From Admissibility to a New Hierarchy of Unification Types

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Abstract

Motivated by the study of admissible rules, a new hierarchy of "exact" unification types is introduced where a unifier is more general than another unifier if all identities unified by the first are unified by the second. A Ghilardi-style algebraic interpretation of this hierarchy is presented that features exact algebras rather than projective algebras. Examples of equational classes distinguishing the two hierarchies are also provided.

1 Introduction

It has long been recognized that the study of admissible rules is inextricably bound up with the study of equational unification (see, e.g., [23, 10, 11]). Indeed, from an algebraic perspective, admissibility in an equational class (variety) of algebras may be viewed as a generalization of unifiability in that class.¹ Let us fix an equational class of algebras \mathcal{V} for a language \mathcal{L} and denote by $\mathbf{Fm}_{\mathcal{L}}(X)$, the formula algebra (absolutely free algebra or term algebra) of \mathcal{L} over a set of variables $X \subseteq \omega$. A substitution (homomorphism) $\sigma: \mathbf{Fm}_{\mathcal{L}}(X) \to \mathbf{Fm}_{\mathcal{L}}(\omega)$ is called a \mathcal{V} -unifier (over X) of a set of \mathcal{L} -identities Σ with variables in X if $\mathcal{V} \models \sigma(\varphi) \approx \sigma(\psi)$ for all $\varphi \approx \psi$ in Σ . A clause $\Sigma \Rightarrow \Delta$ (an ordered pair of finite sets of \mathcal{L} -identities Σ, Δ) is \mathcal{V} -admissible if every \mathcal{V} -unifier of Σ is a \mathcal{V} -unifier of a member of Δ . In particular, Σ is \mathcal{V} -unifiable if and only if $\Sigma \Rightarrow \emptyset$ is not \mathcal{V} -admissible.

In certain cases, \mathcal{V} -admissibility may also be reduced to \mathcal{V} -unifiability. Suppose that the unification type of \mathcal{V} is at most finitary, meaning that every \mathcal{V} -unifier of a set of \mathcal{L} -identities Σ over the variables in Σ is a substitution instance of one of a finite set S of \mathcal{L} -unifiers of Σ . Then a clause $\Sigma \Rightarrow \Delta$ is \mathcal{V} -admissible if each member of S is an \mathcal{L} -unifier of a member of Δ . If there is an algorithm for determining the finite basis set S for Σ and the equational theory of \mathcal{V} is decidable, then checking \mathcal{V} -admissibility is also decidable. This observation, together with the pioneering work of Ghilardi on equational unification for classes of Heyting and modal algebras [10, 11], has led to a wealth of decidability, complexity, and axiomatization results for admissibility in these classes and corresponding modal and intermediate logics [12, 13, 15, 7, 3, 2, 21, 18].

The success of this approach to admissibility appears to rely on considering varieties with at most finitary unification type. That this is not the case, however, is illustrated by the case of MV-algebras, the algebraic semantics of Lukasiewicz infinite-valued logic. Decidability, complexity, and axiomatization results for admissibility in this class have been established by Jeřábek [16, 17, 18] via a similar reduction of finite sets of identities to finite approximating sets of identities. On the other hand, it has been shown by Marra and Spada [20] that the variety of MV-algebras has nullary unification type, which means in particular that there are finite sets of identities for which no finite basis of unifiers exists. Further examples of this discrepancy

¹We refer the reader to [4] and [19] for undefined notions of universal algebra and category theory, respectively.

may be found in [6], including the very simple example of the class of distributive lattices where admissibility and validity of clauses coincide but unification is nullary.

As mentioned above, it is possible to check the \mathcal{V} -admissibility of a clause $\Sigma \Rightarrow \Delta$ by checking that every \mathcal{V} -unifier of Σ in a certain "basis set" \mathcal{V} -unifies Δ . Such a basis set S typically has the property that every other \mathcal{V} -unifier of Σ is a substitution instance of a member of S. The starting point for this paper is the observation that a weaker condition on S suffices, leading potentially to smaller sets. What is really required for checking admissibility is the property that every \mathcal{V} -unifier of Σ \mathcal{V} -unifies all identities \mathcal{V} -unified by some member of S. Then $\Sigma \Rightarrow \Delta$ is \mathcal{V} -admissible if each member of S is a \mathcal{V} -unifier of a member of Δ . This leads to a new ordering of \mathcal{V} -unifiers and hierarchy of exact (unification) types. Moreover, we obtain a Ghilardi-style algebraic characterization making use of exact algebras rather than projective algebras. Crucially, we also show that an equational class can have an exact type that is "better than" its unification type. For example, MV-algebras have finitary exact type and distributive lattices have unitary exact type.

2 Equational Unification and Projective Algebras

Let us first recall some basic notions for equational unification, referring to [1] for further details. We then provide a short overview of the algebraic approach to equational unification developed by Ghilardi in [9].

Let $\mathbf{P} = \langle P, \leq \rangle$ be a preordered set. A *complete* set for \mathbf{P} is a subset $M \subseteq P$ such that for every $x \in P$, there exists $y \in M$ such that $x \leq y$. A complete set M for \mathbf{P} is called a μ -set for \mathbf{P} if $x \not\leq y$ and $y \not\leq x$ for all distinct $x, y \in M$. It is easily seen that if \mathbf{P} has a μ -set, then every μ -set of \mathbf{P} has the same cardinality. \mathbf{P} is said to be *nullary* if it has no μ -sets (type(\mathbf{P}) = 0), *infinitary* if it has a μ -set of infinite cardinality (type(\mathbf{P}) = ∞), *finitary* if it has a finite μ -set of cardinality greater than 1 (type(\mathbf{P}) = ω), and *unitary* if it has a μ -set of cardinality 1 (type(\mathbf{P}) = 1). These types are ordered as follows: $1 < \omega < \infty < 0$.

Now let \mathcal{L} be a language and $X \subseteq \omega$ a set of variables, and consider substitutions $\sigma_i \colon \mathbf{Fm}_{\mathcal{L}}(X) \to \mathbf{Fm}_{\mathcal{L}}(\omega)$ for i = 1, 2. We say that σ_1 is more general than σ_2 (written $\sigma_2 \preccurlyeq \sigma_1$) if there exists a substitution $\sigma' \colon \mathbf{Fm}_{\mathcal{L}}(\omega) \to \mathbf{Fm}_{\mathcal{L}}(\omega)$ such that $\sigma' \circ \sigma_1 = \sigma_2$. Let \mathcal{V} be an equational class of algebras for \mathcal{L} and Σ a finite set of \mathcal{L} -identities with variables denoted by $\operatorname{Var}(\Sigma)$. Then $\mathsf{U}_{\mathcal{V}}(\Sigma)$ is defined as the set of \mathcal{V} -unifiers of Σ over $\operatorname{Var}(\Sigma)$ preordered by \preccurlyeq . For $\mathsf{U}_{\mathcal{V}}(\Sigma) \neq \emptyset$, the \mathcal{V} -unification type of Σ is defined as type $(\mathsf{U}_{\mathcal{V}}(\Sigma))$. The unification type of \mathcal{V} is the maximal type of a \mathcal{V} -unifiable finite set Σ of \mathcal{L} -identities.

We now recall Ghilardi's algebraic account of equational unification. Let $\mathbf{F}_{\mathcal{V}}(X)$ denote the free algebra of \mathcal{L} over a set of variables X and let $h_{\mathcal{V}}: \mathbf{Fm}_{\mathcal{L}}(X) \to \mathbf{F}_{\mathcal{V}}(X)$ be the canonical homomorphism. Given a finite set of \mathcal{L} -identities Σ and a finite $X \supseteq \operatorname{Var}(\Sigma)$, we denote by $\operatorname{Fp}_{\mathcal{V}}(\Sigma, X)$ the algebra in \mathcal{V} finitely presented by Σ and X: that is, the quotient algebra $\mathbf{F}_{\mathcal{V}}(X)/\Theta_{\Sigma}$ where Θ_{Σ} is the congruence generated by $\{(h_{\mathcal{V}}(\varphi), h_{\mathcal{V}}(\psi)): \varphi \approx \psi \in \Sigma\}$. The class of finitely presented algebras in \mathcal{V} is denoted by $\operatorname{FP}(\mathcal{V})$.

Given $\mathbf{A} \in \mathsf{FP}(\mathcal{V})$, a homomorphism $u: \mathbf{A} \to \mathbf{B}$ is called a *unifier* for \mathbf{A} if $\mathbf{B} \in \mathsf{FP}(\mathcal{V})$ is projective in \mathcal{V} (i.e., there exist homomorphisms $i: \mathbf{B} \to \mathbf{F}_{\mathcal{V}}(\omega)$ and $j: \mathbf{F}_{\mathcal{V}}(\omega) \to \mathbf{B}$ such that $j \circ i$ is the identity map on B). Let $u_i: \mathbf{A} \to \mathbf{B}_i$ for i = 1, 2 be unifiers for \mathbf{A} . Then u_1 is more general than u_2 , written $u_2 \leq u_1$, if there exists a homomorphism $f: \mathbf{B}_1 \to \mathbf{B}_2$ such that $f \circ u_1 = u_2$. Let $\mathsf{U}_{\mathcal{V}}(\mathbf{A})$ be the set of unifiers of \mathbf{A} preordered by \leq . For $\mathsf{U}_{\mathcal{V}}(\mathbf{A}) \neq \emptyset$, the unification type of \mathbf{A} in \mathcal{V} is defined as type $(\mathsf{U}_{\mathcal{V}}(\mathbf{A}))$ and the algebraic unification type of \mathcal{V} is the maximal type of \mathbf{A} in $\mathsf{FP}(\mathcal{V})$ such that $\mathsf{U}_{\mathcal{V}}(\mathbf{A}) \neq \emptyset$. In [9], Ghilardi proved that type $(\mathsf{U}_{\mathcal{V}}(\Sigma))$ coincides with type $(\mathsf{U}_{\mathcal{V}}(\mathsf{Fp}_{\mathcal{V}}(\Sigma, \mathsf{Var}(\Sigma))))$, for each \mathcal{V} -unifiable finite set of identities Σ . Hence the algebraic unification type of \mathcal{V} coincides with the unification type of \mathcal{V} .

3 A New Hierarchy

Let us begin by pointing out the relevance of finitely presented algebras for admissibility. We freely identify \mathcal{L} -identities with pairs of \mathcal{L} -formulas and recall that the *kernel* of a homomorphism $h: \mathbf{A} \to \mathbf{B}$ is defined as $\ker(h) = \{(a, b) \in A^2 : h(a) = h(b)\}.$

Lemma 1. The following are equivalent for any \mathcal{L} -clause $\Sigma \Rightarrow \Delta$ with $X = \operatorname{Var}(\Sigma \cup \Delta)$:

- (1) $\Sigma \Rightarrow \Delta$ is admissible in \mathcal{V} .
- (2) If $\sigma \colon \mathbf{Fm}_{\mathcal{L}}(X) \to \mathbf{Fm}_{\mathcal{L}}(\omega)$ satisfies $\Sigma \subseteq \ker(h_{\mathcal{V}} \circ \sigma)$, then $\Delta \cap \ker(h_{\mathcal{V}} \circ \sigma) \neq \emptyset$.

Let X be a set of variables and let $\sigma_i \colon \mathbf{Fm}_{\mathcal{L}}(X) \to \mathbf{Fm}_{\mathcal{L}}(\omega)$ be substitutions for i = 1, 2. We write $\sigma_2 \sqsubseteq \sigma_1$ if all identities \mathcal{V} -unified by σ_1 are \mathcal{V} -unified by σ_2 . More precisely:

$$\sigma_2 \sqsubseteq \sigma_1 \quad \Leftrightarrow \quad \ker(h_{\mathcal{V}} \circ \sigma_1) \subseteq \ker(h_{\mathcal{V}} \circ \sigma_2).$$

Observe immediately that $\sigma_2 \preccurlyeq \sigma_1$ implies $\sigma_2 \sqsubseteq \sigma_1$.

Given an equational class of algebras \mathcal{V} for \mathcal{L} and a finite set Σ of \mathcal{L} -identities, $\mathsf{E}_{\mathcal{V}}(\Sigma, X)$ is defined as the set of \mathcal{V} -unifiers of Σ over $X \supseteq \operatorname{Var}(\Sigma)$ preordered by \sqsubseteq . For $X = \operatorname{Var}(\Sigma)$, we simply write $\mathsf{E}_{\mathcal{V}}(\Sigma)$ instead of $\mathsf{E}_{\mathcal{V}}(\Sigma, X)$.

We define the exact type of Σ in \mathcal{V} to be type($\mathsf{E}_{\mathcal{V}}(\Sigma)$)) (for $\mathsf{E}_{\mathcal{V}}(\Sigma) \neq \emptyset$). Note that, because $\sigma_2 \preccurlyeq \sigma_1$ implies $\sigma_2 \sqsubseteq \sigma_1$, every complete set for $\mathsf{U}_{\mathcal{V}}(\Sigma)$ is also a complete set for $\mathsf{E}_{\mathcal{V}}(\Sigma)$. Therefore, if type($\mathsf{U}_{\mathcal{V}}(\Sigma)$) $\in \{1, \omega\}$, we have

$$\operatorname{type}(\mathsf{E}_{\mathcal{V}}(\Sigma)) \leq \operatorname{type}(\mathsf{U}_{\mathcal{V}}(\Sigma)).$$

We observe also that the choice of $\mathsf{E}_{\mathcal{V}}(\Sigma) = \mathsf{E}_{\mathcal{V}}(\Sigma, \operatorname{Var}(\Sigma))$ to define the exact type of Σ , is not restrictive; that is, for each $X \supseteq \operatorname{Var}(\Sigma)$,

$$type(\mathsf{E}_{\mathcal{V}}(\Sigma)) = type(\mathsf{E}_{\mathcal{V}}(\Sigma, X)).$$

Using Lemma 1, we obtain the desired relationship with admissibility: namely, to check the \mathcal{V} -admissibility of an \mathcal{L} -clause $\Sigma \Rightarrow \Delta$, it suffices to find a complete set (preferably a μ -set) S for $\bigcup_{\mathcal{V}}(\Sigma)$ then check that each $\sigma \in S$ is a \mathcal{V} -unifier of some \mathcal{L} -identity in Δ .

Let us now give the algebraic characterization of exact unification. We call an algebra \mathbf{E} exact in \mathcal{V} if it is isomorphic to a finitely generated subalgebra of $\mathbf{F}_{\mathcal{V}}(\omega)$ (see also [8] for a syntactic characterization). Given $\mathbf{A} \in \mathsf{FP}(\mathcal{V})$, an onto homomorphism $u: \mathbf{A} \to \mathbf{E}$ is called a coexact unifier for \mathbf{A} if \mathbf{E} is exact. Coexact unifiers are ordered in the same way as algebraic unifiers, that is, if $u_i: \mathbf{A} \to \mathbf{E}_i$ for i = 1, 2 are coexact unifiers for \mathbf{A} , then $u_1 \leq u_2$, if there exists a homomorphism $f: \mathbf{E}_1 \to \mathbf{E}_2$ such that $f \circ u_1 = u_2$.

Let $\mathsf{EU}_{\mathcal{V}}(\mathbf{A})$ be the set of coexact unifiers for \mathbf{A} preordered by \leq . If $\mathsf{EU}_{\mathcal{V}}(\mathbf{A}) \neq \emptyset$, then the *exact type* of \mathbf{A} is defined as the type of $\mathsf{EU}_{\mathcal{V}}(\mathbf{A})$. We obtain the following Ghilardi-style result.

Theorem 2. Let \mathcal{V} be an equational class and Σ a finite set of \mathcal{V} -unifiable \mathcal{L} -identities. Then for any $X \supseteq \operatorname{Var}(\Sigma)$,

$$type(\mathsf{E}_{\mathcal{V}}(\Sigma)) = type(\mathsf{E}_{\mathcal{V}}(\Sigma, X)) = type(\mathsf{E}\mathsf{U}_{\mathcal{V}}(\operatorname{Fp}_{\mathcal{V}}(\Sigma, X)).$$

| Class of Algebras | Unification Type | Exact Type |
|-------------------------------|------------------|-----------------------|
| Boolean Algebras | Unitary | Unitary |
| Heyting Algebras | Finitary | Finitary |
| Semigroups | Infinitary | Infinitary or Nullary |
| Modal algebras | Nullary | Nullary |
| Distributive Lattices | Nullary | Unitary |
| Stone Algebras | Nullary | Unitary |
| Bounded Distributive Lattices | Nullary | Finitary |
| Idempotent Semigroups | Nullary | Finitary |
| MV-algebras | Nullary | Finitary |

Table 1: Comparison of unification types and exact types

We define the *exact unification type of* \mathcal{V} to be the maximal exact type of a \mathcal{V} -unifiable finite set Σ of \mathcal{L} -identities. Similarly, the *exact algebraic unification type* of \mathcal{V} is the maximal exact type of \mathbf{A} in \mathcal{V} such that $\mathsf{EU}_{\mathcal{V}}(\mathbf{A}) \neq \emptyset$. By Theorem 2, the exact unification type and the exact algebraic unification type of \mathcal{V} coincide.

The close connection between coexact unifiers and congruences has as a byproduct that if a finitely presented algebra **A** has a finite set of congruences, then type($\mathsf{E}_{\mathcal{V}}(\Sigma)$) is 1 or ω . Hence we obtain the following useful corollary.

Corollary 3. If \mathcal{V} is a locally finite variety, then \mathcal{V} has exact unification type 1 or ω .

4 Examples

Any unitary equational class such as the class of Boolean algebras also has exact unitary type, and any finitary equational class will have unitary or finitary exact type. For example, the class of Heyting algebras is finitary [10] and hence also has finitary exact type: consider the identity $x \lor y \approx \top$ and unifiers σ_1 with $\sigma_1(x) = \top$, $\sigma_1(y) = y$ and σ_2 with $\sigma_2(x) = x$, $\sigma_2(y) = \top$.

Minor changes to the original proofs that the class of semigroups has infinitary unification type [22] and that the class of modal algebras (for the logic K) has nullary unification type [14] establish that the former has infinitary or nullary exact type and the latter has nullary exact type. However, the class of distributive lattices, which is known to have nullary unification type [9], has unitary exact type as all finitely presented distributive lattices are exact. Similarly, the class of Stone algebras has nullary unification type but unitary exact type. The classes of bounded distributive lattices and idempotent semigroups are also both nullary, but because they are locally finite, they have at most – and indeed, it can be shown via suitable cases, precisely – finitary exact type.

In [20] it is proved that the equational class of MV-algebras has nullary unification type. This class is not locally finite so we cannot apply Corollary 3; however, combining results from [17] and [5], we can still prove that MV-algebras have finitary exact type. We observe that because finitely presented MV-algebras admit a presentation of the form { $\alpha \approx \top$ } and [5, Theorem 4.18] proves that every admissible saturated formula ([16, Definition 3.1]) is exact, [17, Theorem 3.8] effectively provides a bound on the exact type of a finitely presented algebra. Note, moreover, that each formula in the admissible saturated approximation (defined in [16, 17]) of a formula

 α corresponds to an exact unifier of the identity $\alpha \approx \top$. Similarly in [6], the current authors present axiomatizations for admissible rules of several locally finite (and hence of finitary exact unification type) equational classes with classical unification type 0. In all these cases a complete description of exact algebras, and the finite exact unification type plays a central (if implicit) role. We therefore expect this approach to be useful for tackling other classes of algebras that have unitary or finitary exact type, independently of their unification type.

These examples are collected in Table 1, noting that we do not know if there are examples of equational classes of (i) finitary unification type that have unitary exact type, (ii) infinitary unification type that have unitary, finitary, or nullary exact type, (iii) nullary unification type that have infinitary exact type.

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