Matching in Quantitative Equational Theories

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RISC

(Quantitative) Equational Theories

Fix a signature Ω and a set of variables X.

- “Classical” setting:
  Equations $s \approx t$ between terms $s, t \in T(\Omega, X)$.

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- Quantitative setting (Mardare-Plotkin-Panangaden 2016):
  Indexed equations $s \approx_\varepsilon t$ for $\varepsilon \in \mathbb{Q}_{\geqslant 0}$
  - Intuition: "s is within ε of t"
    $\rightsquigarrow$ think of metric spaces: $d(s, t) \leqslant \varepsilon$
  - $s \approx_0 t$ corresponds to $s \approx t$
  - Transitivity has to be replaced by the triangle inequality:
    $r \approx_\varepsilon s$ and $s \approx_\delta t$ imply $r \approx_{\varepsilon+\delta} t$.

Inference rules for quantitative equational logic

$$\frac{}{E \vdash t \approx_0 t}\ (\text{Refl.}) \qquad \frac{E \vdash s \approx_\varepsilon t}{E \vdash t \approx_\varepsilon s}\ (\text{Symm.})$$

$$\frac{E \vdash s \approx_\varepsilon t}{E \vdash s\sigma \approx_\varepsilon t\sigma}\ (\text{Subst.}) \qquad \frac{E \vdash s \approx_\varepsilon r \qquad E \vdash r \approx_\delta t}{E \vdash s \approx_{\varepsilon+\delta} t}\ (\text{Triang.})$$

$$\frac{E \vdash s \approx_\varepsilon t}{E \vdash s \approx_{\varepsilon+\delta} t}\ (\text{Max.}) \qquad \frac{E \vdash s \approx_{\varepsilon'} t \mid \varepsilon' > \varepsilon}{E \vdash s \approx_\varepsilon t}\ (\text{Cont.})$$

$$\frac{E \vdash s_1 \approx_\varepsilon t_1, \ldots, E \vdash s_n \approx_\varepsilon t_n}{E \vdash f(s_1, \ldots, s_n) \approx_\varepsilon f(t_1, \ldots, t_n)}\ (\text{NExp.}) \quad \text{for } f : n \in \Omega$$

$$\frac{}{E \vdash s \approx_\varepsilon t}\ (\text{Assum.}) \quad \text{for } s \approx_\varepsilon t \in E$$

Matching Problems

Let $s, t \in T(\Omega, X)$ be terms, E a set of equations.

Matching problem: $s \lesssim_E^? t$

Find a substitution σ such that $E \vdash s\sigma \approx t$.

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Let $s, t \in T(\Omega, X)$ be terms, E a set of indexed equations, $\varepsilon \in \mathbb{Q}_{\geqslant 0}$

Quantitative matching problems

- $s \lesssim^?_\varepsilon t$: Find a substitution σ such that $E \vdash s\sigma \approx_\varepsilon t$.
- $s \lesssim^?_? t$: Find the least $\delta \in \mathbb{Q}_{\geqslant 0}$ such that there exists a substitution σ satisfying $E \vdash s\sigma \approx_\delta t$.

For this talk: Focus on the first problem ("fixed-range matching").

Assumptions

- **Running assumption:** E is finite.
  We may assume that all equations from E have indices in $\mathbb{N}$.
- **Notation:** Write $E = E_0 \sqcup E_+$, where

$$E_0 = \{s \approx_\varepsilon t \in E \mid \varepsilon = 0\} \qquad (\text{“crisp part”}),$$
$$E_+ = \{s \approx_\varepsilon t \in E \mid \varepsilon > 0\} \qquad (\text{“quantitative part”}).$$

- Note: E_0 can be viewed as a classical (non-quantitative) equational theory.
- Assume that E_0 has finitary unification type and that a unification algorithm for E_0 is given.

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$$\begin{aligned}
\sigma \text{ solves } s \lesssim^?_\varepsilon t &\iff E \vdash s\sigma \approx_\varepsilon t \\
&\iff s\sigma \in B_\varepsilon(t) \\
&\iff \exists v \in B_\varepsilon(t) \text{ such that } \sigma \text{ solves } s \lesssim^? v \text{ (syntactically)}
\end{aligned}$$

$\rightsquigarrow$ compute $B_\varepsilon(t)$ and match s against all its elements.

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$\rightsquigarrow$ compute $B_\varepsilon(t)$ and match s against all its elements.

- Problem: $B_\varepsilon(t)$ need not be finite!

Examples

1. $E = \{f(x) \approx_1 g(x, y)\}$, $t = f(a)$, where $a \in \Omega$ is a constant.
   Then: $E \vdash f(a) \approx_1 g(a, y)$
   $\Rightarrow$ every instance of $g(a, y)$ is in $B_1(f(a))$
   $\Rightarrow$ $B_1(t)$ is infinite.

2. $E = \{x \approx_0 f(x)\}$, $t = a$ (constant).
   Then $f^n(a) \in B_0(a)$ for every n
   $\Rightarrow$ $B_0(t)$ is infinite.

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To guarantee finiteness, compute a finite representation $\mathcal{R}_\varepsilon(t)$ of $B_\varepsilon(t)$ that contains:

- non-ground terms from $B_\varepsilon(t)$, but not all of their instances
- representatives of terms up to E_0

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   $\Rightarrow$ every instance of $g(a,y)$ is in $B_1(f(a))$
   $\Rightarrow$ $B_1(t)$ is infinite.
   $\rightsquigarrow$ take $\mathcal{R}_1(t) = \{f(a), g(a,y)\}$ instead!
2. $E = \{x \approx_0 f(x)\}$, $t = a$ (constant).
   Then $f^n(a) \in B_0(a)$ for every n
   $\Rightarrow$ $B_0(t)$ is infinite.
   $\rightsquigarrow$ take $\mathcal{R}_0(a) = \{a\}$ instead!

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Compact representation of the ball

Definition

Define $\mathcal{R}_\varepsilon(x) := \{x\}$ if x is a variable, and otherwise, set

$$\mathcal{R}_\varepsilon(t) = \{t\} \cup \bigcup_{\substack{\zeta \in \mathbb{N}, \\ 0 < \zeta \leqslant \varepsilon, \\ t = f(t_1,\ldots,t_n), \\ s_i \in \mathcal{R}_\zeta(t_i)}} \mathcal{R}_{\varepsilon-\zeta}(f(s_1,\ldots,s_n)) \cup \bigcup_{\substack{l \approxeq_\delta r \in E_+, \\ \delta \leqslant \varepsilon, \\ \sigma \in \mathrm{mcu}_{E_0}(l,t)}} \mathcal{R}_{\varepsilon-\delta}(r\sigma),$$

where

- $l \approxeq_\delta r$ is a fresh, unoriented variant of an equation in E_+
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Remarks

- $\mathcal{R}_\varepsilon(t)$ is finite and defined uniquely up to renaming variables.
- $\mathcal{R}_0(t) = \{t\}$
- If $\varepsilon \leqslant \delta$, then $\mathcal{R}_\varepsilon(t) \subseteq \mathcal{R}_\delta(t)$

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$$\mathcal{R}_2(g(b)) = \{g(b)\} \cup \bigcup_{\substack{\zeta=1,2\\ s\in\mathcal{R}_\zeta(b)}} \mathcal{R}_{2-\zeta}(g(s)) \cup \mathcal{R}_1(f(b,y))$$

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$\rightsquigarrow \sigma = \{x \mapsto b\}$ is a solution.

First results

Proposition

If $E = E_+$ is regular and t is a ground term, then $\mathcal{R}_\varepsilon(t) = B_\varepsilon(t)$.

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Quantitative matching algorithm 1:

Input: Regular $E = E_+$; E-matching problem $s \lesssim_\varepsilon t$ with t ground.
Output: A complete set of solutions.

1. $S \leftarrow \emptyset$
2. Compute $\mathcal{R}_\varepsilon(t)$
3. For each $u \in \mathcal{R}_\varepsilon(t)$:
4. $\quad S \leftarrow S \cup \{\text{syntactic matchers of } s \text{ to } u\}$
5. Return S

Relaxing the assumptions: non-regular E_+

Consider the case where $E = E_+$ need not be regular.

Example 1

$E = \{f(x) \approx_1 g(x, y)\}$; solve $g(x, b) \lesssim_1^? f(a)$.
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The solution $\sigma = \{x \mapsto a\}$ can be found via syntactic **unification** of $g(x, b)$ and $g(a, y)$.

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Example 2

$E = \{a \approx_1 f(g(x), h(x)), g(b) \approx_1 b, h(c) \approx_1 c\}$.
Then $\mathcal{R}_2(a) = \{a\} \cup \mathcal{R}_1(f(g(x), h(x))) \ni f(b, c)$.

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Then $\mathcal{R}_2(a) = \{a\} \cup \mathcal{R}_1(f(g(x), h(x))) \ni f(b, c)$.

But $E \not\models a \approx_2 f(b, c)$!

Relaxing the assumptions: non-empty E_0

Now, consider non-empty E_0.

Recall: $\mathcal{R}_\varepsilon(t)$ represents terms up to equality modulo E_0.

By assumption, we know how to solve unification in E_0. Can we just replace syntactic unification by unification modulo E_0 to solve the matching problem in E?

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$E = \{f(a,x) \approx_1 g(x,a), a \approx_0 b\}$.
Solve $f(b,y) \lesssim_1 g(c,b)$.

$\mathcal{R}_1(g(c,b)) = \{g(c,b), f(a,c)\}$.

$\sigma = \{y \mapsto c\}$ is an E_0-unifier of $f(b,y)$ and $f(a,c)$.

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Example 2

$E = \{f(a, x) \approx_0 g(x), a \approx_1 b\}$; solve $f(b, y) \lesssim_1 g(a)$.
Then $\mathcal{R}_1(g(a)) = \{g(a), g(b)\}$.

There is no E_0-unifier!

To find the solution, one would also need to compute $\tilde{\mathcal{R}}_1(f(b, y)) = \{f(b, y), f(a, y)\}$

Outlook

Possible future work:

- Results for matching in the more general cases, in particular for non-empty E_0 ($\rightsquigarrow$ combining methods)
- Different (e.g., rule-based) approaches for quantitative matching
- Other equational problems in the quantitative setting (unification, anti-unification)
- Different versions of quantitative equational reasoning, e.g. Gavazzo-Di Florio (2023)

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