One-Variable Unification in K

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Overview

Modal logic K
- Syntax
- Kripke Semantics
- Unification

Normal Forms in K

Minimal Formulas

Characterising Unifiability in Special Case

Syntax

Definition

The set $\mathrm{Fm}(X)$ of *formulas* with variables in a set X is defined as the smallest set such that for all $\varphi, \psi \in \mathrm{Fm}(X)$ and $x \in X$,

$$x \in \mathrm{Fm}(X), \qquad \top \in \mathrm{Fm}(X),$$
$$\neg\varphi \in \mathrm{Fm}(X), \qquad \varphi \wedge \psi \in \mathrm{Fm}(X),$$
$$\Box\varphi \in \mathrm{Fm}(X).$$

We denote by $\mathbf{Fm}(X) := \langle \mathrm{Fm}(X), \wedge, \neg, \top, \Box \rangle$ the *formula algebra* over X.
We define the notations $\bot, \vee, \rightarrow, \leftrightarrow$ for formulas as usual.
For $\varphi \in \mathrm{Fm}(X)$ define $\Diamond\varphi := \neg\Box\neg\varphi$.

Substitutions

Definition

A *substitution* is a homomorphism $\sigma\colon \mathbf{Fm}(X) \to \mathbf{Fm}(Y)$, that is, a map $\mathrm{Fm}(X) \to \mathrm{Fm}(Y)$ such that

$$\sigma(\top) = \top, \qquad \sigma(\neg\varphi) = \neg\sigma(\varphi),$$
$$\sigma(\varphi \wedge \psi) = \sigma(\varphi) \wedge \sigma(\psi), \qquad \sigma(\Box\varphi) = \Box\sigma(\varphi).$$

Remark

Each substitution $\mathbf{Fm}(X) \to \mathbf{Fm}(Y)$ *is uniquely determined by its values on* X*.*

Modal Degree

Definition

Define the *modal degree (modal depth)* $\mathsf{md}(\varphi)$ of a formula $\varphi \in \mathrm{Fm}(X)$ recursively:

$$\mathsf{md}(x) := 0, \qquad \mathsf{md}(\top) := 0,$$
$$\mathsf{md}(\neg\varphi) := \mathsf{md}(\varphi), \qquad \mathsf{md}(\varphi \wedge \psi) := \max(\mathsf{md}(\varphi), \mathsf{md}(\psi)),$$
$$\mathsf{md}(\Box\varphi) := \mathsf{md}(\varphi) + 1.$$

I.e., count the maximal number of nested occurrences of $\Box$.
For $n \in \mathbb{N}$ define $\mathrm{Fm}(X, n)$ as the set of formulas of degree *at most* n.
For a substitution $\sigma\colon \mathbf{Fm}(X) \to \mathbf{Fm}(Y)$ define its modal degree as $\mathsf{md}(\sigma) := \sup_{p \in X} \mathsf{md}(\sigma(p))$.

Kripke Semantics

Definition

A *Kripke frame* is a pair $F = \langle W, R \rangle$ where $R \subseteq W \times W$ is a binary relation on the non-empty set W. For $w, v \in W$ with Rwv we say that v is a *successor* of w or that v is *accessible* from w.

A *valuation* V on F with variables in a set X is a map $X \to \mathcal{P}(W)$.

A *Kripke model* is a triple $M = \langle W, R, V \rangle$ where $\langle W, R \rangle$ is a Kripke frame and V is a valuation on $\langle W, R \rangle$.

Definition

Let $M = \langle W, R, V \rangle$ be a Kripke model with variables in X. For $w \in W$ we define recursively, when a formula $\varphi \in \mathrm{Fm}(X)$ is *true at w*, written $M, w \Vdash \varphi$.

$$\begin{aligned} M, w \Vdash x &\iff w \in V(x) \\ M, w \Vdash \top &\text{ holds} \\ M, w \Vdash \neg\varphi &\iff M, w \nVdash \varphi \\ M, w \Vdash \varphi \wedge \psi &\iff M, w \Vdash \varphi \text{ and } M, w \Vdash \psi \\ M, w \Vdash \Box\varphi &\iff \text{for all } v \in W, Rwv \implies M, v \Vdash \varphi \end{aligned}$$

Remark

$$M, w \Vdash \Diamond\varphi \iff \textit{there is } v \in W \textit{ with } Rwv \textit{ and } M, v \Vdash \varphi$$

Unification

Definition

A formula is *valid* if it is true at all points of all Kripke models. A substitution $\sigma\colon \mathbf{Fm}(X) \to \mathbf{Fm}(Y)$ is said to *unify* φ if $\sigma(\varphi)$ is valid. In this case σ is a *unifier* of φ, and φ is *unifiable*.

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Question

Is there an algorithm that given $\varphi \in \mathrm{Fm}(X)$ decides whether φ is unifiable?

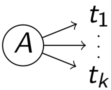
Trees

Definition

Let X be a set of variables. For each $n \in \mathbb{N}$ define inductively the set $T_n^{\mathcal{P}(X)}$ of $\mathcal{P}(X)$-*labelled (commutative idempotent) trees of degree* n by

$$T_0^{\mathcal{P}(X)} := \mathcal{P}(X), \qquad T_{n+1}^{\mathcal{P}(X)} := \mathcal{P}(X) \times \mathcal{P}(T_n^{\mathcal{P}(X)}).$$

We write $\boxed{A}$ for $A \in T_0^{\mathcal{P}(X)}$. For $\langle A, \{t_1, \ldots, t_k\} \rangle \in T_{n+1}^{\mathcal{P}(X)}$ we write

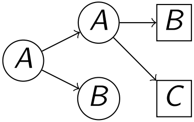


where each t_i is written in this notation.

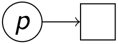
Trees

Example

For $A, B, C \subseteq X$, consider $\langle A, \{\langle A, \{B, C\}\rangle, \langle B, \emptyset\rangle\}\rangle \in T_2^{\mathcal{P}(X)}$. We write:



We adapt the notation if convenient. For $\langle\{p\}, \{\emptyset\}\rangle \in T_1^{\mathcal{P}(\{p\})}$ we write:



Normal Form Theorem

There exist functions $\iota_n \colon T_n^{\mathcal{P}(X)} \to \mathrm{Fm}(X, n)$ such that

Theorem (Fine '75, Ghilardi '95)

Let X be a finite set and $n \in \mathbb{N}$.

- *For each Kripke model M and $w \in M$ there is a unique $t \in T_n^{\mathcal{P}(X)}$ such that $M, w \Vdash \iota_n(t)$.*
- *For each $\varphi \in \mathrm{Fm}(X, n)$ there is a unique $\Phi \subseteq T_n^{\mathcal{P}(X)}$ such that φ is equivalent to $\bigvee \iota_n[\Phi]$.*

We will identify $t \in T_n^{\mathcal{P}(X)}$ and $\iota_n(t) \in \mathrm{Fm}(X, n)$. Similarly, we identify $\Phi \subseteq T_n^{\mathcal{P}(X)}$ and $\bigvee \iota_n[\Phi]$.

Minimal formulas

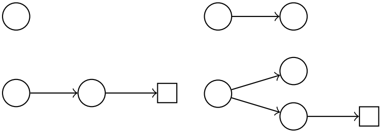
For each ground substitution $\sigma\colon \mathbf{Fm}(X) \to \mathbf{Fm}(\emptyset)$ and $n \in \mathbb{N}$ there exists a computable set $\Phi(X, \sigma, n) \subseteq T_n^{\mathcal{P}(X)}$ such that for all $\Psi \subseteq T_n^{\mathcal{P}(X)}$

$$\Psi \text{ is unified by } \sigma \iff \Phi(X, \sigma, n) \subseteq \Psi.$$

Minimal formulas – Example

There is a concrete way to compute $\Phi(X, \sigma, n)$.
For example let $\sigma\colon \mathbf{Fm}(\{p\}) \to \mathbf{Fm}(\emptyset)$ be defined by $\sigma(p) \coloneqq \Box\bot$. Compute $\Phi(\{p\}, \sigma, 1)$.

1. Write down all elements of $T_2^{\mathcal{P}(\emptyset)}$.



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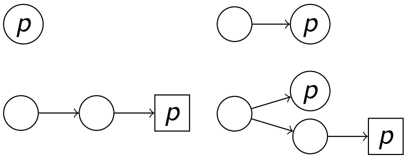
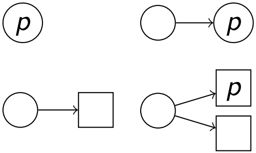
2. Add the label $p \in X$ to a node if $\sigma(p)$ is true at this node, considered as a Kripke frame.



Minimal formulas – Example

There is a concrete way to compute $\Phi(X, \sigma, n)$.
For example let $\sigma\colon \mathbf{Fm}(\{p\}) \to \mathbf{Fm}(\emptyset)$ be defined by $\sigma(p) := \Box\bot$. Compute $\Phi(\{p\}, \sigma, 1)$.

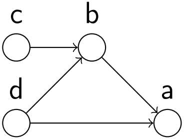
3. Cut away leaves until each tree is of degree 1.



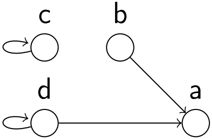
This set is $\Phi(\{p\}, \sigma, 1)$.

Special case

A natural frame structure on $T_2^{\mathcal{P}(\emptyset)}$ is $F(\emptyset, 2)$.
Consider another frame structure F on $T_2^{\mathcal{P}(\emptyset)}$:



(a) $F(\emptyset, 2)$



(b) F

Proposition

$F(\emptyset, 2), t \Vdash \iota_2(t)$ and $F, t \Vdash \iota_2(t)$ for all $t \in T_2^{\mathcal{P}(\emptyset)}$.

Theorem

For all $\varphi \in \mathrm{Fm}(\{p\}, 1)$ the following are equivalent:

1. *φ is unifiable.*
2. *φ is unified by one of the following substitutions: $p \mapsto \top$, $p \mapsto \bot$, $p \mapsto \Box\bot$, $p \mapsto \Diamond\top$.*
3. *There is a valuation V on F such that $\langle F, V\rangle, w \Vdash \varphi$ for all $w \in F$.*

Proof.

2. $\Rightarrow$ 1. is trivial. For 1. $\Rightarrow$ 3. let $\sigma\colon \mathbf{Fm}(\{p\}) \to \mathbf{Fm}(\emptyset)$ be a ground unifier of φ. Set $V_\sigma(p) := \{w \in F \mid F, w \Vdash \sigma(p)\}$. This is well-defined because $\sigma(p)$ is without variables, and hence truth of $\sigma(p)$ does not depend on the valuation. That this valuation is appropriate follows from

$$\langle F, V_\sigma\rangle, w \Vdash \varphi \iff F, w \Vdash \sigma(\varphi).$$

Recall: $\varphi \in \mathrm{Fm}(\{p\}, 1)$ and

2. φ is unified by one of the following substitutions: $p \mapsto \top$, $p \mapsto \bot$, $p \mapsto \Box\bot$, $p \mapsto \Diamond\top$.
3. There is a valuation V on F such that $\langle F, V\rangle, w \Vdash \varphi$ for all $w \in F$.

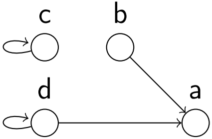
Proof of 3. $\Rightarrow$ 2.

Let $V\colon \{p\} \to \mathcal{P}(F)$ be a valuation on F such that $\langle F, V\rangle, w \Vdash \varphi$ for all $w \in F$. Let $\Phi \subseteq T_1^{\{p\}}$ such that φ is equivalent to Φ.

Approach: Since $\langle F, V\rangle, w \Vdash \Phi$ we can show that certain elements must lie in Φ. By case analysis on we show $\Phi(\{p\}, \sigma, 1) \subseteq \Phi$ for one of the four substitutions $p \mapsto \top$, $p \mapsto \bot$, $p \mapsto \Box\bot$, $p \mapsto \Diamond\top$.

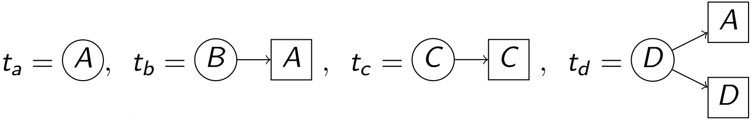
Proof of 3. $\Rightarrow$ 2.

Recall: $\Phi \subseteq T_1^{\{p\}}$ is such that $\langle F, V\rangle, w \Vdash \Phi$ for all $w \in F$.



The frame F

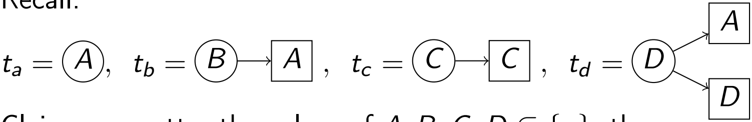
Using Fine's normal form, we get $A, B, C, D \subseteq \{p\}$ and



such that $\langle F, V\rangle, w \Vdash t_w$ and $t_w \in \Phi$ for all $w \in F$.

Proof of 3. ⇒ 2.

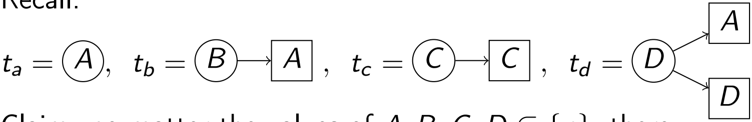
Recall:



Claim: no matter the values of $A, B, C, D \subseteq \{p\}$, there always is some σ, among the four mentioned ones, such that $\Phi(\{p\}, \sigma, 1) \subseteq \{t_a, t_b, t_c, t_d\}$.
Proof by case distinction.

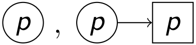
Proof of 3. $\Rightarrow$ 2.

Recall:



Claim: no matter the values of $A, B, C, D \subseteq \{p\}$, there always is some σ, among the four mentioned ones, such that $\Phi(\{p\}, \sigma, 1) \subseteq \{t_a, t_b, t_c, t_d\}$.

Proof by case distinction. For example, if $A = \{p\}$ and $B = \{p\}$ then

$\{t_a, t_b\} = \{$ 

$\} = \Phi(\{p\}, p \mapsto \top, 1).$

Hence in this case $\Phi(\{p\}, p \mapsto \top, 1) \subseteq \{t_a, t_b, t_c, t_d\} \subseteq \Phi$ and Φ is unified by $p \mapsto \top$. $\square$

Limits of this approach

The previous result says: a formula $\varphi \in \mathrm{Fm}(\{p\}, 1)$ is unifiable iff there is a (ground-definable) valuation on F which makes φ true everywhere in F.

Proposition (Jeřábek '23)

The formula $\varphi := (\Box p \to p) \wedge (q \leftrightarrow \neg\Box q)$ is not unifiable and for every finite frame G there is a ground-definable valuation which makes φ true everywhere on G.

I.e., to use the same approach for two variables, the corresponding F would need to be infinite.