Function field analogue of the CRS key exchange

Journées C2 2022

Antoine Leudière Pierre-Jean Spaenlehauer

INRIA Nancy-Grand Est

Number fields and function fields

Number fields	Function fields
$\mathbb{Z}$	$\mathbb{F}_q[X]$
$\mathbb{Q}$	$\mathbb{F}_q(X)$
Number field (finite ext.)	Function field (finite ext.)

Elliptic curves over $\mathbb{F}_q$	Finite Drinfeld modules
$\mathbb{Z}$-module law on $E(\overline{\mathbb{F}_q})$	$\mathbb{F}_q[X]$-module law on $\overline{\mathbb{F}_q}$
Any finite $\mathbb{Z}$-module gives rise to an isogeny	Any finite sub-$\mathbb{F}_q[X]$-module of $\overline{\mathbb{F}_q}$ (+ technical condition) gives rise to an isogeny
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Couveignes showed (1996) that the class field theory of number fields provided a unifying vision for:

- the DLP on multiplicative groups $\mathbb{F}_q^\times$,
- the DLP on elliptic curves $E(\mathbb{F}_q)$,
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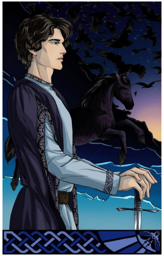
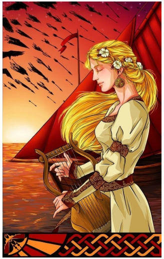
Tristan and Isolde create a private key on a public channel. They choose an abelian group G acting (freely and transitively) on a set X, with an element $x \in X$.





Secure if hard to compute $ab \cdot x$ knowing $x, a \cdot x$ and $b \cdot x$.

Generalizes Diffie-Hellman on a cyclic group H: $G = \mathbb{Z}/\#H\mathbb{Z}$, $X = H$.

CRS and CSIDH are built as hard homogeneous spaces.

Quantum attack in $\exp(c\sqrt{\log(\#G)})$ for some $c > 0$ (Kuperberg, 2005; Bonnetain, Schrottenloher, 2020).

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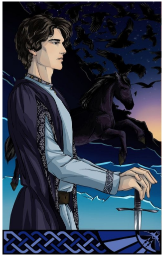

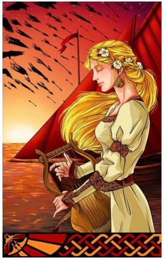

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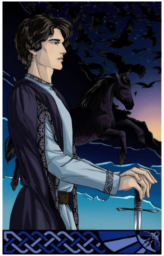

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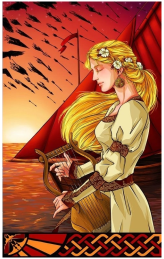

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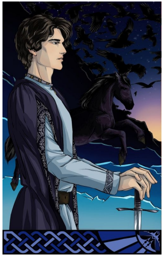

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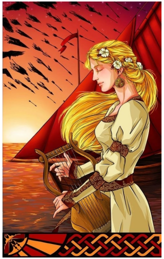

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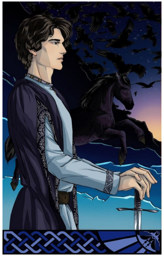

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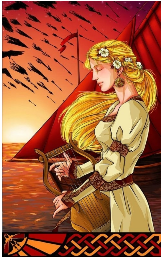

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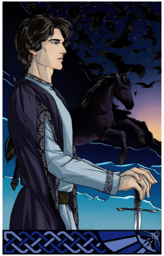

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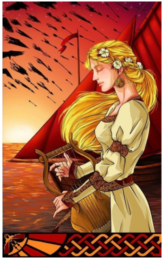

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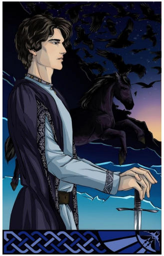

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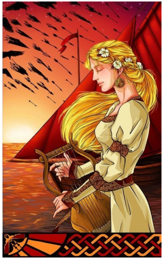

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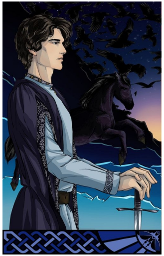

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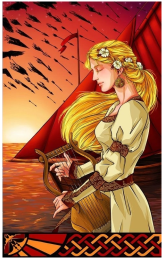

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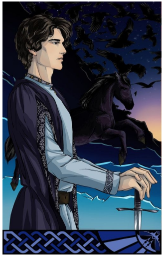

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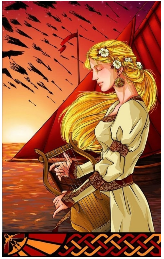

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In ia.cr/2022/349:

- Construction of the function field analogue of CRS.
- Efficient C++ implementation.
- Reduction of the security to the isogeny finding problem.
- Enhancements on the analysis of the recursive algorithm to find isogenies (Joux, Narayanan, 2019; Caranay, Greenberg, Scheidler, 2020).

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Main result

Let $L/\mathbb{F}_q$ be a finite extension with odd degree. Let $\mathcal{H}$ be an imaginary hyperelliptic curve on $\mathbb{F}_q$. Let $\mathbf{A}_{\mathcal{H}}$ be the ring of function of $\mathcal{H}$ regular outside ∞.

Theorem

There is an explicit and computable group action of $\mathrm{Pic}^0(\mathcal{H}) \simeq \mathrm{Cl}(\mathbf{A}_{\mathcal{H}})$ to the set of $\overline{\mathbb{F}_q}$-isomorphism classes of rank 1 $\mathbf{A}_{\mathcal{H}}$-Drinfeld modules defined over L.

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Ore polynomials

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$$L\{\tau\} := \left\{ \sum_{0 \leqslant i \leqslant n} a_n \tau^i \mid n \in \mathbb{Z}_{\geqslant 0}, a_i \in L \right\} \subset \mathrm{End}_{\mathbb{F}_q}(\overline{\mathbb{F}_q})$$

Properties:

- $L\{\tau\}$ is non commutative: $\tau a = a^q \tau, \quad \forall a \in L$.
- $L\{\tau\}$ is left-euclidean, hence notion of rgcd.
- SageMath implementation by X. Caruso.

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Let $L/\mathbb{F}_q$ be a finite extension with odd degree.

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One function field analogue: CRS (3/3)

Representation:

- isomorphism classes of Drinfeld modules are represented by a j-invariant,
- points in $\mathrm{Pic}^0(\mathcal{H})$ are represented by Mumford coordinates.

Input: — A j-invariant $j \in L$.
— Mumford coordinates $(u, v) \in \mathbb{F}_q[X]^2$.

Output: A j-invariant.

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// ω is a global constant
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$\widetilde{u} \leftarrow u(j^{-1}\tau^2 + \tau + \omega) \in L\{\tau\}$;

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$\iota \leftarrow \mathrm{rgcd}(\widetilde{u}, \tau^{[L:\mathbb{F}_q]} - \widetilde{v})$;

$\widehat{g} \leftarrow \iota_0^{-q}(\iota_0 + \iota_1(\omega^q - \omega))$;

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Return $\widehat{g}^{q+1}/\widehat{\Delta}$.

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How secure is it? Not that much...

The security of the protocol reduces to the problem of finding an isogeny between two isogenous Drinfeld modules.

Previous work (Joux, Narayanan, 2019; Caranay, Greenberg, Scheidler, 2020) solve a recursive equation by exploring a research tree with exponential size (in the degree of the desired isogeny). We studied this algorithm and heuristically concluded that it ran in exponential time.

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Recap of the situation

Number fields		Function fields	
Problem	**Security**	**Problem**	**Security**
DLP on $\mathbb{F}_q^\times$	Broken in small characteristic	*Same problem!*	
DLP on $E(\mathbb{F}_q)$	Secure	Analogue with Drinfeld modules	Broken (Scanlon, 1999)
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Function fields / Drinfeld modules analogues of elliptic curve isogeny-based cryptosystems presented here seem very well broken...

It also seems to be the case for CSIDH and SIDH (Joux, Narayanan, 2019).

However, many algorithmic aspects of Drinfeld modules are yet to be explored for cryptographic purposes: higher ranks, abelian varieties...

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