Modal unification step by step

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Overview

We characterize the unification problem in some modal logics as a homomorphism problem for finite graphs.

Syntax of modal logic

The set $F(\mathsf{V})$ of modal formulas over V:

$$\varphi ::= p, q, \ldots \in \mathsf{V} \mid \top \mid \varphi \wedge \varphi \mid \neg\varphi \mid \Box\varphi$$

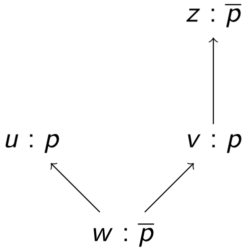
Plus $\Diamond\varphi := \neg\Box\neg\varphi$ and standard definitions for $\bot$, $\vee$, $\rightarrow$ and $\leftrightarrow$.

Semantics of modal logic

Semantics in Kripke models (W, R, v), where W is a set, $R \subseteq W \times W$ and $v : \mathrm{V} \to \mathcal{P}(W)$:

$$[\![p]\!] := v(p) \qquad [\![\top]\!] := W \qquad [\![\varphi \wedge \psi]\!] := [\![\varphi]\!] \cap [\![\psi]\!]$$
$$[\![\neg\varphi]\!] := W \setminus [\![\varphi]\!] \qquad [\![\Box\varphi]\!] := \{w \mid R[w] \subseteq [\![\varphi]\!]\}$$

It follows that $[\![\Diamond\varphi]\!] = \{w \mid R[w] \cap [\![\varphi]\!] \neq \emptyset\}$.



$$[\![\Box p]\!] = \{w, u, z\}$$

$$[\![\Diamond\Box\bot]\!] = \{w, v\}$$

The modal logics **K** and $\mathbf{Alt}_1$

$\varphi \in \mathbf{K}$ iff $[\![\varphi]\!] = W$ holds in all Kripke models (W, R, v).

$\varphi \in \mathbf{Alt}_1$ iff $[\![\varphi]\!] = W$ holds in all Kripke models (W, R, v), with $|R[w]| \leq 1$, for all $w \in W$.

The unifiability problem

A **K**-unifier for a formula $\varphi \in F(\mathsf{V})$ over V is a substitution $\sigma : \mathsf{V} \to F(\emptyset)$ such that $\sigma(\varphi) \in \mathbf{K}$.

The unifiability problem for **K**:
INPUT: a modal formula φ
QUESTION: Is there a **K**-unifier for φ?

Same definitions with $\mathbf{Alt_1}$ in place of **K**.

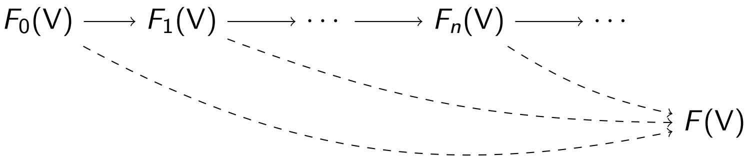
Examples

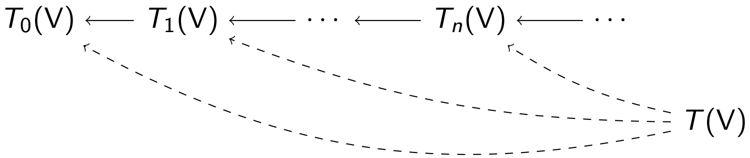
φ	$\sigma(p) = ?$	$\sigma(\varphi)$
$p \to \Box p$	$p \mapsto \top$	$\top \to \Box \top$
$p \leftrightarrow \Box \neg p$	none (why?)	

Some results on unifiability in modal logic

1. Ghilardi (1990's): Decidability for transitive modal logics
2. Baader & Morawska and Baader & Narendran (2000's): Decidability for fragments
3. Wolter & Zakharyaschev (2008): Undecidability for **K** with universal modality
4. Jeřábek (2015): **K** has nullary unification type
5. Balbiani and Tinchev (2016): $\mathbf{Alt}_1$-unifiability is in PSPACE

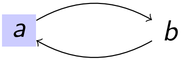
Duality step by step





Characterization for $\mathbf{Alt}_1$

To characterize $\mathbf{Alt}_1$-unifiability we use graphs with a binary relation S and a unary predicate E. Example:



Theorem

The formula φ is $\mathbf{Alt}_1$-unifiable if and only if there is a graph homomorphism $C_n \to P(\varphi)$ for some n.

The "canonical" graphs C_n

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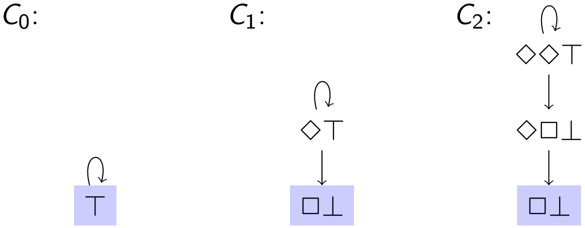
The graphs C_0, C_1 and C_2:



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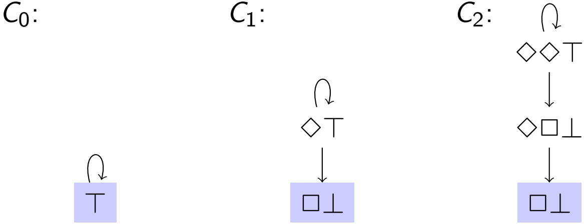
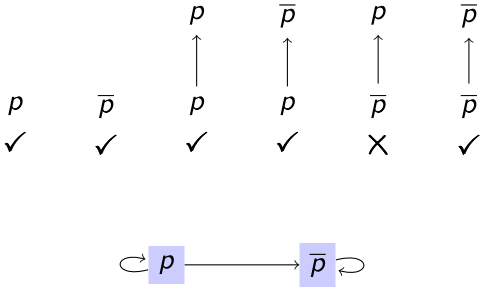


Theorem

The formula φ is $\mathbf{Alt}_1$-unifiable if and only if there is a path $v_0 S v_1 S \ldots S v_n$ in $P(\varphi)$, with $v_0 S v_0$ and $v_n \in E$.

Example: Computing $P(\varphi)$ for $\varphi = p \to \Box p$



New result for $\mathbf{Alt}_1$

Balbiani and Tinchev (2016): $\mathbf{Alt}_1$-unifiability is in PSPACE

Theorem

Unifiability in $\mathbf{Alt}_1$ is PSPACE-complete.

This follows from:

Theorem

The formula φ is $\mathbf{Alt}_1$-unifiable if and only if there is a path $v_0 S v_1 S \ldots S v_n$ in $P(\varphi)$, with $v_0 S v_0$ and $v_n \in E$.

Characterization for **K**

Theorem

The formula φ is **K***-unifiable if and only if there is a $\mathcal{P}$-graph homomorphism $C_n \to P(\varphi)$ for some n.*

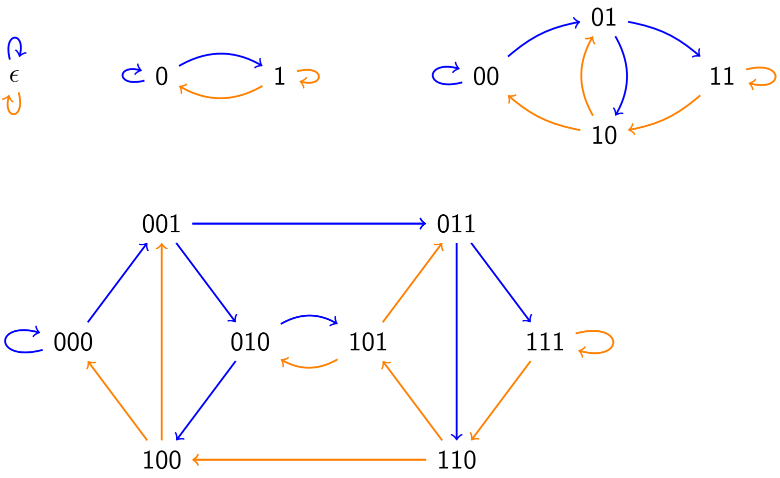
A *$\mathcal{P}$-graph* (X, R) is a set X with a relation $R \subseteq X \times \mathcal{P}(X)$.

A $\mathcal{P}$-graph homomorphism from (X, R) to (X', R') is a function $h : X \to X'$ such that for all $x \in X$ and $U \subseteq X$

$$\text{if } (x, U) \in R \text{ then } (h(x), h[U]) \in R' \; .$$

An intermediate case: de Bruijn graphs

We define a logic for which the “canonical” graphs are:



Conclusions

1. Unifiability problems in modal logic can be reformulated in terms of graph homomorphism.
2. For $\mathbf{Alt}_1$ we obtain a new PSPACE lower bound.
3. For **K** decidability remains difficult.

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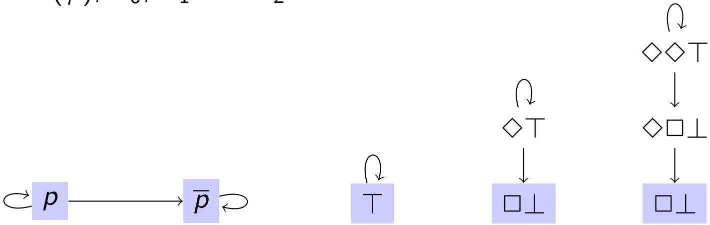
Thank you!

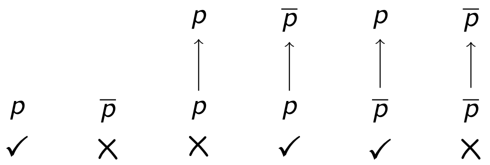
Homomorphisms give rise to unifiers for $\varphi = p \to \Box p$

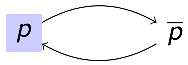
Recall $P(\varphi)$, C_0, C_1 and C_2:



homomorphism	becomes unifier
$C_0 \to P(\varphi)$ with $\top \mapsto p$	$p \mapsto \top$
$C_0 \to P(\varphi)$ with $\top \mapsto \overline{p}$	$p \mapsto \bot$
$C_1 \to P(\varphi)$ with $\Diamond\top \mapsto \overline{p}, \Box\bot \mapsto p$	$p \mapsto \Box\bot$
$C_2 \to P(\varphi)$ with $\Diamond\Diamond\top \mapsto \overline{p}, \Diamond\Box\bot \mapsto p, \Box\bot \mapsto p$	$p \mapsto \Box\Box\bot$
	$(\Box\Box\bot \equiv \Diamond\Box\bot \vee \Box\bot)$

Additional example: $P(\varphi)$ for $\varphi = p \leftrightarrow \Box\neg p$



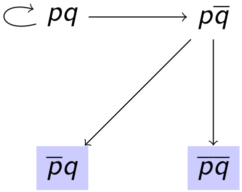


No $C_n \to P(\varphi)$ because $P(\varphi)$ has no reflexive point.
$\Rightarrow p \leftrightarrow \Box\neg p$ is not unifiable!

A more complex example in $\mathbf{Alt}_1$

Consider $\varphi = (\Diamond p \to p \land q) \land (\Diamond\neg q \to p \land \neg q) \land (\Box\bot \to \neg p)$.

The graph $P(\varphi)$:



A unifier is $p \mapsto \Diamond\top, q \mapsto \Diamond\Diamond\top$.