From Admissibility to a New Hierarchy of Unification Types

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Motivation

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- MV-algebras: Nullary type (V. Marra - L. Spada). Axiomatization, decidability and complexity analysis of admissible rules (E. Jeřábek).
- Distributive lattices: Nullary type (S. Ghilardi). Axiomatization and decidability (L.M.C. - G. Metcalfe).

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Framework

Let us fix:

- $\mathcal{L}$:= algebraic language;
- $\mathcal{V}$:= class of $\mathcal{L}$-algebras.

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Let $\mathbf{Fm}_{\mathcal{L}}(X)$ denote the **formula algebra** (also known as term algebra or absolutely free algebra) of $\mathcal{L}$ over a set of variables X.

A substitution (homomorphism)

$$\sigma\colon \mathbf{Fm}_{\mathcal{L}}(X) \to \mathbf{Fm}_{\mathcal{L}}(Y)$$

is called a **$\mathcal{V}$-unifier (over X)** of a set of $\mathcal{L}$-identities Σ with variables in X if

$$\mathcal{V} \models \sigma(\varphi) \approx \sigma(\psi) \text{ for all } \varphi \approx \psi \ \in \Sigma.$$

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Let $\mathrm{U}_{\mathcal{V}}(\Sigma, X)$ denote the set of $\mathcal{V}$-unifiers of Σ over X.

Motivation

Unifiers

If $\sigma_1, \sigma_2 \in \mathrm{U}_{\mathcal{V}}(\Sigma, X)$, we say that σ_1 is **more general** than σ_2

$$\sigma_2 \preceq \sigma_1$$

if there exists a substitution λ defined on the variables of $\sigma_1(X)$ such that $\sigma_2 \cong_{\mathcal{V}} \lambda \circ \sigma_1$.

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A **complete** set for $(\mathrm{U}_{\mathcal{V}}(\Sigma, X), \preceq)$ is a subset $M \subseteq \mathrm{U}_{\mathcal{V}}(\Sigma, X)$ such that for every $\sigma \in \mathrm{U}_{\mathcal{V}}(\Sigma, X)$, there exists $\sigma' \in M$ such that $\sigma \preceq \sigma'$.

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M is called a **μ-set** for $(\mathrm{U}_{\mathcal{V}}(\Sigma, X), \preceq)$ if $\sigma_1 \not\preceq \sigma_2$ and $\sigma_2 \not\preceq \sigma_1$ for all distinct $\sigma_1, \sigma_2 \in M$.

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Unifiers and Admissibility

If Σ, Δ are finite sets of $\mathcal{L}$-identities, the clause $\Sigma \Rightarrow \Delta$ is **$\mathcal{V}$-admissible** if for every $\mathcal{V}$-unifier σ of Σ there exists $\varphi \approx \psi \ \in \Delta$ such that $\mathcal{V} \models \sigma(\varphi) \approx \sigma(\psi)$.

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Let Σ, and Δ be finite sets of $\mathcal{L}$-identities, $X = \mathrm{Var}(\Sigma \cup \Delta)$ and M be a complete (or μ-set) for $\mathrm{U}_{\mathcal{V}}(\Sigma, X)$.

The clause $\Sigma \Rightarrow \Delta$ is $\mathcal{V}$-admissible if for every $\mathcal{V}$-unifier

$$\sigma \in \mathbf{M}$$

there exists $\varphi \approx \psi \ \in \Delta$ such that $\mathcal{V} \models \sigma(\varphi) \approx \sigma(\psi)$.

Motivation

Given a clause $\Sigma \Rightarrow \Delta$ is there any procedure to obtain a **"small"** set M of unifiers of Σ such that:

$$\Sigma \Rightarrow \Delta \text{ is } \mathcal{V}\text{-admissible}$$

$$\text{if } \forall\sigma \in M, \exists\varphi \approx \psi \ \in \Delta \text{ such that } \mathcal{V} \models \sigma(\varphi) \approx \sigma(\psi)?$$

Main Definition

What should we change?

If $\sigma_1, \sigma_2 \in \mathrm{U}_{\mathcal{V}}(\Sigma, X)$, we say that σ_1 is **more general** than σ_2

$$\sigma_2 \preceq \sigma_1$$

if there exists a substitution λ defined on the variables of $\sigma_1(X)$ such that $\sigma_2 \cong_{\mathcal{V}} \lambda \circ \sigma_1$.

A **complete** set for $(\mathrm{U}_{\mathcal{V}}(\Sigma, X), \preceq)$ is a subset $M \subseteq \mathrm{U}_{\mathcal{V}}(\Sigma, X)$ such that for every $\sigma \in \mathrm{U}_{\mathcal{V}}(\Sigma, X)$, there exists $\sigma' \in M$ such that $\sigma \preceq \sigma'$.

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More Exact Unifiers

Let Σ be a finite set of $\mathcal{L}$-equations and σ_1, σ_2 be $\mathcal{V}$-unifiers of Σ. We say that σ_1 is **more exact** than σ_2 (in symbols $\sigma_2 \sqsubseteq \sigma_1$) if σ_1 unifies fewer identities than σ_2.

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More precisely:

$$\sigma_2 \sqsubseteq \sigma_1$$

if

$$\mathcal{V} \models \sigma_2(\varphi) \approx \sigma_2(\psi) \text{ whenever } \mathcal{V} \models \sigma_1(\varphi) \approx \sigma_1(\psi).$$

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Lemma

For each $X \supseteq \mathrm{Var}(\Sigma)$,

$$\mathrm{type}(\mathsf{U}_{\mathcal{V}}(\Sigma, \mathrm{Var}(\Sigma)), \sqsubseteq) = \mathrm{type}(\mathsf{U}_{\mathcal{V}}(\Sigma, X), \sqsubseteq).$$

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We define the **exact type of Σ in $\mathcal{V}$** to be $\mathrm{type}(\mathsf{U}_{\mathcal{V}}(\Sigma, \mathrm{Var}(\Sigma)), \sqsubseteq)$ (for $\mathsf{U}_{\mathcal{V}}(\Sigma, \mathrm{Var}(\Sigma)) \neq \emptyset$).

Main Definition

Consequences

Let Σ, and Δ be finite sets of $\mathcal{L}$-identities, $X = \mathrm{Var}(\Sigma \cup \Delta)$ and

M be a complete (or μ-set) for $(\mathrm{U}_{\mathcal{V}}(\Sigma, X), \sqsubseteq)$.

The clause $\Sigma \Rightarrow \Delta$ is $\mathcal{V}$-admissible if for every $\mathcal{V}$-unifier $\sigma \in M$ there exists $\varphi \approx \psi \in \Delta$ such that $\mathcal{V} \models \sigma(\varphi) \approx \sigma(\psi)$.

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- For each $X \supseteq \mathrm{Var}(\Sigma)$, if M is a complete set for $(\mathsf{U}_{\mathcal{V}}(\Sigma, X), \preccurlyeq)$, then M is a complete set for $(\mathsf{U}_{\mathcal{V}}(\Sigma, X), \sqsubseteq)$.

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Proposition

If we consider the the set of types $\{1, \omega, \infty, 0\}$ *preordered as follows* $1 \leq \omega \leq \infty \leq 0 \leq \infty$*,*

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Proposition

If we consider the the set of types $\{1, \omega, \infty, 0\}$ *preordered as follows* $1 \leq \omega \leq \infty \leq 0 \leq \infty$, *then*

$$\mathrm{type}(\mathsf{U}_{\mathcal{V}}(\Sigma), \sqsubseteq) \leq \mathrm{type}(\mathsf{U}_{\mathcal{V}}(\Sigma), \preccurlyeq).$$

Examples

Class of Algebras	Unification Type	Exact Type
Boolean Algebras	Unitary	Unitary
Heyting Algebras	Finitary	Finitary
Semigroups	Infinitary	Infinitary or Nullary
Modal algebras	Nullary	Nullary
Distributive Lattices	Nullary	Unitary
Stone Algebras	Nullary	Unitary
Idempotent Semigroups	Nullary	Finitary
MV-algebras	Nullary	Finitary

Algebraic Translation

Ghilardi's Algebraic Translation

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$$\begin{tikzcd}
\mathbf{Fm}_{\mathcal{L}}(X) \arrow[r, "\sigma"] \arrow[d, two heads, "/\mathcal{V}"'] & \mathbf{Fm}_{\mathcal{L}}(Y) \arrow[d, two heads, "/\mathcal{V}"] \\
\mathbf{F}_{\mathcal{V}}(X) \arrow[r, "\sigma_{\mathcal{V}}"] & \mathbf{F}_{\mathcal{V}}(Y)
\end{tikzcd}$$

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\mathbf{F}_{\mathcal{V}}(X) \arrow[r, "\sigma_{\mathcal{V}}"] \arrow[d, two heads, "/\Sigma"'] & \mathbf{F}_{\mathcal{V}}(Y) \arrow[d, two heads, bend left] \\
\mathbf{F}_{\mathcal{V}}(X)/(\Sigma) \arrow[r, "h"] & P \arrow[u, hook, bend left]
\end{tikzcd}
$$

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Pre-order:

$$\begin{tikzcd} A \arrow[r, "h_1"] \arrow[dr, "h_2"'] & P_1 \arrow[d, "f"] \\ & P_2 \end{tikzcd}$$

Theorem (S. Ghilardi)

For each $\mathcal{V}$-unifiable finite set of identities Σ,

$$\mathrm{type}(\mathsf{U}_{\mathcal{V}}(\Sigma), \preccurlyeq) = \mathrm{type}(\mathsf{U}_{\mathcal{V}}(\mathbf{F}_{\mathcal{V}}(X)/(\Sigma)), \leq)$$

We call an algebra **E** **exact in $\mathcal{V}$** if it is finitely generated and embeds into $\mathbf{F}_{\mathcal{V}}(X)$ for some set X.

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Algebraic Co-Exact Unifiers

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Algebraic Co-Exact Unifiers

Theorem

Let $\mathcal{V}$ be an equational class Σ a finite set of $\mathcal{V}$-unifiable $\mathcal{L}$-identities and A the algebra finitely presented by Σ. Let $\mathsf{EU}_{\mathcal{V}}(A)$ *denote the preorder set of co-exact unifiers of A. Then*

$$\mathrm{type}(\mathsf{U}_{\mathcal{V}}(\Sigma), \sqsubseteq) = \mathrm{type}(\mathsf{EU}_{\mathcal{V}}(A)).$$

Algebraic Translation

Algebraic Co-Exact Unifiers

Corollary

If A the finitely presented algebra by Σ has a finitely many congruences, then $\mathrm{type}(\mathsf{U}_{\mathcal{V}}(\Sigma), \sqsubseteq)$ *is unitary or finitary.*

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Corollary

If $\mathcal{V}$ is a locally finite variety, then $\mathcal{V}$ has exact unification type unitary or finitary.

Future Work

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- Procedures to determine μ-sets.
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- Purpose designed types.

Thank you for your attention!

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