Hard Homogeneous Spaces from the Class Field Theory of Imaginary Hyperelliptic Function Fields

JNCF 2022

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Shor's quantum algorithm breaks this problem for any group.

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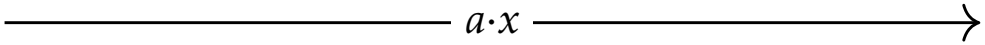

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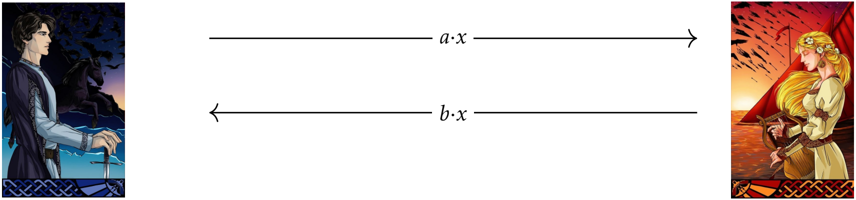

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Quantum attack in $\exp(c\sqrt{\log(\#G)})$ for some $c > 0$ (Kuperberg, 2005).

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CSIDH (Castryck, Lange, Martindale, Panny, Renes, 2018) does better, but the structure of thei G is very hard to compute.

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Two families: ordinary and supersingular	

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- SageMath library for finite Drinfeld modules; we aim for an integration in SageMath.
- Numerical experiments suggest that the inverse problem of the action is hard.

Ore polynomials (1/2)

Fix $\mathbb{F}_q \hookrightarrow L \hookrightarrow \overline{\mathbb{F}_q}$ a finite field extension and

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Definition (Ore, 1933)

The set

$$L\{\tau\} := \left\{ \sum_{0 \leqslant i \leqslant n} a_n \tau^i \mid n \in \mathbb{Z}_{\geqslant 0}, a_i \in L \right\} \subset \mathrm{End}_{\mathbb{F}_q}(\overline{\mathbb{F}_q})$$

is a ring for addition and composition, called the *ring of Ore polynomials*.

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- SageMath implementation by X. Caruso.

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An *$\mathbb{F}_q[X]$-Drinfeld module over L* is an $\mathbb{F}_q$-algebra morphism

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such that

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Efficiently computed (Musleh, Schost, 2019).

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Theorem

If $2 \nmid [L : \mathbb{F}_q]$, the map $(I, \phi) \mapsto I \star \phi$ extends to a group action of

$$\mathrm{Cl}(\mathbb{F}_q[X][Y]/\xi)$$

to

$$\left\{ \mathrm{Isom}_{\overline{\mathbb{F}_q}}(\phi) \mid \mathrm{Rank}(\phi) = 2, \mathrm{CharPol}(\phi) = \xi \right\}.$$

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Assume ξ defines an hyperelliptic curve $\mathcal{H}$.

Representation:

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Input: — A j-invariant $j \in L$.
— Mumford coordinates $(u, v) \in \mathbb{F}_q[X]^2$.

Output: A j-invariant.

$\widetilde{u} \leftarrow u(j^{-1}\tau^2 + \tau + \omega) \in L\{\tau\}$;
$\widetilde{v} \leftarrow v(j^{-1}\tau^2 + \tau + \omega) \in L\{\tau\}$;
$\iota \leftarrow \mathrm{rgcd}(\widetilde{u}, \tau_L - \widetilde{v})$;
$\widehat{g} \leftarrow \iota_0^{-q}(\iota_0 + \iota_1(\omega^q - \omega))$;
$\widehat{\Delta} \leftarrow j^{-q^{\deg_\tau(\iota)}}$;
Return $\widehat{g}^{q+1}/\widehat{\Delta}$.

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Our action is realized with $\mathbf{A} = \mathbb{F}_q[X][Y]/\xi$ and $r = 1$.

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- The structure of the group is calculated in 53 hours in our case (Kedlaya-Vercauteren alg.); 52 CPU-years for CSIDH-512 (Beullens, Kleinjung, Vercauteren, 2019).

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