Typed Unification: when failure may not be wrong

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In logic programming, terms are:

- syntactic objects interpreted as semantic values that are trees in the Herbrand interpretation.

Unification, therefore:
- is syntactic,
- either returns a most general unifier (MGU) or fails.
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Types are associated with domains. We can assume:

- a function symbol that builds terms has a functional type (n input types and one output type)
- now terms themselves can contain type errors, while in the Herbrand interpretation there is a single type (TERM), so there can be no type errors in a term.
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Terms and Their Types

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- Unification only makes sense for values of the same domain.
- So we will perform unification while also checking if the terms belong to the same type (and are well-typed).

Since not all terms are ground and, in particular, interesting unification cases include non-ground terms, we need to define a type for a non-ground term.
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Non-Ground Terms and Their Types

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We will give different names (type variables) to the types of different variables just to denote constraints for each one.
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Types for well-typed non-ground terms can be types containing type variables, which we will call polymorphic types.
Examples of a Well-Typed Term

Let \( t = f(2, g(1, a, b)) \) be a ground term.
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- If we know that the types for the function symbols $f$ and $g$ are $\text{int} \times \text{int} \rightarrow \text{atom}$ and $\text{int} \times \text{atom} \times \text{atom} \rightarrow \text{int}$, respectively.
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- In fact, with just information for \( f \) we could have known what the type for \( t \) could be, if it was well-typed.
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- We can conclude that the type for term \( t \) is \( \text{atom} \).
- In fact, with just information for \( f \) we could have known what the type for \( t \) could be, if it was well-typed.
- We call \( f \) the principal functor of \( t \), and the type of a well-typed term is the output type for its principal functor.
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- Then we can note that no ground instance of this term is well-typed.
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- If we know that the type for the function symbols \( f \) is \( \text{int} \times \text{atom} \rightarrow \text{int} \).
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- The type for \( X \) has to be simultaneously \( \text{int} \) an \( \text{atom} \).
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- If we know that the type for the function symbols $f$ is $\text{int} \times \text{atom} \rightarrow \text{int}$.
- Then we can note that no ground instance of this term is well-typed.
- The type for $X$ has to be simultaneously $\text{int}$ an $\text{atom}$.
- Therefore we say that the term is ill-typed.
If we assume a type discipline for terms, now when we perform unification we can have three output values:
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- false - if the terms do not unify but are well-typed (and have unifiable types)
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- false - if the terms do not unify but are well-typed (and have unifiable types)
- wrong - we cannot simultaneously unify the terms and have them be well-typed.
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If we assume a type discipline for terms, now when we perform unification we can have three output values:

- an MGU - if the terms unify and are well-typed
- false - if the terms do not unify but are well-typed (and have unifiable types)
- wrong - we cannot simultaneously unify the terms and have them be well-typed.

We do not yet have a proof that our algorithm behaves correctly, but we are working on it currently.
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- $C \rightarrow$ a set of equality constraints that initially contains only $t_1 = t_2$
- $S \rightarrow$ a set of type equality constraints and membership constraints that consists on the following constraints:
  - let $t_1 = f(s_1, \ldots, s_n)$ and $t_2 = g(u_1, \ldots, u_m)$
  - let $f$ have type $\tau_1 \times \cdots \times \tau_n \rightarrow \tau$ and $g$ have $\tau'_1 \times \cdots \times \tau'_m \rightarrow \tau'$
  - $\{ \tau \equiv \alpha, \tau' \equiv \alpha, s_1 \in \tau_1, \ldots, s_n \in \tau_n, u_1 \in \tau'_1, \ldots, u_m \in \tau'_m \}$
Example

The initial set set built for terms $f(1, h(Y, a), X)$ and $g(h(Z, Z))$, considering $f$ has type $\text{int} \times \text{int} \times \text{int} \rightarrow \text{int}$ and $g$ has type $\text{int} \rightarrow \text{int}$ is:

\[
C = \{ f(1, h(Y, a), X) = g(h(Z, Z)) \}
\]
\[
S = \{ \text{int} \rightarrow \alpha, \text{int} \rightarrow \alpha, 1 \in \text{int}, h(Y, a) \in \text{int}, X \in \text{int}, h(Z, Z) \in \text{int} \}
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\[
\text{output} = (C, S)
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- output $= (C, S)$
The algorithm applies certain rules in order.
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1. \((C, \{f(t_1, \ldots, t_n) \in \tau\} \cup \text{Rest}) \rightarrow (C, \{t_1 \in \tau'_1, \ldots, t_n \in \tau'_n, \\
   \tau'_1 \doteq \tau_1, \ldots, \tau'_n \doteq \tau_n\} \cup \text{Rest}),\) where the type for \(f\) is \(f : \tau_1 \times \cdots \times \tau_n \rightarrow \tau\), and \(\tau'_i\) are the types for the principal functors of \(t_i\), respectively.
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2. \((C, \{X_i \in \alpha_i\} \cup \text{Rest}) \rightarrow (C, \text{Rest})\)
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2. \((C, \{X_i \in \alpha_i\} \cup Rest) \rightarrow (C, Rest)\)

3. \((C, \{c \in \tau\} \cup Rest) \rightarrow (C, Rest)\)

These reduce the number of membership constraints to zero, while generating type equality constraints.
The next few steps are as follows:

4. \( (C, \{ f(\tau_1, \ldots, \tau_n) = f(\tau'_1, \ldots, \tau'_n) \} \cup \text{Rest}) \rightarrow (C, \{ \tau_1 = \tau'_1, \ldots, \tau_n = \tau'_n \} \cup \text{Rest}) \)

5. \( (C, \{ \tau = \tau \} \cup \text{Rest}) \rightarrow (C, \text{Rest}) \)

6. \( (C, \{ f(\tau_1, \ldots, \tau_n) = g(\tau'_1, \ldots, \tau'_m) \} \cup \text{Rest}) \rightarrow \text{wrong}, \text{if} f \neq g \text{ or } n \neq m \)

7. \( (C, \{ \alpha = \tau \} \cup \text{Rest}) \rightarrow (C, \{ \alpha = \tau \} \cup \text{Rest}) \), \( \alpha \) is not a type variable

8. \( (C, \{ \alpha = \tau \} \cup \text{Rest}) \rightarrow (C, \{ \alpha = \tau \} \cup \text{Rest}) \), \( \alpha \rightarrow \tau \), \( \alpha \) does not occur in \( \tau \)

9. \( (C, \{ \alpha = \tau \} \cup \text{Rest}) \rightarrow \text{wrong}, \text{if} \alpha \) occurs in \( \tau \)

These correspond to the Martelli-Montanari algorithm for unification, but on types.
Algorithm Steps - 2

The next few steps are as follows:

4. \((C, \{ f(\tau_1, \ldots, \tau_n) \doteq f(\tau'_1, \ldots, \tau'_n) \} \cup \text{Rest}) \rightarrow (C, \{ \tau_1 \doteq \tau'_1, \ldots, \tau_n \doteq \tau'_n \} \cup \text{Rest})\)

5. \((C, \{ \tau \doteq \tau \} \cup \text{Rest}) \rightarrow (C, \text{Rest})\)

6. \((C, \{ f(\tau_1, \ldots, \tau_n) \doteq g(\tau'_1, \ldots, \tau'_m) \} \cup \text{Rest}) \rightarrow \text{wrong}, \text{ if } f \neq g \text{ or } n \neq m\)

7. \((C, \{ \tau \doteq \alpha \} \cup \text{Rest}) \rightarrow (C, \{ \alpha \doteq \tau \} \cup \text{Rest}), \text{ } \tau \text{ is not a type variable}\)

8. \((C, \{ \alpha \doteq \tau \} \cup \text{Rest}) \rightarrow (C, \{ \alpha \doteq \tau \} \cup \text{Rest}[\alpha \mapsto \tau]), \text{ if } \alpha \text{ does not occur in } \tau\)

9. \((C, \{ \alpha \doteq \tau \} \cup \text{Rest}) \rightarrow \text{wrong}, \text{ if } \alpha \text{ occurs in } \tau\)

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5. \((C, \{\tau = \tau\} \cup Rest) \rightarrow (C, Rest)\)

6. \((C, \{f(\tau_1, \ldots, \tau_n) = g(\tau'_1, \ldots, \tau'_m)\} \cup Rest) \rightarrow \text{wrong}, \text{ if } f \neq g \text{ or } n \neq m\)

7. \((C, \{\tau = \alpha\} \cup Rest) \rightarrow (C, \{\alpha = \tau\} \cup Rest), \tau \text{ is not a type variable}\)

8. \((C, \{\alpha = \tau\} \cup Rest) \rightarrow (C, \{\alpha = \tau\} \cup Rest[\alpha \mapsto \tau]), \text{ if } \alpha \text{ does not occur in } \tau\)

9. \((C, \{\alpha = \tau\} \cup Rest) \rightarrow \text{wrong}, \text{ if } \alpha \text{ occurs in } \tau\)

These correspond to the Martelli-Montanari algorithm for unification, but on types.
Algorithm Steps - 3

The last few steps are as follows:
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10. \( \{ f(t_1, \ldots, t_n) = f(s_1, \ldots, s_n) \} \cup \text{Rest}, T \rightarrow (\{ t_1 = s_1, \ldots, t_n = s_n \} \cup \text{Rest}, T) \)

11. \( \{ t = t \} \cup \text{Rest}, T \rightarrow (\text{Rest}, T) \)

12. \( \{ f(t_1, \ldots, t_n) = g(s_1, \ldots, s_m) \} \cup \text{Rest}, T \rightarrow \text{false}, \text{ if } f \neq g \text{ or } n \neq m \)

13. \( \{ t = X \} \cup \text{Rest}, T \rightarrow (\{ X = t \} \cup \text{Rest}, T), t \text{ is not a variable} \)

14. \( \{ X = t \} \cup \text{Rest}, T \rightarrow (\{ X = t \} \cup \text{Rest}[X \leftarrow t], T), \text{ if } X \text{ does not occur in } t \)

15. \( \{ X = t \} \cup \text{Rest}, T \rightarrow \text{false}, \text{ if } X \text{ occurs in } t. \)
Algorithm Steps - 3

The last few steps are as follows:

10. \( \{ f(t_1, \ldots, t_n) = f(s_1, \ldots, s_n) \} \cup Rest, T ) \rightarrow ( \{ t_1 = s_1, \ldots, t_n = s_n \} \cup Rest, T ) \)

11. \( \{ t = t \} \cup Rest, T ) \rightarrow (Rest, T ) \)

12. \( \{ f(t_1, \ldots, t_n) = g(s_1, \ldots, s_m) \} \cup Rest, T ) \rightarrow false, \) if \( f \neq g \) or \( n \neq m \)

13. \( \{ t = X \} \cup Rest, T ) \rightarrow (\{ X = t \} \cup Rest, T ), \) \( t \) is not a variable

14. \( \{ X = t \} \cup Rest, T ) \rightarrow (\{ X = t \} \cup Rest[X \mapsto t], T ), \) if \( X \) does not occur in \( t \)

15. \( \{ X = t \} \cup Rest, T ) \rightarrow false, \) if \( X \) occurs in \( t \).

These also correspond to the Martelli-Montanari algorithm for unification for terms.
Details in the algorithm

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- We can always apply one of the cases for the membership constraints.
- When we fail unification on types, notice that we output wrong.
- When we fail unification on terms, we output false.
- Since the steps are applied in order, we can only return false if we do not output wrong, which means we were able to unify the types for both terms, and did not find a type error.
Let $t_1 = f(1, f(X, 1))$ and $t_2 = f(Y, f(2, Y))$, and $f$ have type $\text{int} \times \text{int} \rightarrow \text{int}$. 
Example of Unification

Let \( t_1 = f(1, f(X, 1)) \) and \( t_2 = f(Y, f(2, Y)) \), and \( f \) have type \( int \times int \rightarrow int \).

The initial tuple is: \( (\{ f(1, f(X, 1)) = f(Y, f(2, Y)) \}, \{ f(1, f(X, 1)) \in int, f(Y, f(2, Y)) \in int, int \models \alpha, int \models \alpha \}) \)
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Let $t_1 = f(1, f(X, 1))$ and $t_2 = f(Y, f(2, Y))$, and $f$ have type $\text{int} \times \text{int} \to \text{int}$.

The initial tuple is: $(\{f(1, f(X, 1)) = f(Y, f(2, Y))\},
\{f(1, f(X, 1)) \in \text{int}, f(Y, f(2, Y)) \in \text{int}, \text{int} \doteq \alpha, \text{int} \doteq \alpha\})$

We can apply the rules for the membership constraints step-by-step.
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The initial tuple is: \( \{ \{ f(1, f(X, 1)) = f(Y, f(2, Y)) \} \), \( \{ f(1, f(X, 1)) \in \text{int}, f(Y, f(2, Y)) \in \text{int}, \text{int} \doteq \alpha, \text{int} \doteq \alpha \} \) \)

We can apply the rules for the membership constraints step-by-step.

\( \{ f(1, f(X, 1)) = f(Y, f(2, Y)) \}, \{ 1 \in \text{int}, f(X, 1) \in \text{int}, \text{int} \doteq \text{int}, \text{int} \doteq \text{int}, f(Y, f(2, Y)) \in \text{int}, \text{int} \doteq \alpha, \text{int} \doteq \alpha \} \)
Example of Unification

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\{(f(1, f(X, 1)) = f(Y, f(2, Y)) \}, \{1 \in \text{int}, f(X, 1) \in \text{int}, \text{int} \vdash int, \text{int} \vdash int, f(Y, f(2, Y)) \in \text{int}, \text{int} \vdash \alpha, \text{int} \vdash \alpha\}\}
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\[\{ f(1, f(X, 1)) = f(Y, f(2, Y)) \}, \{ 1 \in \text{int}, f(X, 1) \in \text{int}, \text{int} = \text{int}, \text{int} = \text{int}, f(Y, f(2, Y)) \in \text{int}, \text{int} = \alpha, \text{int} = \alpha \} \]

\[\{ f(1, f(X, 1)) = f(Y, f(2, Y)) \}, \{ X \in \alpha_X, 1 \in \text{int}, \alpha_X = \text{int}, \text{int} = \text{int}, \text{int} = \text{int}, \text{int} = \text{int}, f(Y, f(2, Y)) \in \text{int}, \text{int} = \alpha, \text{int} = \alpha \} \]
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We do the same for the other membership constraint:
Example of Unification

\[
\{ f(1, f(X, 1)) = f(Y, f(2, Y)) \}, \{ 1 \in \text{int}, f(X, 1) \in \text{int}, \text{int} \vdash \text{int}, \text{int} \vdash \text{int}, f(Y, f(2, Y)) \in \text{int}, \text{int} \vdash \alpha, \text{int} \vdash \alpha \}\)

\[
\{ f(1, f(X, 1)) = f(Y, f(2, Y)) \}, \{ X \in \alpha_X, 1 \in \text{int}, \alpha_X \vdash \text{int}, \text{int} \vdash \text{int}, \text{int} \vdash \text{int}, f(Y, f(2, Y)) \in \text{int}, \text{int} \vdash \alpha, \text{int} \vdash \alpha \}\)

We do the same for the other membership constraint:

\[
\{ f(1, f(X, 1)) = f(Y, f(2, Y)) \}, \{ \alpha_X \vdash \text{int}, \text{int} \vdash \text{int}, \text{int} \vdash \text{int}, \alpha_X \vdash \text{int}, \alpha_Y \vdash \text{int}, \text{int} \vdash \alpha, \text{int} \vdash \alpha \}\)
Example of Unification

Now we start applying the set of rules dealing with type equality.
Now we start applying the set of rules dealing with type equality.

\[
(f(1, f(X, 1)) = f(Y, f(2, Y))), \ {\{\alpha_X \doteq int, \ int \doteq int, \ int \dot{=} int, \\
int \doteq int, \alpha_Y \doteq int, \ int \doteq int, \alpha_Y \doteq int, \ int \doteq \alpha, \ int \doteq \alpha}\}}
\]
Now we start applying the set of rules dealing with type equality.

\[
\{ f(1, f(X, 1)) = f(Y, f(2, Y)) \}, \{ \alpha_X \doteq int, \text{int} \doteq \text{int}, \text{int} \doteq \text{int}, \\
\text{int} \doteq \text{int}, \alpha_Y \doteq \text{int}, \text{int} \doteq \alpha, \text{int} \doteq \alpha \} 
\]

\[
\{ f(1, f(X, 1)) = f(Y, f(2, Y)) \}, \{ \alpha_X \doteq int, \alpha_Y \doteq int, \alpha_Y \doteq \text{int}, \text{int} \doteq \alpha, \text{int} \doteq \alpha \} 
\]
Example of Unification

Now we start applying the set of rules dealing with type equality.

\[
(\{ f(1, f(X, 1)) = f(Y, f(2, Y)) \}, \{ \alpha_X \doteq int, int \doteq int, int \doteq int, int \doteq int, \alpha_Y \doteq int, int \doteq int, \alpha_Y \doteq int, int \doteq \alpha, int \doteq \alpha \})
\]

\[
(\{ f(1, f(X, 1)) = f(Y, f(2, Y)) \}, \{ \alpha_X \doteq int, \alpha_Y \doteq int, \alpha_Y \doteq int, \alpha \doteq \alpha, int \doteq \alpha \})
\]

\[
(\{ f(1, f(X, 1)) = f(Y, f(2, Y)) \}, \{ \alpha_X \doteq int, \alpha_Y \doteq int, \alpha_Y \doteq int, \alpha \doteq int, \alpha \doteq int, int \doteq \alpha \})
\]
Example of Unification

\[
\{ f(1, f(X, 1)) = f(Y, f(2, Y)) \}, \{ \alpha_X \doteq int, \alpha_Y \doteq int, \alpha_Y \doteq int, \\
\alpha \doteq int, \text{int} \doteq \alpha \}\]

No more rules apply, we did not halt with wrong, therefore there is no type error. We move on to the equality constraints.
Example of Unification

$$\{ f(1, f(X, 1)) = f(Y, f(2, Y)) \}, \{ \alpha_X \doteq \text{int}, \alpha_Y \doteq \text{int}, \alpha_Y \doteq \text{int}, \alpha \doteq \text{int}, \text{int} \doteq \alpha \}$$

$$\{ f(1, f(X, 1)) = f(Y, f(2, Y)) \}, \{ \alpha_X \doteq \text{int}, \alpha_Y \doteq \text{int}, \text{int} \doteq \text{int}, \alpha \doteq \text{int}, \text{int} \doteq \text{int} \}$$
Example of Unification

\[
\{f(1, f(X, 1)) = f(Y, f(2, Y))\}, \{ \alpha_X \doteq int, \alpha_Y \doteq int, \alpha_Y \doteq int, \\
\alpha \doteq int, int \doteq \alpha \}\)
\]

\[
\{f(1, f(X, 1)) = f(Y, f(2, Y))\}, \{ \alpha_X \doteq int, \alpha_Y \doteq int, int \doteq int, \\
\alpha \doteq int, int \doteq int \}\)
\]

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\{f(1, f(X, 1)) = f(Y, f(2, Y))\}, \{ \alpha_X \doteq int, \alpha_Y \doteq int, \alpha \doteq int \}\)
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Example of Unification

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(f(1, f(X, 1))) = f(Y, f(2, Y)), \{\alpha_X \doteq int, \alpha_Y \doteq int, \alpha \doteq int, int \doteq \alpha\}
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Example of Unification

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\]

\[
\{ 1 = Y, f(X, 1) = f(2, Y) \}, \{ \alpha_X \doteq int, \alpha_Y \doteq int, \alpha \doteq int \} 
\]
Example of Unification

\[
\{ f(1, f(X, 1)) = f(Y, f(2, Y)) \}, \{ \alpha_X \doteq int, \alpha_Y \doteq int, \alpha \doteq int \} \]

\[
\{ 1 = Y, f(X, 1) = f(2, Y) \}, \{ \alpha_X \doteq int, \alpha_Y \doteq int, \alpha \doteq int \} \]

\[
\{ 1 = Y, X = 2, 1 = Y \}, \{ \alpha_X \doteq int, \alpha_Y \doteq int, \alpha \doteq int \} \]
Example of Unification

\[
\{ f(1, f(X, 1)) = f(Y, f(2, Y)) \}, \{ \alpha_X \doteq int, \alpha_Y \doteq int, \alpha \doteq int \}\)
\]

\[
\{ 1 = Y, f(X, 1) = f(2, Y) \}, \{ \alpha_X \doteq int, \alpha_Y \doteq int, \alpha \doteq int \}\)
\]

\[
\{ 1 = Y, X = 2, 1 = Y \}, \{ \alpha_X \doteq int, \alpha_Y \doteq int, \alpha \doteq int \}\)
\]

\[
\{ Y = 1, X = 2, Y = 1 \}, \{ \alpha_X \doteq int, \alpha_Y \doteq int, \alpha \doteq int \}\)
\]
Example of Unification

\( \{ f(1, f(X, 1)) = f(Y, f(2, Y)) \}, \{ \alpha_X = int, \alpha_Y = int, \alpha = int \} \) 

\( \{ 1 = Y, f(X, 1) = f(2, Y) \}, \{ \alpha_X = int, \alpha_Y = int, \alpha = int \} \) 

\( \{ 1 = Y, X = 2, 1 = Y \}, \{ \alpha_X = int, \alpha_Y = int, \alpha = int \} \) 

\( \{ Y = 1, X = 2, Y = 1 \}, \{ \alpha_X = int, \alpha_Y = int, \alpha = int \} \) 

\( \{ Y = 1, X = 2, 1 = 1 \}, \{ \alpha_X = int, \alpha_Y = int, \alpha = int \} \)
Example of Unification

\[
\begin{align*}
\{ f(1, f(X, 1)) &= f(Y, f(2, Y)) \}, \{ \alpha_X \doteq int, \alpha_Y \doteq int, \alpha \doteq int \} \\
\{ 1 = Y, f(X, 1) = f(2, Y) \}, \{ \alpha_X \doteq int, \alpha_Y \doteq int, \alpha \doteq int \} \\
\{ 1 = Y, X = 2, 1 = Y \}, \{ \alpha_X \doteq int, \alpha_Y \doteq int, \alpha \doteq int \} \\
\{ Y = 1, X = 2, Y = 1 \}, \{ \alpha_X \doteq int, \alpha_Y \doteq int, \alpha \doteq int \} \\
\{ Y = 1, X = 2, 1 = 1 \}, \{ \alpha_X \doteq int, \alpha_Y \doteq int, \alpha \doteq int \} \\
\{ Y = 1, X = 2 \}, \{ \alpha_X \doteq int, \alpha_Y \doteq int, \alpha \doteq int \}
\end{align*}
\]
Example of Unification

\[
(\{f(1, f(X, 1)) = f(Y, f(2, Y))\}, \{\alpha_X \doteq int, \alpha_Y \doteq int, \alpha \doteq int\})
\]

\[
(\{1 = Y, f(X, 1) = f(2, Y)\}, \{\alpha_X \doteq int, \alpha_Y \doteq int, \alpha \doteq int\})
\]

\[
(\{1 = Y, X = 2, 1 = Y\}, \{\alpha_X \doteq int, \alpha_Y \doteq int, \alpha \doteq int\})
\]

\[
(\{Y = 1, X = 2, Y = 1\}, \{\alpha_X \doteq int, \alpha_Y \doteq int, \alpha \doteq int\})
\]

\[
(\{Y = 1, X = 2, 1 = 1\}, \{\alpha_X \doteq int, \alpha_Y \doteq int, \alpha \doteq int\})
\]

\[
(\{Y = 1, X = 2\}, \{\alpha_X \doteq int, \alpha_Y \doteq int, \alpha \doteq int\})
\]

\[Y = 1, X = 2\] is an MGU of both terms.
Example of Unification

Let $t_1 = g(1, a, h(X))$ and $t_2 = h(g(Y, b, Y))$, $g$ have type $\text{int} \times \text{atom} \times \text{int} \rightarrow \text{int}$, and $\text{int} \rightarrow \text{int}$. 
Example of Unification

Let $t_1 = g(1, a, h(X))$ and $t_2 = h(g(Y, b, Y))$, $g$ have type $\text{int} \times \text{atom} \times \text{int} \to \text{int}$, and $\text{int} \to \text{int}$.

The initial tuple is: $(\{g(1, a, h(X)) = h(g(Y, b, Y))\}, \{g(1, a, h(X)) \in \text{int}, h(g(Y, b, Y)) \in \text{int}, \text{int} \doteq \alpha, \text{int} \doteq \alpha\})$
Example of Unification

Let $t_1 = g(1, a, h(X))$ and $t_2 = h(g(Y, b, Y))$, $g$ have type $int \times atom \times int \to int$, and $int \to int$.

The initial tuple is: $\langle \{g(1, a, h(X)) = h(g(Y, b, Y))\}, \{g(1, a, h(X)) \in int, h(g(Y, b, Y)) \in int, int \doteq \alpha, int \doteq \alpha\} \rangle$

We can immediately see that the terms do not unify, syntactically. But if we replace $X$ by any integer and $Y$ by any integer, the terms are well-typed. The algorithm should return $false$. 
Let $t_1 = g(1, a, h(X))$ and $t_2 = h(g(Y, b, Y))$, $g$ have type $\text{int} \times \text{atom} \times \text{int} \to \text{int}$, and $\text{int} \to \text{int}$.

The initial tuple is: $(\{g(1, a, h(X)) = h(g(Y, b, Y))\}, \{g(1, a, h(X)) \in \text{int}, h(g(Y, b, Y)) \in \text{int}, \text{int} \doteq \alpha, \text{int} \doteq \alpha\})$

We can immediately see that the terms do not unify, syntactically. But if we replace $X$ by any integer and $Y$ by any integer, the terms are well-typed. The algorithm should return \textit{false}.

$(\{g(1, a, h(X)) = h(g(Y, b, Y))\}, \{g(1, a, h(X)) \in \text{int}, h(g(Y, b, Y)) \in \text{int}, \text{int} \doteq \alpha, \text{int} \doteq \alpha\})$
Example of Unification

\{(g(1, a, h(X)) = h(g(Y, b, Y)) \}, \\
\{g(1, a, h(X)) \in \text{int}, h(g(Y, b, Y)) \in \text{int}, \text{int} \doteq \alpha, \text{int} \doteq \alpha \}\)
Example of Unification

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\]

\[
\left(\{g(1, a, h(X)) = h(g(Y, b, Y))\}, \{1 \in \text{int}, a \in \text{atom}, X \in \alpha_X, \text{int} \doteq \text{int}, \text{atom} \doteq \text{atom}, \alpha_X \doteq \text{int}, Y \in \alpha_Y, b \in \text{atom}, \alpha_Y \doteq \text{int}, \text{atom} \doteq \text{atom}, \text{int} \doteq \alpha, \text{int} \doteq \alpha\}\right)
\]
Example of Unification

\[
\begin{align*}
\{g(1, a, h(X)) = h(g(Y, b, Y))\}, \\
\{g(1, a, h(X)) \in \text{int}, h(g(Y, b, Y)) \in \text{int}, \text{int} \doteq \alpha, \text{int} \doteq \alpha\}
\end{align*}
\]

\[
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\{g(1, a, h(X)) = h(g(Y, b, Y))\}, \\
\{1 \in \text{int}, a \in \text{atom}, X \in \alpha_X, \text{int} \doteq \text{int}, \text{atom} \doteq \text{atom}, \alpha_X \doteq \text{int}, \\
Y \in \alpha_Y, b \in \text{atom}, \alpha_Y \doteq \text{int}, \text{atom} \doteq \text{atom}, \text{int} \doteq \alpha, \text{int} \doteq \alpha\}
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\]

\[
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\{g(1, a, h(X)) = h(g(Y, b, Y))\}, \\
\{\text{int} \doteq \text{int}, \text{atom} \doteq \text{atom}, \alpha_X \doteq \text{int}, \alpha_Y \doteq \text{int}, \text{atom} \doteq \text{atom}, \\
\text{int} \doteq \alpha, \text{int} \doteq \alpha\}
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\[
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\[
\{ g(1, a, h(X)) = h(g(Y, b, Y)) \},
\{ 1 \in \text{int}, a \in \text{atom}, X \in \alpha_X, \text{int} \doteq \text{int}, \text{atom} \doteq \text{atom}, \alpha_X \doteq \text{int},
Y \in \alpha_Y, b \in \text{atom}, \alpha_Y \doteq \text{int}, \text{atom} \doteq \text{atom}, \text{int} \doteq \alpha, \text{int} \doteq \alpha \}\)

\[
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\begin{align*}
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\{\alpha_X \triangleright int, \alpha_Y \triangleright int, int \triangleright \alpha, int \triangleright \alpha\}
\end{align*}
\]

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\{g(1, a, h(X)) &= h(g(Y, b, Y))\}, \\
\{\alpha_X \triangleright int, \alpha_Y \triangleright int, \alpha \triangleright int\}
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false
We want to make sure that our algorithm has the expected behaviour. In particular:
Soundness

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- We always detect ill-typed terms.
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We want to make sure that our algorithm has the expected behaviour. In particular:

- We always detect ill-typed terms.
- We return an MGU if there is a substitution for which the terms are well-typed, have the same type and are equal.
- We return wrong when there is no substitution for which the terms can have the same type and be well-typed.
- We return false if there is a substitution for which the terms are well-typed and have the same type, but they cannot be unified.
Proof Progress

We state the following propositions, that lead to the soundness proof.
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- The algorithm terminates. (proved)
- If the algorithm terminates it outputs either a unifier, false or wrong. (proved)
- If the algorithm outputs a unifier, the unifier is a MGU. (use Martelli-Montanari proof)
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- If the algorithm outputs \textit{wrong}, then either there is a type error in one of the terms, or there is no substitution for which both terms have the same type.
- If the algorithm outputs \textit{false}, then either there is a substitution for which the terms have the same type, but they do not unify.
Thank you!