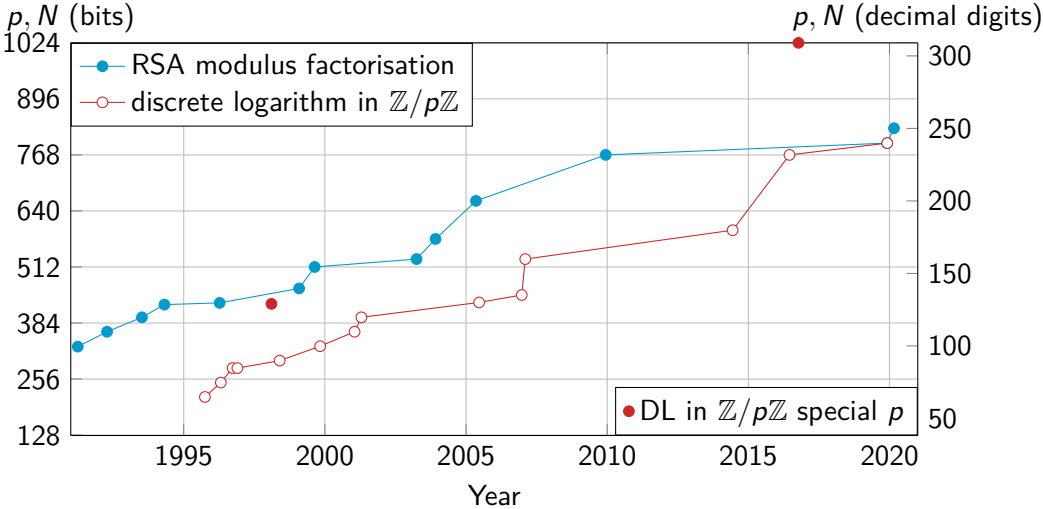
$(64/9)^{1/3} \simeq 1.923$	NFS
special p:	
$(32/9)^{1/3} \simeq 1.526$	SNFS

medium characteristic $p = L_{p^n}(\alpha),\ 1/3 < \alpha < 2/3$:

$(96/9)^{1/3} \simeq 2.201$	prime n NFS-HD (Conjugation)
$(48/9)^{1/3} \simeq 1.747$	composite n, best case of TNFS: when parameters fit perfectly
special p:	
$(64/9)^{1/3} \simeq 1.923$	NFS-HD+Joux–Pierrot'13
$(32/9)^{1/3} \simeq 1.526$	composite n, best case of STNFS


log2 cost
512
448
384
320
256
192
128
64
1024
2048
3072
4096
5120
6144
7168
8192
log2 N
log2 p
√N, √ℓ (naive facto. N, generic DL nod p, ℓ = (p − 1)/2)
e∛(64/9(ln p)(ln ln p)²)/2^10.17 (DL-240 ↔ 2^67.51)
e∛(64/9(ln N)(ln ln N)²)/2^11.91 (RSA-240 ↔ 2^65.77)

Record computations



Outline

Cryptographic pairing: black-box properties

$(\mathbf{G}_1, +), (\mathbf{G}_2, +), (\mathbf{G}_T, \cdot)$ three cyclic groups of large prime order r
Bilinear Pairing: map $e : \mathbf{G}_1 \times \mathbf{G}_2 \to \mathbf{G}_T$

1. bilinear: $e(P_1 + P_2,\ Q) = e(P_1, Q) \cdot e(P_2, Q)$, $e(P, Q_1 + Q_2) = e(P, Q_1) \cdot e(P, Q_2)$
2. non-degenerate: $e(g_1, g_2) \neq 1$ for $\langle g_1 \rangle = \mathbf{G}_1$, $\langle g_2 \rangle = \mathbf{G}_2$
3. efficiently computable.

Mostly used in practice:

$$e([a]P, [b]Q) = e([b]P, [a]Q) = e(P, Q)^{ab} \ .$$

$\rightsquigarrow$ Many applications in asymmetric cryptography
(identity-based encryption, short signatues, NIZK, ZK-SNARK...)

Pairing-based cryptography

Weil or Tate pairing on an elliptic curve

Discrete logarithm problem with one more dimension.

$$e \ : \ E(\mathbb{F}_p)[\ell] \times E(\mathbb{F}_{p^n})/\ell E(\mathbb{F}_{p^n}) \longrightarrow \mathbb{F}_{p^n}^*, \quad e([a]P, [b]Q) = e(P, Q)^{ab}$$

Pairing-based cryptography

Weil or Tate pairing on an elliptic curve

Discrete logarithm problem with one more dimension.

$$e \ : \ E(\mathbb{F}_p)[\ell] \times E(\mathbb{F}_{p^n})/\ell E(\mathbb{F}_{p^n}) \longrightarrow \mathbb{F}_{p^n}^*, \quad e([a]P, [b]Q) = e(P, Q)^{ab}$$

Attacks

Pairing-based cryptography

Weil or Tate pairing on an elliptic curve

Discrete logarithm problem with one more dimension.

$$e \ : \ E(\mathbb{F}_p)[\ell] \times E(\mathbb{F}_{p^n})/\ell E(\mathbb{F}_{p^n}) \longrightarrow \mathbb{F}_{p^n}^*, \quad e([a]P, [b]Q) = e(P, Q)^{ab}$$

Attacks

- discrete logarithm computation in $E(\mathbb{F}_p)$: hard problem (exponential, in $O(\sqrt{\ell})$)

Pairing-based cryptography

Weil or Tate pairing on an elliptic curve

Discrete logarithm problem with one more dimension.

$$e \ : \ E(\mathbb{F}_p)[\ell] \times E(\mathbb{F}_{p^n})/\ell E(\mathbb{F}_{p^n}) \longrightarrow \mathbb{F}_{p^n}^*, \quad e([a]P, [b]Q) = e(P, Q)^{ab}$$

Attacks

- discrete logarithm computation in $E(\mathbb{F}_p)$: hard problem (exponential, in $O(\sqrt{\ell})$)
- inversion of e : hard problem (exponential)

Pairing-based cryptography

Weil or Tate pairing on an elliptic curve

Discrete logarithm problem with one more dimension.

$$e \ : \ E(\mathbb{F}_p)[\ell] \times E(\mathbb{F}_{p^n})/\ell E(\mathbb{F}_{p^n}) \longrightarrow \mathbb{F}_{p^n}^*, \quad e([a]P, [b]Q) = e(P, Q)^{ab}$$

Attacks

- discrete logarithm computation in $E(\mathbb{F}_p)$: hard problem (exponential, in $O(\sqrt{\ell})$)
- inversion of e : hard problem (exponential)
- discrete logarithm computation in $\mathbb{F}_{p^n}^*$: **easier, subexponential** $\rightarrow$ take a large enough field

Pairing-friendly curves are special

$\ell \mid p^n - 1$, $\mathbf{G}_T \subset \mathbb{F}_{p^n}$, n is minimal : **embedding degree**
Tate Pairing: $e : \mathbf{G}_1 \times \mathbf{G}_2 \to \mathbf{G}_T$

When n is small, the curve is *pairing-friendly*.
This is very rare: usually $\log n \sim \log \ell$ ([Balasubramanian Koblitz]).

Barreto-Naehrig (BN), $n = 12$:
$r(x) = 36x^4 + 36x^3 + 18x^2 + 6x + 1$, $p(x) = 36x^4 + 36x^3 + 24x^2 + 6x + 1$
Barreto–Lynn–Scott (BLS), $n = 12$:
$r(x) = x^4 - x^2 + 1$, $p(x) = (x-1)^2 r(x)/3 + x$
$D = -3$, $j = 0$, $\mathbf{G}_T \subset \mathbb{F}_{p^{12}}$

p is special
Examples:
Euthereum blockchain has a 254-bit BN curve with $x =$`0x44e992b44a6909f1`,
Zcash cryptocurrency has a 381-bit BLS curve with $x =$`-0xd201000000010000`.

Outline

Discrete Log in $\mathbb{F}_{p^n}$

$\mathbb{F}_{p^n}$ much less investigated than $\mathbb{F}_p$ or integer factorization.
Much better results in pairing-related fields

Discrete Log in $\mathbb{F}_{p^n}$

$\mathbb{F}_{p^n}$ much less investigated than $\mathbb{F}_p$ or integer factorization.
Much better results in pairing-related fields

- Special NFS in $\mathbb{F}_{p^n}$: Joux–Pierrot 2013
- Tower NFS (TNFS): Barbulescu Gaudry Kleinjung 2015
- Extended Tower NFS: Kim–Barbulescu, Kim–Jeong, Sarkar–Singh 2016
- Tower of number fields

Use more structure: subfields

Special Tower NFS

$\mathbb{F}_{p^{12}}$, subfield $\mathbb{F}_{p^2}$ defined by $y^2 + 1$
Idea: $a + bx$ in NFS $\rightarrow$ $(a_0 + a_1 y) + (b_0 + b_1 y)x$ in TNFS
Integers to factor are **much smaller**

- factors integer $\text{Norm}_f = \text{Res}(\text{Res}(\mathbf{a} + \mathbf{b}x, f_y(x)), y^2 + 1)$
- factors integer $\text{Norm}_g = \text{Res}(\text{Res}(\mathbf{a} + \mathbf{b}x, g_y(x)), y^2 + 1)$

Res = resultant of polynomials

Special Tower NFS

$\mathbb{F}_{p^{12}}$, subfield $\mathbb{F}_{p^2}$ defined by $y^2 + 1$
Idea: $a + bx$ in NFS $\rightarrow$ $(a_0 + a_1 y) + (b_0 + b_1 y)x$ in TNFS
Integers to factor are **much smaller**

- factors integer $\text{Norm}_f = \text{Res}(\text{Res}(\mathbf{a} + \mathbf{b}x, f_y(x)), y^2 + 1)$
- factors integer $\text{Norm}_g = \text{Res}(\text{Res}(\mathbf{a} + \mathbf{b}x, g_y(x)), y^2 + 1)$

Res = resultant of polynomials
$p = p(s)$ is special

Special Tower NFS

$\mathbb{F}_{p^{12}}$, subfield $\mathbb{F}_{p^2}$ defined by $y^2 + 1$
Idea: $a + bx$ in NFS $\rightarrow$ $(a_0 + a_1 y) + (b_0 + b_1 y)x$ in TNFS
Integers to factor are **much smaller**

- factors integer $\text{Norm}_f = \text{Res}(\text{Res}(\mathbf{a} + \mathbf{b}x, f_y(x)), y^2 + 1)$
- factors integer $\text{Norm}_g = \text{Res}(\text{Res}(\mathbf{a} + \mathbf{b}x, g_y(x)), y^2 + 1)$

Res = resultant of polynomials
$p = p(s)$ is special

Index calculus in the 80's: implemented *before* complexity known
TNFS: complexity known, implementation just started for GF(p^6)
`https://listserv.nodak.edu/cgi-bin/wa.exe?A2=1302&L=NMBRTHRY&D=0&P=3444089`
Discrete logarithm in GF(p^6) of 521 bits with the Tower NFS algorithm
Gabrielle De Micheli, Pierrick Gaudry, Cécile Pierrot

Complexities

large characteristic $p = L_{p^n}(\alpha),\ \alpha > 2/3$:

$(64/9)^{1/3} \simeq 1.923$	NFS
special p:	
$(32/9)^{1/3} \simeq 1.526$	SNFS

medium characteristic $p = L_{p^n}(\alpha),\ 1/3 < \alpha < 2/3$:

$(96/9)^{1/3} \simeq 2.201$	prime n NFS-HD (Conjugation)
$(48/9)^{1/3} \simeq 1.747$	composite n, best case of TNFS: when parameters fit perfectly
special p:	
$(64/9)^{1/3} \simeq 1.923$	NFS-HD+Joux–Pierrot'13
$(32/9)^{1/3} \simeq 1.526$	composite n, best case of STNFS

Ranking polynomials: Murphy's α and E

B. A. Murphy, 1999

Input: irreducible polynomials f, g, sharing root m mod p ($p \mid \mathrm{Res}(f, g)$)

- $\alpha(f, B_0)$: bias in smoothness between norms and integers (up to some bound B_0)
  $\alpha(f, B_0), \alpha(g, B_0) < 0$ wanted
- $E(f, g, B_f, B_g, \text{area} = A^2/2)$: estimation of the yield of polynomials
  B_f, B_g smoothness bounds of f, g sides
  How many relations would (f, g) produce?
- Rank many (f_i, g_i), choose the best pair

Ranking polynomials: Murphy's α and E

B. A. Murphy, 1999

Input: irreducible polynomials f, g, sharing root m mod p ($p \mid \mathrm{Res}(f, g)$)

- $\alpha(f, B_0)$: bias in smoothness between norms and integers (up to some bound B_0)
  $\alpha(f, B_0), \alpha(g, B_0) < 0$ wanted
- $E(f, g, B_f, B_g, \text{area} = A^2/2)$: estimation of the yield of polynomials
  B_f, B_g smoothness bounds of f, g sides
  How many relations would (f, g) produce?
- Rank many (f_i, g_i), choose the best pair

Generalization to the TNFS setting:

- $\alpha(h, f)$, $\alpha(h, g)$
  SageMath & Magma code, generalization from `cado-nfs` α
  (Bai, Gaudry, Hanrot, Thomé, Zimmermann)
- Monte-Carlo simulation for Murphy's E

Polynomial selection

- 10% total time in a record computation
- Which criteria?

Wanted: norms more often B-smooth than integers of the same size
Very fast choice

```
from tnfs.alpha.alpha2d import alpha2d
ZZx.<x> = ZZ[]
alpha2d(x^2-2, 2000)
alpha2d(x^2+1, 2000)
alpha2d(x^2+2, 2000)
```

f	$\alpha(f, 2000)$
$x^2 - 2$	1.752860
$x^2 + 1$	1.366415
$x^2 + 2$	1.116395
...	
$8x^2 + 3x + 10$	-0.097068

Smoothness probabilities: definition of α

$$\alpha(f, B) = \sum_{\substack{\ell \text{ prime} \\ \ell \le B}} \ln \ell \Big(\mathbb{E}(\mathrm{val}_\ell(n),\ n \in \mathbb{Z}) - \mathbb{E}(\mathrm{val}_\ell(\mathrm{Res}_x(f(x), a + bx),\ (a, b) \in I \times J) \Big)$$

f with many roots mod many small primes ℓ wanted

If $B \to \infty$, α converges (Th. Barbulescu–Lachand 2017)

Smoothness probabilities: definition of α

$$\alpha(f, B) = \sum_{\substack{\ell \text{ prime} \\ \ell \leq B}} \ln \ell \Big(\mathbb{E}(\mathrm{val}_\ell(n),\ n \in \mathbb{Z}) - \mathbb{E}(\mathrm{val}_\ell(\mathrm{Res}_x(f(x), a + bx),\ (a, b) \in I \times J) \Big)$$

f with many roots mod many small primes ℓ wanted
If $B \to \infty$, α converges (Th. Barbulescu–Lachand 2017)

Average Valuation at $\ell = 2$ for random integers n :

- $1/2$ chance to be even, if even: $\mathrm{val}_2\, n \geq 1$
- $1/4$ chance to be multiple of 4, in this case: $\mathrm{val}_2\, n \geq 2$
- $1/2^i$ chance to be multiple of 2^i, in this case : $\mathrm{val}_2\, n \geq i$

Smoothness probabilities: definition of α

$$\alpha(f, B) = \sum_{\substack{\ell \text{ prime} \\ \ell \le B}} \ln \ell \Big(\mathbb{E}(\mathrm{val}_\ell(n),\ n \in \mathbb{Z}) - \mathbb{E}(\mathrm{val}_\ell(\mathrm{Res}_x(f(x), a + bx),\ (a, b) \in I \times J) \Big)$$

f with many roots mod many small primes ℓ wanted
If $B \to \infty$, α converges (Th. Barbulescu–Lachand 2017)

Average Valuation at $\ell = 2$ for random integers n :

- $1/2$ chance to be even, if even: $\mathrm{val}_2\, n \ge 1$
- $1/4$ chance to be multiple of 4, in this case: $\mathrm{val}_2\, n \ge 2$
- $1/2^i$ chance to be multiple of 2^i, in this case : $\mathrm{val}_2\, n \ge i$

With $\mathrm{val}_\ell(n)$ a random variable in $\mathbb{N}$,

$$\mathbb{E}(\mathrm{val}_\ell(n),\ n \in \mathbb{Z}) = \sum_{i=1}^{\infty} i \Pr[\mathrm{val}_\ell(n) = i] = \sum_{i=1}^{\infty} \Pr[\mathrm{val}_\ell(n) \ge i]$$

Smoothness probabilities: definition of α

$$\alpha(f,B) = \sum_{\substack{\ell \text{ prime} \\ \ell \le B}} \ln \ell \Big(\mathbb{E}(\mathrm{val}_\ell(n),\ n \in \mathbb{Z}) - \mathbb{E}(\mathrm{val}_\ell(\mathrm{Res}_x(f(x), a+bx),\ (a,b) \in I \times J) \Big)$$

f with many roots mod many small primes ℓ wanted
If $B \to \infty$, α converges (Th. Barbulescu–Lachand 2017)

Average Valuation at $\ell = 2$ for random integers n :

- $1/2$ chance to be even, if even: $\mathrm{val}_2\, n \ge 1$
- $1/4$ chance to be multiple of 4, in this case: $\mathrm{val}_2\, n \ge 2$
- $1/2^i$ chance to be multiple of 2^i, in this case : $\mathrm{val}_2\, n \ge i$

With $\mathrm{val}_\ell(n)$ a random variable in $\mathbb{N}$,

$$\mathbb{E}(\mathrm{val}_\ell(n),\ n \in \mathbb{Z}) = \sum_{i=1}^{\infty} i \Pr[\mathrm{val}_\ell(n) = i] = \sum_{i=1}^{\infty} \Pr[\mathrm{val}_\ell(n) \ge i]$$

$$= \frac{1}{\ell} + \frac{1}{\ell^2} + \frac{1}{\ell^3} + \ldots + \frac{1}{\ell^k} + \ldots = \sum_{k=1}^{\infty} \frac{1}{\ell^k} = \lim_{k \to \infty} \frac{1 - 1/\ell^k}{1 - 1/\ell} - 1 = \frac{1}{\ell - 1} \text{ with } \ell \text{ prime}$$

Computing $\alpha(f)$

$$\mathbb{E}(\mathrm{val}_\ell(\mathrm{Res}_x(f, a+bx),\ (a,b) \in I \times J))$$

Let n_{ℓ^k} the number of distinct roots of f mod ℓ^k

$$\mathrm{val}_\ell(f) = \mathrm{val}_\ell(\mathrm{Res}_x(f, a+bx)) = \sum_{k=1}^{\infty} \frac{\ell}{\ell+1} \frac{n_{\ell^k}}{\ell^k}$$

Lemma (Hensel)

Let $f(x)$ a polynomial of degree ≥ 2, r a simple root mod ℓ ($f'(r) \neq 0 \bmod \ell$). For all $k > 1$, $f(x)$ has a unique root $r_k \in \mathbb{Z}/\ell^k\mathbb{Z}$ above r.

If $n_{\ell^k} = n_\ell$ for all $k \geq 1$, $\mathrm{val}_\ell(f) = \ell n_\ell/(\ell^2-1)$.
If $\ell \nmid \mathrm{disc}(f)$,

$$\alpha_\ell(f) = \left(\frac{1}{\ell-1} - \frac{n_\ell \ell}{\ell^2-1}\right) \log \ell$$

Examples: α looks like computing the factor bases

ℓ	$f = x^2 - 2 \bmod \ell$
2	x^2 bad prime
7	$(x-3)(x+3)$
17	$(x-6)(x+6)$
23	$(x-5)(x+5)$
31	$(x-8)(x+8)$
41	$(x-17)(x+17)$
47	$(x-7)(x+7)$
71	$(x-12)(x+12)$
73	$(x-32)(x+32)$
79	$(x-9)(x+9)$
89	$(x-25)(x+25)$
97	$(x-14)(x+14)$

$\alpha(f, 100) = 1.5981$

ℓ	$f = x^2 + 1 \bmod \ell$
2	$(x+1)^2$ bad prime
5	$(x+2)(x-2)$
13	$(x+5)(x-5)$
17	$(x+4)(x-4)$
29	$(x+12)(x-12)$
37	$(x+6)(x-6)$
41	$(x+9)(x-9)$
53	$(x+23)(x-23)$
61	$(x+11)(x-11)$
73	$(x+27)(x-27)$
89	$(x+34)(x-34)$
97	$(x+22)(x-22)$

$\alpha(f, 100) = 1.2752$

ℓ	$f = 8x^2 + 3x + 10 \bmod \ell$
2	x^2 projective root
3	$(x+1)(x-1)$
5	$x(x-4)$
7	$(x-3)(x-1)$
13	$(x-8)(x-3)$
47	$(x-22)(x-7)$
53	$(x-31)(x-15)$
67	$(x-51)(x-24)$
73	$(x-70)(x-30)$
79	$(x-43)(x-6)$
83	$(x-75)(x-18)$
89	$(x-63)(x-59)$

$\alpha(f, 100) = -0.1432$

α for bad primes (multiple roots)

If $\ell \mid \mathrm{disc}(f)$, ℓ is a *bad prime*.
Exists e such that $n_{\ell^k} = n_{\ell^e}$ for all $k \geq e$.
Computes such e and n_{ℓ^k} for all $k \leq e$.
Reverse-engineering of `cado-nfs/polyselect/auxiliary.c`

Ugly details in the appendix of the paper.

Simulation without sieving

Polynomial selection: for many pairs (f, g)

- compute $\alpha(h, f), \alpha(h, g)$ (w.r.t. subfield) **bias in smoothness**
- select polys f, g with negative bias $\alpha(f), \alpha(g)$ if possible
- **Monte-Carlo** simulation with 10^6 random samples from $\mathcal{S} = \{(a_0 + a_1 y + \ldots + a_d y^d) + (b_0 + b_1 y + \ldots + b_d y^d)x,\ |a_i|, |b_j| < A\}$ For each sample:
  1. compute its algebraic norm N_f, N_g in each number field
  2. smoothness probability (N_f, α_f), (N_g, α_g) with Dickman-ρ
- Average smoothness probability of samples
  $\rightarrow$ estimation of the total number of possible relations in $\mathcal{S}$
  $\rightarrow$ **Murphy's E for TNFS**

Simulation without sieving

Polynomial selection: for many pairs (f, g)

- compute $\alpha(h, f), \alpha(h, g)$ (w.r.t. subfield) **bias in smoothness**
- select polys f, g with negative bias $\alpha(f), \alpha(g)$ if possible
- **Monte-Carlo** simulation with 10^6 random samples from $\mathcal{S} = \{(a_0 + a_1 y + \ldots + a_d y^d) + (b_0 + b_1 y + \ldots + b_d y^d)x,\ |a_i|, |b_j| < A\}$ For each sample:
  1. compute its algebraic norm N_f, N_g in each number field
  2. smoothness probability (N_f, α_f), (N_g, α_g) with Dickman-ρ
- Average smoothness probability of samples
  $\rightarrow$ estimation of the total number of possible relations in $\mathcal{S}$
  $\rightarrow$ **Murphy's E for TNFS**

dichotomy to approach the best balanced parameters

smoothness bound B, coefficient bound A.
$\rightarrow$ refinement of Barbulescu–Duquesne technique [BD19 Journal of Cryptology]

Murphy's α function

$\alpha(f)$ for NFS estimates the bias in smoothness

Algebraic norms in $K_f = \mathbb{Q}[x]/(f(x))$ of $\log_2 N_f$ bits have same smoothness proba as integers of $\log_2 N_f + \alpha(f)/\log(2)$ bits
$\rightarrow \alpha(f) < 0$ wanted
$\alpha(f)$ computes the exact number of roots of $f(x)$ mod q^k
for all primes $q < 2000$ (say)
Easy prime $q \nmid \mathrm{disc}(f)$, tricky prime $q \mid \mathrm{disc}(f)$

Implementation for TNFS

Reverse-engineering of `cado-nfs/polyselect/{auxiliary.c,alpha.sage}`
Magma and SageMath `https://gitlab.inria.fr/tnfs-alpha/alpha`
Same algorithm, prime $q \rightarrow$ prime ideal of norm q

Example : Barreto-Naehrig curve, p 254 bits

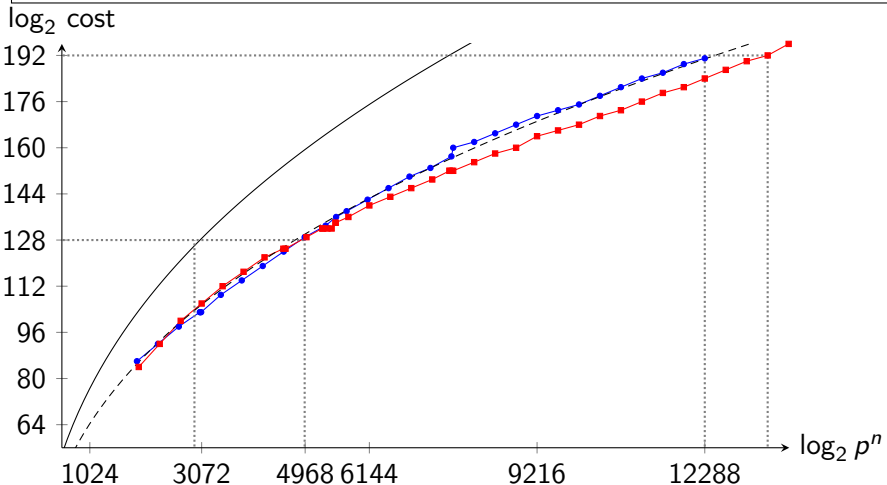
$p = 36s^4 + 36s^3 + 24s^2 + 6s + 1$ where $s = -(2^{62} + 2^{55} + 1)$

$f = 36x^8 + 36yx^6 + 24y^2x^4 + 6y^3x^2 + y^4$

$g = x^2 + sy = x^2 + 4647714815446351873y$

$B = 2000$

h	$1/\zeta_{K_h}(2)$	$\alpha(h, f, B)$	$\alpha(h, g, B)$	$\alpha_f + \alpha_g$
$y^6 + y^5 - y^2 - y - 1$	0.953	2.042	2.479	4.521
$y^6 - y^4 + y^3 + y^2 - 1$	0.917	1.288	1.740	3.028
$y^6 + y^3 + y^2 - y - 1$	0.917	2.419	2.876	5.295
$y^6 + y^5 - y^3 + y - 1$	0.909	0.278	2.357	2.636
$y^6 + y^5 + y^4 + y^3 + y^2 + y - 1$	0.883	2.341	2.033	4.374
$y^6 + y^4 + y^3 + y - 1$	0.867	0.899	2.526	3.425
$y^6 + y^4 + y^2 + y + 1$	0.836	1.955	1.141	3.095
$y^6 + y^5 + y^2 - y + 1$	0.763	0.891	1.264	2.155
$y^6 + y^5 - y^4 + y^3 + y^2 + y - 1$	0.756	0.956	1.177	2.133
$y^6 + y^5 + y - 1$	0.736	1.925	2.108	4.032
$y^6 + y^5 + y^3 - y^2 + y - 1$	0.732	1.729	2.099	3.828
$y^6 + y^3 + y - 1$	0.728	-0.250	1.191	0.941
$y^6 + y^3 - y + 1$	0.720	1.605	1.348	2.952
$y^6 + y^3 + y^2 + 1$	0.718	1.151	1.294	2.445
$y^6 - y^4 + y^3 - y^2 - y - 1$	0.710	0.406	2.278	2.684
$y^6 + y^5 - y^3 + y^2 - y + 1$	0.697	1.572	0.818	2.390
$y^6 + y^4 + y + 1$	0.679	1.319	1.683	3.002


Simul. in $\mathbb{F}_{p^{12}}$, BN, STNFS deg $h = 6, 4$
Simul. in $\mathbb{F}_{p^{12}}$, BLS12, STNFS deg $h = 12, 6$
$L^0_{p^n}(1/3, 1.923)/2^{10.17}$ (DL theoretical re-scaled DL-240dd $\leftrightarrow 2^{67.51}$)
$L^0_{p^n}(1/3, 1.526)/2^{4.5}$ (SNFS theoretical re-scaled SDL-1024 $\leftrightarrow 2^{64.4}$)
$\log_2$ cost
192
176
160
144
128
112
96
80
64
1024
3072
4968
6144
9216
12288
$\log_2 p^n$

Numerical example: BLS12-446 bits

$p(x) = (x-1)^2(x^4 - x^2 + 1)/3 + x$
$r(x) = x^4 - x^2 + 1$
$s = -(2^{74} + 2^{73} + 2^{63} + 2^{57} + 2^{50} + 2^{17} + 1)$
seed with `enumerate_sparse_T.sage` [G. Masson Thomé]
`https://gitlab.inria.fr/smasson/cocks-pinch-variant` $p = p(s)$ of 446 bits,
twist-secure subgroup-secure curve
p^k 5352 bits
$h = Y^6 - Y^4 + Y^3 - Y + 1$

$f_y = X^{12} - 2yX^{10} + 2y^3X^6 + y^5X^2 + y^4 - y^3 + y - 1$
$g_y = X^2 - uy = X^2 + 28343567510342708887553y$
$A = 968$, $B = 2^{68.2}$

Estimated cost: $\approx 2^{132}$

Key size for pairings

	cost DL 2^{128}		cost DL 2^{192}	
$\mathbb{F}_{p^n}$, curve	$\log_2 p$	$\log_2 p^n$	$\log_2 p$	$\log_2 p^n$
$\mathbb{F}_p$	3072–3200		7400–8000	
$\mathbb{F}_{p^6}$, MNT	$\approx$672	$\approx$ 4032	$\approx$ 1536	$\approx$ 9216
$\mathbb{F}_{p^{12}}$, BN	$\approx$448	$\approx$5376	$\approx$ 1024	$\approx$ 12288
$\mathbb{F}_{p^{12}}$, BLS	$\approx$448	$\approx$5376	$\approx$ 1120	$\approx$ 13440
$\mathbb{F}_{p^{16}}$, KSS	330	5280	$\approx$ 768	$\approx$ 12288
$\mathbb{F}_{p^{18}}$, KSS	348	6264	$\approx$ 640	$\approx$ 11520
$\mathbb{F}_{p^{24}}$, BLS	318	7632	$\approx$ 512	$\approx$ 12288

- BN-254 $\approx 2^{103}$
- BN-382 $\approx 2^{123}$, BLS12-381 $\approx 2^{126}$
- BN-446 and BLS12-446 $\approx 2^{132}$
- BN-462 and BLS12-461 $\approx 2^{135}$

Other curves:

- Fotiadis-Martindale [FM19] $k = 12$ with $r = r_{\mathsf{BN}}$ like BLS12
- modified Cocks-Pinch with $k = 8$ and $\rho = 2.125$ [GMT19]

References


Aurore Guillevic and Shashank Singh.
On the alpha value of polynomials in the tower number field sieve algorithm.
Mathematical Cryptology, 1(1), Feb. 2021.
https://journals.flvc.org/mathcryptology/article/view/125142
ePrint 2019/885

Source code: `https://gitlab.inria.fr/tnfs-alpha/alpha`
Webpage on pairing-friendly curves
`https://members.loria.fr/AGuillevic/pairing-friendly-curves/`

Bibliography I


S. Bai, R. P. Brent, and E. Thomé.
Root optimization of polynomials in the number field sieve.
Math. Comp., 84(295):2447–2457, 2015.


R. Barbulescu and S. Duquesne.
Updating key size estimations for pairings.
Journal of Cryptology, 32(4):1298–1336, Oct. 2019.


R. Barbulescu, P. Gaudry, and T. Kleinjung.
The tower number field sieve.
In T. Iwata and J. H. Cheon, editors, *ASIACRYPT 2015, Part II*, volume 9453 of *LNCS*, pages 31–55. Springer, Heidelberg, Nov. / Dec. 2015.


R. Barbulescu and A. Lachand.
Some mathematical remarks on the polynomial selection in NFS.
Math. Comp., 86(303):397–418, 2017.


F. Boudot, P. Gaudry, A. Guillevic, N. Heninger, E. Thomé, and P. Zimmermann.
Comparing the difficulty of factorization and discrete logarithm: A 240-digit experiment.
In D. Micciancio and T. Ristenpart, editors, *CRYPTO 2020, Part II*, volume 12171 of *LNCS*, pages 62–91. Springer, Heidelberg, Aug. 2020.

Bibliography II


A. Guillevic.
A short-list of pairing-friendly curves resistant to special TNFS at the 128-bit security level.
In A. Kiayias, M. Kohlweiss, P. Wallden, and V. Zikas, editors, *PKC 2020, Part II*, volume 12111 of *LNCS*, pages 535–564. Springer, Heidelberg, May 2020.


T. Kim and R. Barbulescu.
Extended tower number field sieve: A new complexity for the medium prime case.
In M. Robshaw and J. Katz, editors, *CRYPTO 2016, Part I*, volume 9814 of *LNCS*, pages 543–571. Springer, Heidelberg, Aug. 2016.


B. A. Murphy.
Polynomial selection for the number field sieve integer factorisation algorithm.
Phd thesis, Australian National University, Australia, 1999.
`http://maths-people.anu.edu.au/~brent/pd/Murphy-thesis.pdf`.