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Second-order unification and functional arity

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  introduces a unification constraint.
- In STLC:

$$X_M \doteq X_N \to X_{MN}$$

  where X_M, X_N, X_{MN} are unification variables.

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- This introduces a unification constraint

$$F_{\mathtt{M}} \mathtt{A}_1 \ldots \mathtt{A}_n \doteq X_{\mathtt{N}} \to X_{\mathtt{M}\ \mathtt{A}_1 \ldots \mathtt{A}_n \mathtt{N}}$$

where $F_{\mathtt{M}}$ is a second-order unification variable, $X_{\mathtt{N}}, X_{\mathtt{M}\ \mathtt{A}_1 \ldots \mathtt{A}_n \mathtt{N}}$ are unification variables.

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- Constraints fall within the second-order unification language.
- If we are (un)lucky – second-order abstract syntax is necessary.

What kind of programming language?

$$M, N ::= x \mid \lambda x.M \mid \Lambda X.M \mid MA \mid MN$$

Type application critical for unification.
Type application omitted $\Rightarrow$ seminunification.

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- Example (J.B.Wells):

$$
\begin{array}{l}
b : \forall\gamma.(\gamma \to \gamma) \to \beta, \\
c : \forall.(\mu_1 \to \delta_1) \to (\delta_2 \to \mu_2) \to (\tau_2 \to \tau_2), \\
\vdash \\
b(\lambda x.cxx)
\end{array}
$$

where $\tau_1 \leq \mu_1, \quad \tau_2 \leq \mu_2$ is an instance of the seminunification problem.

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  - $\text{arity}(c) = 0$ for a constant c and $\text{arity}(A_1 \to \cdots \to A_n \to c) = \max(\text{arity}(A_1), \ldots, \text{arity}(A_n), n)$,

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  - $\mathrm{rank}(c) = 0$ for a constant c, and $\mathrm{rank}(A \to B) = \max(\mathrm{rank}(A) + 1, \mathrm{rank}(B))$.

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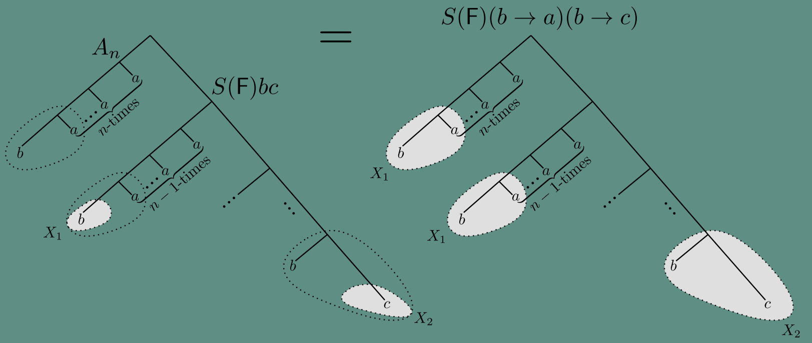
Example

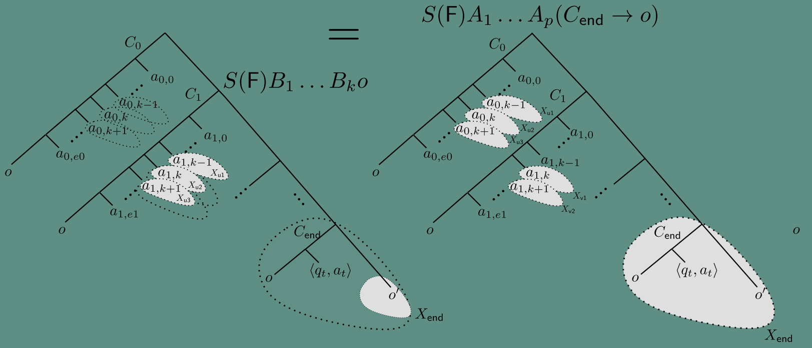
Consider SOU instance

$$A_n \to \mathsf{F}bc \quad \doteq \quad \mathsf{F}(b \to a)(b \to c)$$



(Misleading) Machine simulation

$$C_0 \to \mathsf{F} B_1 \dots B_k o \doteq \mathsf{F} A_1 \dots A_p (C_{n-1} \to o)$$



Verification of machine consistency

$$(\mathsf{F}\mathsf{s}1 \ldots \mathsf{s}p\, o') \to \mathsf{G}D'_1 \ldots D'_l \doteq \mathsf{G}D_1 \ldots D_l$$

$\mathsf{s} \to \langle b, o\rangle \rightsquigarrow \langle \mathsf{s}, 1\rangle$ in case $\pi_3(\mathsf{s}) = b$,
$\mathsf{s} \to o \rightsquigarrow \langle \mathsf{s}, 1\rangle$ in case $\pi_3(\mathsf{s}) = \bullet$, or $\pi_2(\mathsf{s}) = \bullet$,
$\mathsf{s} \to \langle \mathsf{s}', 1\rangle \rightsquigarrow \langle \mathsf{s}, 2\rangle$ in case $\pi_3(\mathsf{s}) = \pi_2(\mathsf{s}')$ and $\pi_1(\mathsf{s}') = \pi_2(\mathsf{s})$,
$\mathsf{s} \to \langle \mathsf{s}', 2\rangle \rightsquigarrow \langle \mathsf{s}, 3\rangle$ in case $\pi_3(\mathsf{s}) = \pi_2(\mathsf{s}')$ and $\pi_1(\mathsf{s}') = \pi_2(\mathsf{s})$,
$a \to \langle \mathsf{s}, 3\rangle \rightsquigarrow \langle a, 4\rangle$ in case $\pi_1(\mathsf{s}) = a$, $a \to \langle b, 4\rangle \rightsquigarrow \langle a, 4$

Conclusions

- Restriction to bounded types leads to decidable type-checking, type reconstruction.
- Restriction to types with bounded arity/rank may lead to undecidable type-checking, type reconstruction.

The End