Pairing-friendly elliptic curves, design, implementation, and discrete logarithm computations

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`https://members.loria.fr/AGuillevic/files/talks/23_Rennes.pdf`

Bilinear pairing in cryptography

As a black-box:
$(\mathbb{G}_1, +), (\mathbb{G}_2, +), (\mathbb{G}_T, \cdot)$ three cyclic groups of large prime order r

Bilinear pairing: map $e : \mathbb{G}_1 \times \mathbb{G}_2 \to \mathbb{G}_T$

1. bilinear: $e(P_1 + P_2,\ Q) = e(P_1, Q) \cdot e(P_2, Q)$, $e(P, Q_1 + Q_2) = e(P, Q_1) \cdot e(P, Q_2)$
2. non-degenerate: $e(G_1, G_2) \neq 1$ for $\langle G_1 \rangle = \mathbb{G}_1$, $\langle G_2 \rangle = \mathbb{G}_2$
3. efficiently computable

Mostly used in practice:

$$e([a]P, [b]Q) = e([b]P, [a]Q) = e(P, Q)^{ab}$$

Examples of applications

- 1984: idea of identity-based encryption (IBE) by Shamir
- 1999: first practical identity-based cryptosystem of Sakai-Ohgishi-Kasahara
- 2000: constructive pairings, Joux's tri-partite key-exchange
- 2001: IBE of Boneh-Franklin, short signatures Boneh-Lynn-Shacham

...

- Broadcast encryption, re-keying
- aggregate signatures
- zero-knowledge (ZK) proofs
  - non-interactive ZK proofs (NIZK)
  - zk-SNARK (Z-cash, Zexe...)
- tool in isogeny-based post-quantum cryptography, different setting (not in this talk)

Bilinear pairings

Security relies on

- Discrete Log Problem (DLP):

$$\text{given } g, h \in \mathbb{G}, \text{ compute } x \text{ s.t. } g^x = h$$

- Diffie-Hellman Problem (DHP):

$$\text{given } g, g^a, g^b \in \mathbb{G}, \text{ compute } g^{ab}$$

- bilinear DLP and DHP
- pairing inversion problem

Open the black-box: torsion points

Curve25519 : $y^2 = x^3 + \underbrace{486662}_{A} x^2 + x$ over GF(p), $p = 2^{255} - 19$

order $\#E(\mathbb{F}_p) = 8r$, 253-bit prime r

2-torsion points $= \{P \in E,\ 2P = \mathcal{O} \iff y_P = 0\}$

- 2-torsion over $\mathbb{F}_p$: $\{\mathcal{O}, (0,0)\}$
- full 2-torsion over $\mathbb{F}_{p^2}$: $\{\mathcal{O}, (0,0), (\lambda, 0), (\mu, 0)\}$, $x^2 + Ax + 1 = (x - \lambda)(x - \mu)$

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For an integer ℓ, the ℓ-torsion $E[\ell]$ has order ℓ^2

- $\#E[2] = 4 \subset E(\mathbb{F}_{p^2})$
- $\#E[4] = 16 \subset E(\mathbb{F}_{p^2})$
- $\#E[8] = 64 \subset E(\mathbb{F}_{p^2})$
- $\#E[r] = r^2 \subset E(\mathbb{F}_{p^k})$, $k = (r-1)/6$ of 250 bits for Curve25519

Pairing-friendly curves should be designed on purpose

In cryptographic setting: $E[r]$ has structure $\mathbb{Z}_r \times \mathbb{Z}_r$ denoted $\mathbb{G}_1 \times \mathbb{G}_2$

128-, resp. 192-bit security level:

- r large prime ~ 256, resp. 384 bits
- $\#E(\mathbb{F}_p) = h \cdot r$, h small **cofactor**, $\mathbb{G}_1 = E(\mathbb{F}_p)[r]$
- $E[r] \subset E(\mathbb{F}_{p^k})$ and $1 \leq k \leq 54$, $\mathbb{G}_2 \subset E(\mathbb{F}_{p^k})[r]$
  k embedding degree
- $\mathbb{G}_T \subset \mathbb{F}_{p^k}^*$ multiplicative subgroup of order r

Usually $\log k \sim \log r$ (Balasubramanian Koblitz [BK98]).

Plain curves (25519, NIST curves) are never pairing-friendly

Pairing-based cryptography

Weil or Tate pairing on an elliptic curve

Discrete logarithm problem with one more dimension

$$e\colon E(\mathbb{F}_p)[r] \times \mathbb{G}_2 \longrightarrow \mathbb{G}_T \subset \mathbb{F}_{p^k}^*, \quad e([a]P, [b]Q) = e(P, Q)^{ab}$$

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Attacks

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$$\uparrow$$

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- inversion of e : hard problem (exponential)

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- inversion of e : hard problem (exponential)
- discrete logarithm computation in $E(\mathbb{F}_p)$: hard problem (exponential, in $O(\sqrt{r})$)

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Attacks

- inversion of e : hard problem (exponential)
- discrete logarithm computation in $E(\mathbb{F}_p)$: hard problem (exponential, in $O(\sqrt{r})$)
- discrete logarithm computation in $\mathbb{F}_{p^k}^*$: **easier, subexponential** $\to$ take a large enough field

Pairing-friendly curves are special

1st ones were *supersingular*, not in this talk.

Ordinary curves:

- 2001: Miyaji–Nakabayashi–Takano curves, $k \in \{3, 4, 6\}$, prime order [MNT01]
- Cocks-Pinch technique
- Barreto–Lynn–Scott curves, $3 \mid k$, $18 \nmid k$ [BLS03]
- Brezing–Weng construction [BW05]
- Freeman $k = 10$ [Fre06], Barreto–Naehrig curves $k = 12$, prime order [BN06]
- Kachisa–Schaefer–Scott curves, $k \in \{8, 16, 18, 32, 36, 40\}$ [KSS08]
- Freeman–Scott–Teske Taxonomy [FST10]
- Scott–G, $k = 54$ [SG18]
- Gasnier–G, $k = 20, 22$ [GG23]

Why Barreto–Naehrig'2005 curves were so popular?

$k = 12,\ j = 0,\ D = -3,$

$$E\colon y^2 = x^3 + b$$

$$p(x) = 36x^4 + 36x^3 + 24x^2 + 6x + 1$$
$$r(x) = 36x^4 + 36x^3 + 18x^2 + 6x + 1$$

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$\mathbb{G}_T \subset \mathbb{F}_{p^{12}}$ of $12 \log p \approx 3048$ bits
≈ 3072 bits expected to offer 128 bits of security for RSA and Discrete Log in the 2000's

$\Longrightarrow$ BN curves were the perfect match

Choosing pairing-friendly curves

Pairing-based cryptography needs **secure, efficient, compact** pairing-friendly curves

- secure against discrete log in $E(\mathbb{F}_p)$, $E(\mathbb{F}_{p^k})$, $\mathbb{F}_{p^k}$
- efficient for scalar multiplication in E, exponentiation in $\mathbb{F}_{p^k}$, pairing
- compact: key sizes as small as possible

Which curves are the best options?

Discrete Log in $\mathbb{F}_{p^k}$

$\mathbb{F}_{p^k}$ much less investigated than $\mathbb{F}_p$ or integer factorization
Much better results in pairing-related fields

- Special NFS in $\mathbb{F}_{p^k}$: Joux–Pierrot 2013 [JP14]
- Tower NFS (TNFS): Barbulescu–Gaudry–Kleinjung 2015 [BGK15]
- Extended Tower NFS: Kim–Barbulescu [KB16], Kim–Jeong [KJ17], Sarkar–Singh 2016 [SS16]

Use more structure: subfields

Choosing key sizes: Lenstra–Verheul [LV01] extrapolation

Initially for RSA modulus size

For DL in $\mathbb{F}_Q$ of length(Q) bits
n bits of security $\leftrightarrow$ the best (mathematical) attack should take at least 2^n steps

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- $+o(1)$ not known (or diverges [LG21])

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$$e^{\sqrt[3]{(64/9+0)(\ln Q_{\text{DL-240}})(\ln\ln Q_{\text{DL-240}})^2}} = 2^{77.68}$$

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- DL-240 in $2^{67.51}$ operations [BGG$^+$20] $\to 2^{67.51}/2^{77.68} = 2^{-10.17}$

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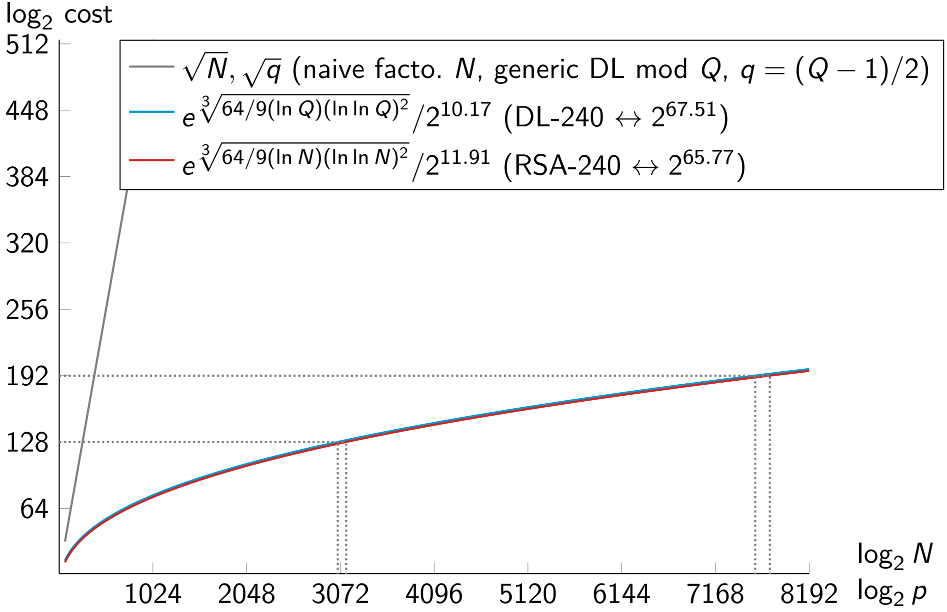
DL in prime field: Replace unknown $+o(1)$ by scaling factor $2^{-10.17}$



RSA-240: 953 core-years, Intel Xeon Gold 6130 CPUs as a reference (2.1GHz) $\approx 953 \cdot 365.25 \cdot 24 \cdot 60 \cdot 60 \cdot 2.1 \cdot 10^9 \approx 2^{65.77}$

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https://members.loria.fr/AGuillevic/files/Enseignements/UBS/slides.pdf

Estimating key sizes for DL in $\mathbb{F}_{p^k}$

- Latest variants of TNFS (Kim–Barbulescu, Kim–Jeong) seem most promising for $\mathbb{F}_{p^k}$ where k is composite
- The asymptotic complexities do not correspond to a fixed k, but to a ratio between k and p
- We need record computations if we want to extrapolate from asymptotic complexities

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Discrete logarithm in GF(p^6) with Tower-NFS [DGP21]

- $Q = p^6$ of 521 bits, total time 24798 core-hours (2.83 core-years) $\leftrightarrow 2^{57.37}$
- Tower-NFS-Conjugation $e^{\sqrt[3]{(48/9+o(1))(\ln Q)(\ln\ln Q)^2}}$
- $e^{\sqrt[3]{(48/9+0)(\ln Q_{\text{DL-521}})(\ln\ln Q_{\text{DL-521}})^2}} = 2^{58.52}$

DL in non-special $\mathbb{F}_{p^6}$ field: too early to apply Lenstra–Verheul extrapolation

Largest record computations in $\mathbb{F}_{p^k}$ with NFS and its variants[1]

Finite field	Size of p^k	Cost: CPU days	Authors	sieving dim
		Tower-NFS		
$\mathbb{F}_{p^6}$	521	1,033	[DGP21] De Micheli et al.'21	6, Tower
$\mathbb{F}_{p^4}$	512	2244	[Rob22] Robinson'22	4, Tower
		NFS and NFS-HD		
$\mathbb{F}_{p^{12}}$	203	11	[HAKT13, HAKT15]	7
$\mathbb{F}_{p^6}$	423	3,400	[MR20]	3
$\mathbb{F}_{p^5}$	324	386	[GGM17]	3
$\mathbb{F}_{p^4}$	392	510	[BGGM15a]	2
$\mathbb{F}_{p^3}$	593	8,400	[GGM16, GMT16]	2
$\mathbb{F}_{p^2}$	595	175	[BGGM15b]	2
$\mathbb{F}_p$	768	1,935,825	[KDLPS17]	2
$\mathbb{F}_p$	795	1,132,275	[BGGHTZ19]	2

[1] Data extracted from DiscreteLogDB by L.Grémy

Estimating key sizes for DL in $\mathbb{F}_{p^k}$

Simulation tool at `https://gitlab.inria.fr/tnfs-alpha/alpha` from [GS21]

- SageMath
- MIT License

Can select polynomials for (S)TNFS and estimates the running-time
Estimated cost of De Micheli et al. record: 2^{50} (real time: $2^{57.37}$)

We used that tool to estimate the security level in $GF(p^k)$ for many curves

Special Tower NFS

Find two number fields that can be mapped to $\mathbb{F}_{p^k}$ with reduction modulo p
Sharing a subfield $\mathbb{F}_{p^\kappa}$, $1 < \kappa \mid k$

$K_f = K_h[x]/(f_\tau(x))$ $\qquad$ $K_g = K_h[x]/(g_\tau(x))$

$\eta = k/\kappa \leq \deg f$ $\qquad$ $\deg g \geq \eta = k/\kappa$

$K_h = \mathbb{Q}[y]/(h(y))$

$\kappa = \deg h$

$\mathbb{Q}$

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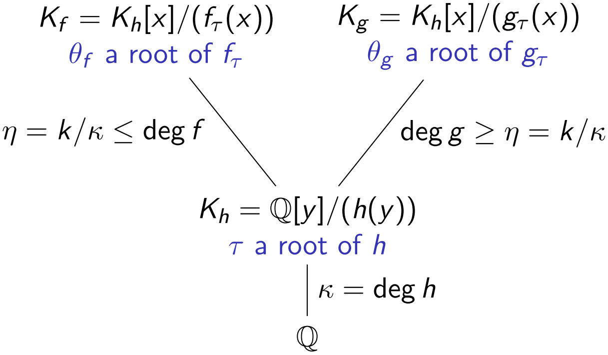

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$K_g = K_h[x]/(g_\tau(x))$ — θ_g a root of g_τ

$\eta = k/\kappa \leq \deg f$

$\deg g \geq \eta = k/\kappa$

$K_h = \mathbb{Q}[y]/(h(y))$ — τ a root of h

$\kappa = \deg h$

$\mathbb{Q}$

- f_τ, g_τ share a common gcd polynomial $\phi_\tau(x)$ irreducible of degree $\eta = k/\kappa$
- θ_f, θ_g map to the same θ_p in $\mathbb{F}_{p^k}$
- h is irreducible modulo p

Special Tower NFS

1. Polynomial selection: choose 3 polynomials h, f, g
2. Relation collection: obtain many smooth norms of
   $\boldsymbol{a} + \boldsymbol{b}\theta_f = (a_0 + a_1\tau + \ldots + a_i\tau^i) + (b_0 + b_1\tau + \ldots + b_i\tau^i)\theta_f$, $\boldsymbol{a} + \boldsymbol{b}\theta_g$
3. Filtering step of the matrix (apply Galois automorphisms if any)
4. Linear algebra
5. Individual discrete logarithm

Are the norms as smooth as integers of the same size?
Bias $\rightarrow \alpha(f), \alpha(g)$
TNFS: $\alpha(h, f), \alpha(h, g)$

Simulation without sieving

Polynomial selection: for many pairs (f, g)

- compute $\alpha(h, f), \alpha(h, g)$ (w.r.t. subfield) **bias in smoothness**
- select polys f, g with negative bias $\alpha(f), \alpha(g)$ if possible
- **Monte-Carlo** simulation with 10^6 random samples from
  $\mathcal{S} = \{(a_0 + a_1 y + \ldots + a_d y^d) + (b_0 + b_1 y + \ldots + b_d y^d)x,\ |a_i|, |b_j| < A\}$
  For each sample:
  1. compute its algebraic norm N_f, N_g in each number field
  2. smoothness probability (N_f, α_f), (N_g, α_g) with Dickman-ρ
- Average smoothness probability of samples
  $\rightarrow$ estimation of the total number of possible relations in $\mathcal{S}$
  $\rightarrow$ **Murphy's E for TNFS**

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dichotomy to approach the best balanced parameters

smoothness bound B, coefficient bound A.

$\rightarrow$ refinement of Barbulescu–Duquesne technique [BD19]

Murphy's α function

$\alpha(f)$ for NFS estimates the bias in smoothness

Algebraic norms in $K_f = \mathbb{Q}[x]/(f(x))$ of $\log_2 N_f$ bits have same smoothness proba as integers of $\log_2 N_f + \alpha(f)/\log(2)$ bits
$\rightarrow \alpha(f) < 0$ wanted
$\alpha(f)$ computes the exact number of roots of $f(x)$ mod q^k
for all primes $q < 2000$ (say)
Easy prime $q \nmid \mathrm{disc}(f)$, tricky prime $q \mid \mathrm{disc}(f)$

Implementation for TNFS

Reverse-engineering of `cado-nfs/polyselect/{auxiliary.c,alpha.sage}`
Magma and SageMath `https://gitlab.inria.fr/tnfs-alpha/alpha`
Same algorithm, prime $q \rightarrow$ prime ideal of norm q

Example : Barreto-Naehrig curve, p 254 bits

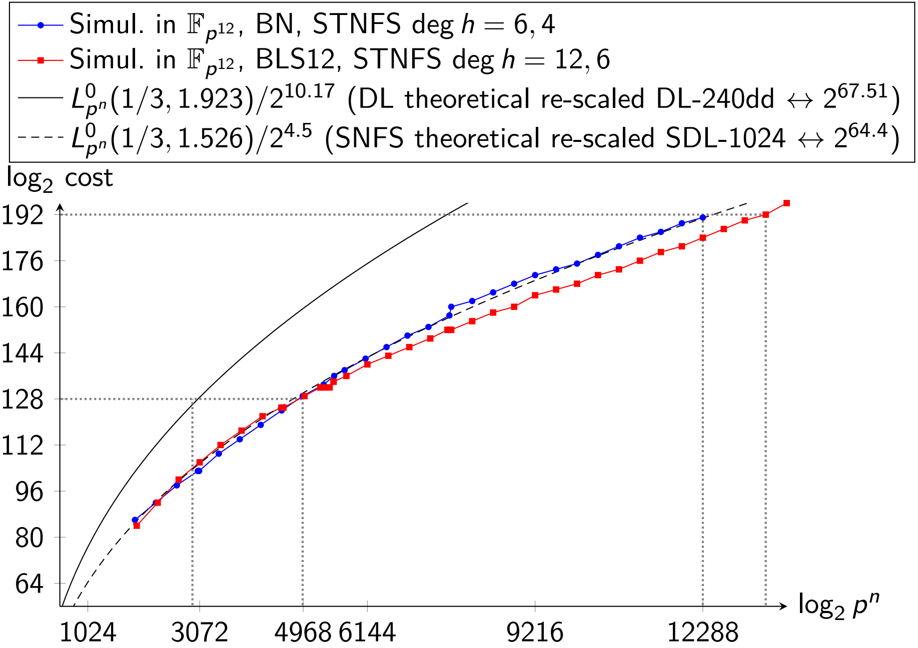
$p = 36s^4 + 36s^3 + 24s^2 + 6s + 1$ where $s = -(2^{62} + 2^{55} + 1)$

$f = 36x^8 + 36yx^6 + 24y^2x^4 + 6y^3x^2 + y^4$

$g = x^2 + sy = x^2 + 4647714815446351873y$

$B = 2000$

h	$1/\zeta_{K_h}(2)$	$\alpha(h, f, B)$	$\alpha(h, g, B)$	$\alpha_f + \alpha_g$
$y^6 + y^5 - y^2 - y - 1$	0.953	2.042	2.479	4.521
$y^6 - y^4 + y^3 + y^2 - 1$	0.917	1.288	1.740	3.028
$y^6 + y^3 + y^2 - y - 1$	0.917	2.419	2.876	5.295
$y^6 + y^5 - y^3 + y - 1$	0.909	0.278	2.357	2.636
$y^6 + y^5 + y^4 + y^3 + y^2 + y - 1$	0.883	2.341	2.033	4.374
$y^6 + y^4 + y^3 + y - 1$	0.867	0.899	2.526	3.425
$y^6 + y^4 + y^2 + y + 1$	0.836	1.955	1.141	3.095
$y^6 + y^5 + y^2 - y + 1$	0.763	0.891	1.264	2.155
$y^6 + y^5 - y^4 + y^3 + y^2 + y - 1$	0.756	0.956	1.177	2.133
$y^6 + y^5 + y - 1$	0.736	1.925	2.108	4.032
$y^6 + y^5 + y^3 - y^2 + y - 1$	0.732	1.729	2.099	3.828
$y^6 + y^3 + y - 1$	0.728	−0.250	1.191	0.941
$y^6 + y^3 - y + 1$	0.720	1.605	1.348	2.952
$y^6 + y^3 + y^2 + 1$	0.718	1.151	1.294	2.445
$y^6 - y^4 + y^3 - y^2 - y - 1$	0.710	0.406	2.278	2.684
$y^6 + y^5 - y^3 + y^2 - y + 1$	0.697	1.572	0.818	2.390
$y^6 + y^4 + y + 1$	0.679	1.319	1.683	3.002



Simul. in $\mathbb{F}_{p^{12}}$, BN, STNFS deg $h = 6, 4$
Simul. in $\mathbb{F}_{p^{12}}$, BLS12, STNFS deg $h = 12, 6$
$L^0_{p^n}(1/3, 1.923)/2^{10.17}$ (DL theoretical re-scaled DL-240dd $\leftrightarrow 2^{67.51}$)
$L^0_{p^n}(1/3, 1.526)/2^{4.5}$ (SNFS theoretical re-scaled SDL-1024 $\leftrightarrow 2^{64.4}$)
$\log_2$ cost
192
176
160
144
128
112
96
80
64
$\log_2 p^n$
1024
3072
4968
6144
9216
12288

Numerical example: BLS12-446 bits

$p(x) = (x-1)^2(x^4 - x^2 + 1)/3 + x$
$r(x) = x^4 - x^2 + 1$
$s = -(2^{74} + 2^{73} + 2^{63} + 2^{57} + 2^{50} + 2^{17} + 1)$
seed with `enumerate_sparse_T.sage` [GMT20]
`https://gitlab.inria.fr/smasson/cocks-pinch-variant`
$p = p(s)$ of 446 bits, twist-secure curve
p^k 5352 bits
$h = Y^6 - Y^4 + Y^3 - Y + 1$

$f_y = X^{12} - 2yX^{10} + 2y^3X^6 + y^5X^2 + y^4 - y^3 + y - 1$
$g_y = X^2 - uy = X^2 + 283435675103427088875 53y$
$A = 968,\ B = 2^{68.2}$

Estimated cost: $\approx 2^{132}$

Differences

- Barbulescu–Duquesne [BD19] (curve name, prime field GF(p) bitzise):
  - BN-462 (p^{12}: 5544 bits), BLS12-461 (p^{12}: 5532 bits) for the 128-bit security level
  - BLS24-559 (p^{24} 13416 bits) for the 192-bit security level
- Guillevic–Singh [GS21]:
  - BN-446, BLS12-446 (p^{12} 5352 bits), 64-bit machine-word aligned
  - BLS24-509 (p^{12} 12216 bits)

Differences

- Barbulescu–Duquesne [BD19] (curve name, prime field GF(p) bitzise):
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  - BLS24-559 (p^{24} 13416 bits) for the 192-bit security level
- Guillevic–Singh [GS21]:
  - BN-446, BLS12-446 (p^{12} 5352 bits), 64-bit machine-word aligned
  - BLS24-509 (p^{12} 12216 bits)
- shorter p bitsize, one 64-bit machine-word less $\rightarrow$ faster $\mathbb{F}_p$-multiplication, ratio of $(2s^2 + s)/(2s_0^2 + s_0)$, $s = \lceil p/64 \rceil$ [AFK$^+$13, Sect. 8]
  462-bit $\rightarrow$ 446-bit: $\mathbf{m}_{446} = 0.77\mathbf{m}_{462}$
  559-bit $\rightarrow$ 509-bit: $\mathbf{m}_{509} = 0.8\mathbf{m}_{559}$
- faster pairing, faster group operations, shorter keysizes

Differences

Keysize recommendation difference:
[BD19] assumes there exists *optimal* polynomial h
and the attacker knows how to select it

BLS24
There exists $h(y)$ of degree 24 such that

- $\|h\|_\infty = 1$ i.e. $h_i \in \{0, 1, -1\}$
- h irreducible mod p of a BLS24 curve
- h has cyclic Galois group of order 24

Open problem: *Does it exist? How to find such $h(y)$?*
Ideas are welcome

Ongoing work

Active branches

automorphisms

fab46aea · taking into account special automorphisms for cyclotomic polynomials h. Tested... · 3 weeks ago

master default protected

378f61dd · comment on BLS24 seeds · 1 month ago

Ongoing work

Finding curve seeds of low Hamming weight

```
sage -python -m tnfs.gen.generate_sparse_curve --bls \
    -k 24 -r 254 256 --2NAF --find_all_w_up_to -w 4
cat \
test_vector_sparse_bls24_rnbits_254_256_u_1_4_mod_6_unbits_33_Hw2naf_6.py
test_vector_sparse_bls24 = [
    {'u':-0xeffff000, ... 'label':"-2^32+2^28+2^12 Hw2naf 3"},
```

With high 2-valuation of $p-1$ and $r-1$ for Youssef El Housni

```
sage -python -m tnfs.gen.compute_test_vector_curve --bls \
    -k 24 -r 254 256 --find_all_u --valuation 16
cat \
test_vector_bls24_rnbits_254_256_val2_16_r_prime_pos_u__u_1_4_mod_6.py
# BLS24 curves with seed u = [1, 4] mod 6 s.t. r has 254 to 256 bits
test_vector_BLS24 = [
  {'u':0xe19c0001, 'u_mod_4':1, 'b': 1, 'pnbits':317,'rnbits':255, \
```

Previous work: 128-bit security level

Webpage at
`https://members.loria.fr/AGuillevic/pairing-friendly-curves/`

k	curve	seed	$\log_2 Q$	$\log_2 r$	ρ	bit sec. $GF(p^k)$
		Curves with fast pairing				
12	BN-382	$-(2^{94}+2^{78}+2^{67}+2^{64}+2^{48}+1)$	382	382	1.0	123
12	BN-446	$2^{110}+2^{36}+1$	446	446	1.0	132
12	BLS12-381	$-(2^{63}+2^{62}+2^{60}+2^{57}+2^{48}+2^{16})$	381	254	1.5	126
12	BLS12	see gitlab	440–448	295–300	1.5	132
		Curves with smallest possible $\mathbb{G}_1$ [CDS20]				
13	`BW13-P310`	-0x8b0=-2224	310	267	1.167	140
19	`BW19-P286`	-0x91=-145	286	259	1.111	160
		Curves for SNARK $2^L \mid p-1,\ r-1$				
12	BLS12-377	$2^{63}+2^{58}+2^{56}+2^{51}+2^{47}+2^{46}+1$	377	252	1.5	126
24	BLS24-315	$-2^{32}+2^{30}+2^{22}-2^{20}+1$	315	253	1.25	160

Choosing curves: criteria

- $384 \leq \log_2 r$ for the 192-bit security level
- $12 \leq k$
- adjust $\rho = \log_2 p / \log_2 r$

Lessons learned from the 128-bit short list:

- Too many curve families
- High degree twist is important for fast pairing
- Best curve choice varies from use-cases

Our choices:

→ restrict to $j = 0$ and $3 \mid k$, $6 \mid k$
or $j = 1728$ and $4 \mid k$

→ ρ varies up to 2 (Fotiadis et al. [FK19])

Pre-selected curves

k	curve	seed	$\log p$	$\log r$	ρ	$\log p^k$	secu
16	KSS16	$2^{78} - 2^{76} - 2^{28} + 2^{14} + 2^{7} + 1$	766	605	1.25	12256	194
	FM23	$2^{48} - 2^{44} - 2^{38} + 2^{31}$	765	384	2	12240	196
	AFG16	$-(2^{48} - 2^{44} + 2^{37})$	765	384	2	12240	196
18	KSS18	$2^{80} + 2^{77} + 2^{76} - 2^{61} - 2^{53} - 2^{14}$	638	474	1.33	11484	193
	SG18	$-(2^{63} + 2^{54} + 2^{16})$	638	383	1.66	11484	187
	FM25	$-2^{64} + 2^{33} + 2^{30} + 2^{20} + 1$	768	384	2	13824	197
20	FST 6.4	$-2^{56} + 2^{44} + 1$	670	448	1.5	13400	193
	SG20	$-2^{47} - 2^{45} + 2^{15} + 2^{13}$	670	383	1.75	13400	203
	GG20b	$2^{49} + 2^{46} - 2^{41} + 2^{35} + 2^{30} - 1$	575	379	1.52	11500	196

small $\mathbb{G}_1$

k	curve	seed	$\log p$	$\log r$	ρ	$\log p^k$	secu
21	BLS21	$-2^{32} + 2^{25} + 2^{6} + 2$	511	384	1.33	10731	199
24	BLS24	$-2^{51} - 2^{28} + 2^{11} - 1$	509	409	1.25	12216	193
27	BLS27	$-2^{21} - 2^{19} - 2^{15} + 2^{10} + 2^{4} + 2^{2} + 1$	426	383	1.11	11529	218
28	FST 6.4	$2^{32} - 2^{25} + 2^{22} + 2^{15} + 1$	510	384	1.33	14280	209

Why varying $\rho = \log_2 p / \log_2 r$?

Pairing computation $e(P, Q)$: Miller loop + final exponentiation to $(p^k - 1)/r$

Miller loop: evaluate a function $f_{m,P}$ at point Q [Jou04, Ver10]
Contains a scalar multiplication

$$[m]P \text{ where } \log_2 m \approx \frac{\log_2 r}{\varphi(k)} = \frac{log_2 r}{\deg \Phi_k}$$

Φ_k the k-th cyclotomic polynomial
SageMath: `euler_phi(k)`
$\varphi(12) = 4$, $\varphi(16) = 8$, $\varphi(18) = 6$, $\varphi(20) = 8$, $\varphi(24) = 8$

At fixed k, reducing r gives a faster Miller loop

Pairing: Miller loop and final exponentiation

Algorithm 6.1: $\textsc{MillerFunction}(u, P, Q)$

Input: $E, \mathbb{F}_p, \mathbb{F}_{p^k}$, k even, $P \in E(\mathbb{F}_p)[r]$, $Q \in E(\mathbb{F}_{p^k})[r]$ in affine coord.,
$\pi_p(Q) = [p]Q$, $c \in \mathbb{N}$.

Result: $f = f_{c,Q}(P)$

$f \leftarrow 1$; $R \leftarrow Q$;
for *b from the second most significant bit of c to the least* **do**
  $\ell_0 \leftarrow \ell_{R,R}(P)$; $R \leftarrow [2]R$; // Dbl step, tangent line
  $f \leftarrow f^2$; // $\mathbf{s}_k$
  if $b = 1$ **then**
    $\ell_1 \leftarrow \ell_{R,Q}(P)$; $R \leftarrow R + Q$; // Add step, chord line
    $f \leftarrow f \cdot (\ell_0 \cdot \ell_1)$; // $\mathbf{m}_k +$ sparse-sparse-$\mathbf{m}_k$
  else
    $f \leftarrow f \cdot \ell_0$; // full-sparse-$\mathbf{m}_k$
return f;

Pairing: Miller loop and final exponentiation

Raise to

$$\frac{p^k - 1}{r} = \underbrace{\frac{q^k - 1}{\Phi_k(q)}}_{\text{easy}} \underbrace{\frac{\phi_k(q)}{r}}_{\text{hard}}$$

1st comparison: timing estimates in $\mathbb{F}_p$-multiplications

Estimate the number of multiplications **m** in $\mathbb{F}_p$ needed for

- $\mathbf{m}_k$ multiplication in $\mathbb{F}_{p^k}$
- $\mathbf{s}_k$ squaring in $\mathbb{F}_{p^k}$
- $\mathbf{f}_k$ Frobenius power $x \mapsto x^p$ in $\mathbb{F}_{p^k}$
- $\mathbf{i}_k$ inversion in $\mathbb{F}_{p^k}$
- $\mathbf{s}_k^{\text{cyclo}}$ squaring in the cyclotomic subgroup of $\mathbb{F}_{p^k}^*$ of order $\Phi_k(q)$ (subgroup of norm 1, inversion is free)

Relative cost: multiplication $\mathbf{m}_k$ squaring $\mathbf{s}_k$ Frobenius $\mathbf{f}_k$ inversion $\mathbf{i}_k$ $\mathbb{F}_{p^k}$

k	$\mathbf{m}_k$	$\mathbf{s}_k$	$\mathbf{f}_k$	$\mathbf{s}_k^{\text{cyclo}}$	$\mathbf{i}_k - \mathbf{i}_1$	$\mathbf{i}_k,\ \mathbf{i}_1 = 25\mathbf{m},\ \mathbf{s} = \mathbf{m}$
1	$\mathbf{m}$	$\mathbf{s}$	0		0	$25\mathbf{m}$
2	$3\mathbf{m}$	$2\mathbf{m}$	0	$2\mathbf{s}$	$2\mathbf{m} + 2\mathbf{s}$	$29\mathbf{m}$
3	$6\mathbf{m}$	$2\mathbf{m} + 3\mathbf{s}$ [CH07]	$2\mathbf{m}$		$9\mathbf{m} + 3\mathbf{s}$	$37\mathbf{m}$
4	$9\mathbf{m}$	$2\mathbf{m}_2 = 6\mathbf{m}$	$2\mathbf{m}$	$2\mathbf{s}_2 = 4\mathbf{m}$	$12\mathbf{m} + 2\mathbf{s}$	$39\mathbf{m}$
5	$13\mathbf{m}$	$13\mathbf{s}$ [Mon05]	$4\mathbf{m}$		$48\mathbf{m}$	$73\mathbf{m}$
6	$18\mathbf{m}$	$2\mathbf{m}_2 + 3\mathbf{s}_2 = 12\mathbf{m}$	$4\mathbf{m}$	$6\mathbf{m}$ [GS10]	$34\mathbf{m}$	$59\mathbf{m}$
7	$22\mathbf{m}$	$22\mathbf{s}$	$6\mathbf{m}$		$104\mathbf{m}$	$129\mathbf{m}$
12	$54\mathbf{m}$	$2\mathbf{m}_6 = 36\mathbf{m}$	$10\mathbf{m}$	$6\mathbf{m}_2 = 18\mathbf{m}$	$97\mathbf{m}$	$119\mathbf{m}$
16	$81\mathbf{m}$	$2\mathbf{m}_8 = 54\mathbf{m}$	$14\mathbf{m}$	$2\mathbf{s}_8 = 36\mathbf{m}$	$134\mathbf{m}$	$159\mathbf{m}$
18	$108\mathbf{m}$	$2\mathbf{m}_9 = 72\mathbf{m}$	$16\mathbf{m}$	$6\mathbf{m}_3 = 36\mathbf{m}$	$232\mathbf{m}$	$257\mathbf{m}$
20	$117\mathbf{m}$	$2\mathbf{m}_{10} = 78\mathbf{m}$	$18\mathbf{m}$	$2\mathbf{s}_{10} = 52\mathbf{m}$	$255\mathbf{m}$	$280\mathbf{m}$
21	$132\mathbf{m}$	$110\mathbf{m}$	$20\mathbf{m}$		$393\mathbf{m}$	$418\mathbf{m}$
24	$162\mathbf{m}$	$2\mathbf{m}_{12} = 108\mathbf{m}$	$22\mathbf{m}$	$6\mathbf{m}_4 = 54\mathbf{m}$	$318\mathbf{m}$	$343\mathbf{m}$
27	$216\mathbf{m}$	$153\mathbf{m}$	$26\mathbf{m}$		$511\mathbf{m}$	$536\mathbf{m}$
28	$198\mathbf{m}$	$132\mathbf{m}$	$26\mathbf{m}$	$88\mathbf{m}$	$437\mathbf{m}$	$462\mathbf{m}$

Estimated cost in $\mathbb{F}_p$-multiplications **m** but p varies

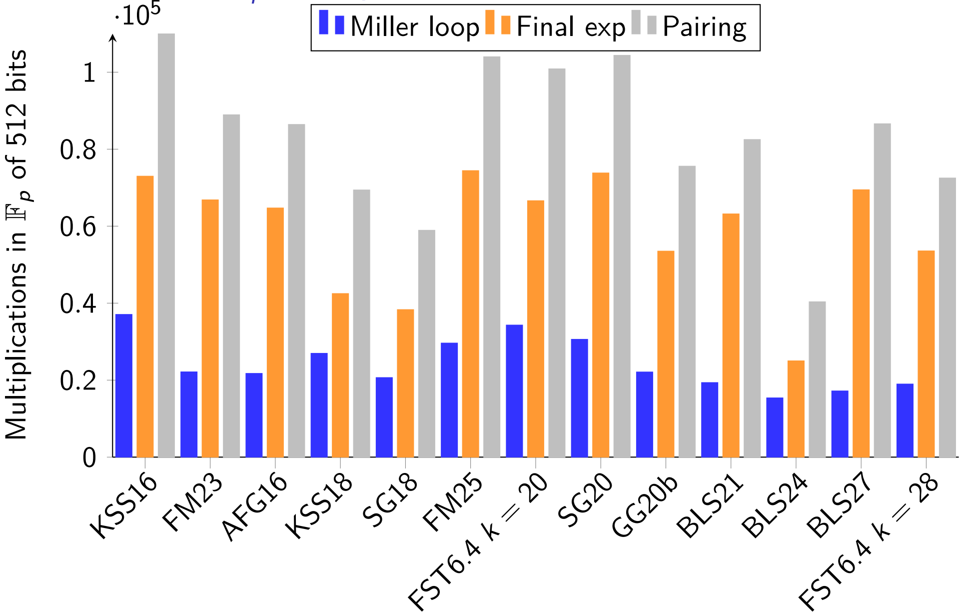
k	curve	p bits	r bits	Miller loop optimal ate	final exp easy	final exp hard	final exp total	pairing total
16	KSS16	766	605	16784**m**	240**m**	32826**m**	33066**m**	49850**m**
	FM23	765	384	10020**m**	255**m**	30024**m**	30279**m**	40299**m**
	AFG16	765	384	9838**m**	255**m**	29067**m**	29322**m**	39160**m**
18	KSS18	638	474	17433**m**	480**m**	27008**m**	27488**m**	44921**m**
	SG18	638	383	13351**m**	480**m**	24308**m**	24788**m**	38139**m**
	FM25	768	384	13410**m**	464**m**	33256**m**	33720**m**	47130**m**
20	FST 6.4	670	448	18416**m**	507**m**	35276**m**	35783**m**	54199**m**
	SG20	670	383	16427**m**	507**m**	39152**m**	39659**m**	56086**m**
	GG20b	575	379	17554**m**	507**m**	42017**m**	42524**m**	60078**m**
21	BLS21	511	384	19321**m**	717**m**	62426**m**	63143**m**	82464**m**
24	BLS24	509	409	15345**m**	658**m**	24310**m**	24968**m**	40313**m**
27	BLS27	426	383	22212**m**	1185**m**	88438**m**	89907**m**	112119**m**
28	FST 6.4	510	384	18940**m**	859**m**	52670**m**	53529**m**	72469**m**

Estimated cost in $\mathbb{F}_p$-multiplications $\mathbf{m}_{512}$, normalized

Rule of thumb Aranha et al. [AFK+13, Sect. 8]

- $\mathbb{F}_p$-elements represented with $\ell = 1 + \lfloor \log_2 p \rfloor$ bits
- packed in $w = \lceil \ell/64 \rceil$ 64-bit machine-words
- Montgomery representation
- $\rightarrow$ $\mathbf{m}$ with reduction in $\mathbb{F}_p$ has complexity $O(2w^2 + w)$
  - $m_{426} = 0.772 m_{512}$
  - $m_{576} = 1.257 m_{512}$
  - $m_{640} = 1.544 m_{512}$
  - $m_{704} = 1.860 m_{512}$
  - $m_{768} = 2.205 m_{512}$

Estimated cost in $\mathbb{F}_p$-multiplications $\mathbf{m}_{512}$, normalized



Other embedding degrees and quadratic twists are not promising

k	curve	p bits	r bits	Miller loop optimal ate	final exp easy	final exp hard	final exp total	pairing total
20	FST 6.4	670	448	18416**m**	507**m**	35276**m**	35783**m**	54199**m**
	SG20	670	383	16427**m**	507**m**	39152**m**	39659**m**	56086**m**
	GG20b	575	379	17554**m**	507**m**	42017**m**	42524**m**	60078**m**
	FST 6.6	527	384	28703**m**	507**m**	37621**m**	38128**m**	66831**m**
22	GG $D = 7$	457	383	41154**m**	789**m**	72352**m**	73141**m**	114295**m**
	FST 6.3	544	420	39707**m**	789**m**	65604**m**	66393**m**	106100**m**
24	BLS24	509	409	15345**m**	658**m**	24310**m**	24968**m**	40313**m**
27	BLS27	426	383	22212**m**	1185**m**	88438**m**	89907**m**	112119**m**

Benchmarks: Timings, clock cycles, RELIC toolkit

https://github.com/relic-toolkit/relic/

Intel Kaby Lake Core i7-7700 CPU machine with 64GB of RAM running single-threaded at 3.6GHz, with Turbo Boost and HT disabled to reduce measurement variability.

k	curve	p bits	r bits	Miller loop optimal ate	final exp	pairing total
16	KSS16	766	605	11855126	26977632	38832758
	AFG16	765	384	7343697	27093958	34443913
18	KSS18	638	474	9327153	15607334	24971803
	SG18	638	383	7135510	13628040	20763550
24	BLS24	509	409	5429826	9670702	15100528

Benchmarks: Timings, clock cycles, RELIC toolkit

https://github.com/relic-toolkit/relic/

Intel Kaby Lake Core i7-7700 CPU machine with 64GB of RAM running single-threaded at 3.6GHz, with Turbo Boost and HT disabled to reduce measurement variability.

Curve	BLS12-381 (ref 128)	KSS16-766	AFG16-765	KSS18-638	SG18-638	BLS24-509
Exp. in $\mathbb{G}_1$	394115	3392210	2210385	1718671	1414755	1066576
Exp. in $\mathbb{G}_2$	843175	18433905	12200469	7450643	6101680	5106474
Exp. in $\mathbb{G}_T$	1202601	13300120	9140193	11748224	9437487	7656674
Hash to $\mathbb{G}_1$	275816	1759828	4269969	1115238	1490162	498829
Hash to $\mathbb{G}_2$	962008	24355512	31849925	8894196	15130315	5804460
Test $P \in \mathbb{G}_1$	254753	3060100	1525930	1808786	1018560	797969
Test $Q \in \mathbb{G}_2$	311478	6880667	2909464	1927367	1663177	1068349
Test $z \in \mathbb{G}_T$	357063	5895541	2280958	2359975	9878582	1294991
Miller Loop	1396749	11855126	7343697	9327153	7135510	5429826
Final Exp	1740115	26977632	27093958	15607334	13628040	9670702
Pairing	3110112	38832758	34443913	24971803	20763550	15100528

Outcomes

- BLS12 is the best at the 128-bit security level
- BLS24 is the best at the 192-bit security level
- Fast Hashing to $\mathbb{G}_1$, $\mathbb{G}_2$, $\mathbb{G}_T$ matters too
- Preprint soon

Thank you.

`https://gitlab.inria.fr/tnfs-alpha/alpha`
`https://gitlab.inria.fr/zk-curves/snark-2-chains`

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Complexities $L_{p^k}(\alpha, c) = \exp\Big((c + o(1))(\ln p^k)^\alpha (\ln \ln p^k)^{1-\alpha}\Big)$

large characteristic $p = L_{p^k}(\alpha_p),\ \alpha_p > 2/3$: $L_{p^k}(1/3, c)$	
$c = (64/9)^{1/3} \simeq 1.923$	NFS
special p:	
$c = (32/9)^{1/3} \simeq 1.526$	SNFS

medium characteristic $p = L_{p^k}(\alpha_p),\ 1/3 < \alpha_p < 2/3$: $L_{p^k}(1/3, c)$	
$c = (96/9)^{1/3} \simeq 2.201$	prime n NFS-HD (Conjugation)
$c = (48/9)^{1/3} \simeq 1.747$	composite n, best case of TNFS: when parameters fit perfectly
special p:	
$c = (64/9)^{1/3} \simeq 1.923$	NFS-HD+Joux–Pierrot'13
$c = (32/9)^{1/3} \simeq 1.526$	composite n, best case of STNFS