Modal unification step by step

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Overview

We characterize the unification problem in some modal logics as a homomorphism problem for finite graphs.
Syntax of modal logic

The set $F(V)$ of modal formulas over $V$:

$$
\varphi ::= p, q, \ldots \in V \mid T \mid \varphi \land \varphi \mid \neg \varphi \mid \Box \varphi
$$

Plus $\Diamond \varphi ::= \neg \Box \neg \varphi$ and standard definitions for $\bot, \lor, \rightarrow$ and $\leftrightarrow$.  

Semantics of modal logic

Semantics in Kripke models \((W, R, v)\), where \(W\) is a set, \(R \subseteq W \times W\) and \(v : V \rightarrow P(W)\):

\[
[p] := v(p) \quad [\top] := W \quad [\varphi \land \psi] := [\varphi] \cap [\psi] \\
[\neg \varphi] := W \setminus [\varphi] \quad [\Box \varphi] := \{w \mid R[w] \subseteq [\varphi]\}
\]

It follows that \([\Diamond \varphi] = \{w \mid R[w] \cap [\varphi] \neq \emptyset\}\).

\[
\begin{aligned}
&z : \overline{p} \\
\uparrow \\
&u : p \\
&\quad \downarrow \quad \downarrow
\\
&w : \overline{p} \\
&v : \overline{p}
\end{aligned}
\]

\[
[\Box p] = \{w, u, z\} \quad [\Diamond \Box \bot] = \{w, v\}
\]
The modal logics $\mathbf{K}$ and $\mathbf{Alt}_1$

$\varphi \in \mathbf{K}$ iff $[\varphi] = W$ holds in all Kripke models $(W, R, v)$.

$\varphi \in \mathbf{Alt}_1$ iff $[\varphi] = W$ holds in all Kripke models $(W, R, v)$, with $|R[w]| \leq 1$, for all $w \in W$. 
The unifiability problem

A $K$-unifier for a formula $\varphi \in F(V)$ over $V$ is a substitution $\sigma : V \rightarrow F(\emptyset)$ such that $\sigma(\varphi) \in K$.

The unifiability problem for $K$:
INPUT: a modal formula $\varphi$
QUESTION: Is there a $K$-unifier for $\varphi$?

Same definitions with $\text{Alt}_1$ in place of $K$. 
Examples

<table>
<thead>
<tr>
<th>$\varphi$</th>
<th>$\sigma(p) = ?$</th>
<th>$\sigma(\varphi)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p \rightarrow \Box p$</td>
<td>$p \leftrightarrow \top$</td>
<td>$\top \rightarrow \Box \top$</td>
</tr>
<tr>
<td>$p \leftrightarrow \Box \neg p$</td>
<td>none (why?)</td>
<td></td>
</tr>
</tbody>
</table>


Some results on unifiability in modal logic

1. Ghilardi (1990’s): Decidability for transitive modal logics
3. Wolter & Zakharyaschev (2008): Undecidability for $\mathbf{K}$ with universal modality
4. Jeřábek (2015): $\mathbf{K}$ has nullary unification type
5. Balbiani and Tinchev (2016): $\mathbf{Alt}_1$-unifiability is in PSPACE
Duality step by step

\[ F_0(V) \rightarrow F_1(V) \rightarrow \cdots \rightarrow F_n(V) \rightarrow \cdots \]

\[ T_0(V) \leftarrow T_1(V) \leftarrow \cdots \leftarrow T_n(V) \leftarrow \cdots \]
Characterization for $\text{Alt}_1$

To characterize $\text{Alt}_1$-unifiability we use graphs with a binary relation $S$ and a unary predicate $E$. Example:

$$a \rightarrow b$$

**Theorem**

The formula $\varphi$ is $\text{Alt}_1$-unifiable if and only if there is a graph homomorphism $C_n \rightarrow P(\varphi)$ for some $n$. 
The “canonical” graphs $C_n$

Theorem
The formula $\varphi$ is $\text{Alt}_1$-unifiable if and only if there is a graph homomorphism $C_n \to P(\varphi)$ for some $n$.

The graphs $C_0$, $C_1$ and $C_2$:
The “canonical” graphs $C_n$

**Theorem**

The formula $\varphi$ is $\text{Alt}_1$-unifiable if and only if there is a graph homomorphism $C_n \rightarrow P(\varphi)$ for some $n$.

The graphs $C_0$, $C_1$ and $C_2$:

$C_0$: 

$C_1$: 

$C_2$: 

**Theorem**

The formula $\varphi$ is $\text{Alt}_1$-unifiable if and only if there is a path $v_0Sv_1S\ldots Sv_n$ in $P(\varphi)$, with $v_0Sv_0$ and $v_n \in E$. 
Example: Computing $P(\varphi)$ for $\varphi = p \rightarrow \Box p$
New result for Alt$_1$

Balbiani and Tinchev (2016): Alt$_1$-unifiability is in PSPACE

Theorem

Unifiability in Alt$_1$ is PSPACE-complete.

This follows from:

Theorem

The formula $\varphi$ is Alt$_1$-unifiable if and only if there is a path $v_0Sv_1S\ldots Sv_n$ in $P(\varphi)$, with $v_0Sv_0$ and $v_n \in E$. 
Characterization for $\mathbf{K}$

Theorem

The formula $\varphi$ is $\mathbf{K}$-unifiable if and only if there is a $\mathcal{P}$-graph homomorphism $C_n \rightarrow \mathcal{P}(\varphi)$ for some $n$.

A $\mathcal{P}$-graph $(X, R)$ is a set $X$ with a relation $R \subseteq X \times \mathcal{P}(X)$. A $\mathcal{P}$-graph homomorphism from $(X, R)$ to $(X', R')$ is a function $h : X \rightarrow X'$ such that for all $x \in X$ and $U \subseteq X$

$$\text{if } (x, U) \in R \text{ then } (h(x), h[U]) \in R'$$.
An intermediate case: de Bruijn graphs

We define a logic for which the “canonical” graphs are:
Conclusions

1. Unifiability problems in modal logic can be reformulated in terms of graph homomorphism.
2. For $\text{Alt}_1$ we obtain a new PSPACE lower bound.
3. For $\mathbf{K}$ decidability remains difficult.
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Thank you!
Homomorphisms give rise to unifiers for $\varphi = p \rightarrow \Box p$

Recall $P(\varphi)$, $C_0$, $C_1$ and $C_2$:

- $C_0 \rightarrow P(\varphi)$ with $\top \mapsto p$
- $C_0 \rightarrow P(\varphi)$ with $\top \mapsto \overline{p}$
- $C_1 \rightarrow P(\varphi)$ with $\Diamond \top \mapsto \overline{p}$, $\Box \bot \mapsto p$
- $C_2 \rightarrow P(\varphi)$ with $\Diamond \Diamond \top \mapsto \overline{p}$, $\Diamond \Box \bot \mapsto p$, $\Box \bot \mapsto p$

The homomorphism becomes unifier

- $p \mapsto \top$
- $p \mapsto \bot$
- $p \mapsto \Box \bot$
- $p \mapsto \Box \Box \bot$

($(\Box \Box \bot \equiv \Diamond \bot \lor \Diamond \bot)$
Additional example: $P(\varphi)$ for $\varphi = p \iff \Box \neg p$

$\begin{array}{cccccc}
p & \bar{p} & p & p & \bar{p} \\
\checkmark & \times & \times & \checkmark & \checkmark & \times
\end{array}$

No $C_n \rightarrow P(\varphi)$ because $P(\varphi)$ has no reflexive point.
$\Rightarrow p \iff \Box \neg p$ is not unifiable!
A more complex example in \textbf{Alt}_1

Consider $\varphi = (\Diamond p \rightarrow p \land q) \land (\Diamond \neg q \rightarrow p \land \neg q) \land (\Box \bot \rightarrow \neg p)$.

The graph $P(\varphi)$:

A unifier is $p \mapsto \Diamond \top, q \mapsto \Diamond \Diamond \top$. 