Combining Decision Procedures

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Lecture 3

Outline

The Combination Problem

Verification conditions typically are in combination of many data-structures/theories

- arithmetic
- uninterpreted function symbols
- arrays
- lists
- records
- sets
- ...

Uninterpreted Functions + Arithmetic: An Example

Consider T_{UF} the theory of equality, and $T_{\mathbb{R}}$ the theory of linear of arithmetic (over the Reals)

The $T_{UF} \cup T_{\mathbb{R}}$-satisfiability problem

$$f(c) + 1 \neq 1 + f(d), c \leq d, d + a \leq c, a + b = 1, b = 1 + a$$

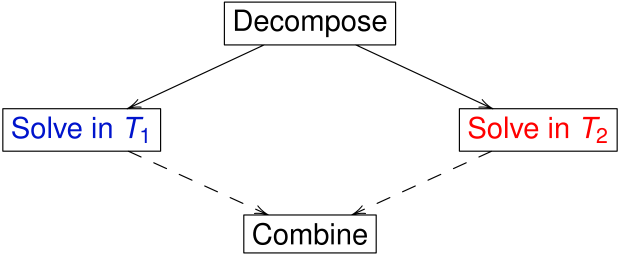
can be equivalently transformed into

$x = f(c), y = f(d)$,

$x + 1 \neq 1 + y, c \leq d, d + a \leq c, a + b = 1, b = 1 + a$

How to solve this conjunction of pure formulas in a modular way by combining a T_{UF}-satisfiability procedure and a $T_{\mathbb{R}}$-satisfiability procedure?

Combination: General Principle



Decompose: purify the problem, by introducing existentially quantified new variables to denote aliens
Solve: existence of models?
Combine: build a combined model from the models of T_1 and T_2

Decomposition

Purification performed via Variable Abstraction

VA
$$\varphi \cup \{p(\ldots, s[t]_\omega, \ldots)\} \vdash \varphi \cup \{p(\ldots, s[x]_\omega, \ldots), x = t\}$$

IEQ
$$\varphi \cup \{s = t\} \vdash \varphi \cup \{x = s, x = t\}$$

IDEQ
$$\varphi \cup \{s \neq t\} \vdash \varphi \cup \{x = s, y = t, x \neq y\}$$

where s is a Σ_i-term, t is a Σ_j-term, $i \neq j$

Union of Theories: Example

- Two theories:

$$T_1 = \text{Linear Rational Arithmetic}$$

$$T_2 = \begin{cases} \mathrm{r}(\mathrm{w}(v,i,e),i) = e \\ i \neq j \Rightarrow \mathrm{r}(\mathrm{w}(v,i,e),j) = \mathrm{r}(v,j) \end{cases}$$

- $T_1 \cup T_2$-satisfiability of Φ :

$$e = \mathrm{r}(\mathrm{v},j) \wedge e' = \mathrm{r}(\mathrm{v},i) \wedge \mathrm{r}(\mathrm{w}(\mathrm{v},i,e),i) \neq e' \wedge i+j \leq 2j \wedge j+4i \leq 5i$$

Decomposition: Example

Naive approach :

- Decompose Φ into $\Phi_1 \wedge \Phi_2$:

$$\Phi_1 = \quad i+j \leq 2j \ \wedge \ j+4i \leq 5i$$

$$\Phi_2 = \quad e = \mathrm{r}(\mathrm{v}, j) \ \wedge \ e' = \mathrm{r}(\mathrm{v}, i) \ \wedge \ \mathrm{r}(\mathrm{w}(\mathrm{v}, i, e), i) \neq e'$$

- Φ_1 is T_1-satisfiable
- Φ_2 is T_2-satisfiable
- Return *satisfiable* ???

Decomposition (cont'd)

- In fact : *unsatisfiable*. . .

$$i + j \leq 2j \ \wedge \ j + 4i \leq 5i \Rightarrow i = j$$

$$e = \mathrm{r}(\mathrm{v}, j) \ \wedge \ e' = \mathrm{r}(\mathrm{v}, i) \ \wedge \ \mathrm{r}(\mathrm{w}(\mathrm{v}, i, e), i) \neq e' \ \wedge \ i = j \Rightarrow \bot$$

- Problems :
  - Shared variables
  - Shared equality predicate
- Possible solution: *propagation of shared equalities between variables*. . .

A Family Picture: Outline

- State of the art: *combination of disjoint theories*
  - Nelson-Oppen: *conjunction of ground literals*
  - Shostak: *conjunction of ground equalities and inequalities*
- **This lecture**:
  - **New:** rational reconstruction of combination methods
    ➽ classification of component theories

Combination Methods for the Satisfiability Problem

- **Nelson-Oppen Approach**
  - Unions of disjoint theories
  - Combination of satisfiability procedures
  - General brute-force method, easy to understand, but no interest in practice...
- **Shostak Approach**
  - Union of disjoint theories including the theory of equality
  - Combination of Congruence Closure (for the theory of equality) and specific procedures for other theories
  - Efficient method, difficult to prove, but implemented in many systems...

A Rational Reconstruction

Address the following problems:

- Which theories to ensure correctness of combination methods?
- Which procedures to combine?
- How to specify the combination methods to ease understanding and correctness?

Combination of Satisfiability Procedures

Combination procedure: under some conditions,
$\Phi_1 \cup \Phi_2$ is $T_1 \cup T_2$-satisfiable iff there exists some finite set Φ_0 of shared literals such that
$\Phi_1 \cup \Phi_0$ is T_1-satisfiable,
$\Phi_2 \cup \Phi_0$ is T_2-satisfiable,
and Φ_0 states which shared variables are equal or not.

- When is this procedure complete?
  if T_1 and T_2 are signature-disjoint and **stably infinite**

  T is **stably infinite** if for any T-satisfiable Φ, there exists a model of T satisfying Φ and whose domain is infinite.
- How to find Φ_0? Thanks to **Convexity**
  T is **convex** if for any set of literals Φ, we have
  $T \models (\Phi \Rightarrow \bigvee_{i=1}^{n} x_i = y_i)$ if and only if
  $T \models (\Phi \Rightarrow x_j = y_j)$ for some $j \in \{1, ..., n\}$

Arrangements

Guess a shared formula Φ_0 as an arrangement defined by an equivalence relation over a set of variables V.

Notations

Given an equivalence relation $\equiv$ on V, the equivalence class of $x \in V$ w.r.t $\equiv$ is denoted by $[x]_\equiv$.
Let $r : V/\equiv \;\rightarrow\; V$ such that for any $x \in V$, $r([x]_\equiv) \equiv x$. The mapping r returns a representative element for each equivalence class w.r.t $\equiv$.
An equivalence relation $\equiv$ on V uniquely defines a (consistent) set of (shared) V-literals called an **arrangement over** V:

$$\begin{aligned} \Phi_0 \quad = \quad & \{x = r([x]_\equiv) \mid x \in V, x \neq r([x]_\equiv)\} \\ & \cup \{r([x]_\equiv) \neq r([y]_\equiv) \mid x, y \in V,\ r([x]_\equiv) \neq r([y]_\equiv)\} \end{aligned}$$

Example

Let $V = \{x, y, u, v\}$ and $\equiv$ the equivalence relation over V whose equivalence classes are $\{x, y\}, \{u, v\}$. Then $\Phi_0 = \{x = y, u = v, y \neq v\}$.
For $\equiv$ defined by $\{x, y, u, v\}$, $\Phi_0 = \{x = v, y = v, u = v\}$
For $\equiv$ defined by $\{x\}, \{y, u, v\}$, $\Phi_0 = \{y = v, u = v, x \neq v\}$
For $\equiv$ defined by $\{x\}, \{y\}, \{u\}, \{v\}$, $\Phi_0 = \{x \neq y, x \neq u, x \neq v, y \neq u, y \neq v, u \neq v\}$

Non-Deterministic Nelson-Oppen Algorithm

Identification $(\Phi_1 \cup \Phi_2)_{T_1 \cup T_2}$
$\vdash$
$\bigvee_{\Phi_0}((\Phi_1 \cup \Phi_0)_{T_1} \wedge (\Phi_2 \cup \Phi_0)_{T_2})$
where Φ_0 is any arrangement over $Var(\Phi_1) \cap Var(\Phi_2)$

Decision – True	$(\Phi_i)_{T_i}$	$\vdash$	$\top$	if Φ_i is T_i-satisfiable
Decision – False	$(\Phi_i)_{T_i}$	$\vdash$	$\bot$	if Φ_i is T_i-unsatisfiable

Combination Procedure: Completeness

Completeness proof is based on this *Combination Lemma*:

Lemma

Let T_1 be a Σ_1-theory and T_2 a Σ_2-theory such that Σ_1 and Σ_2 are disjoint signatures.

A formula $\Phi_1 \wedge \Phi_2$ is $T_1 \cup T_2$-satisfiable iff there exists an arrangement Φ_0 over the set of shared variables $Var(\Phi_1) \cap Var(\Phi_2)$ such that

- *$\Phi_1 \cup \Phi_0$ is satisfiable in a model $\mathcal{A}_1$ of T_1, and*
- *$\Phi_2 \cup \Phi_0$ is satisfiable in a model $\mathcal{A}_2$ of T_2, and*
- $|A_1| = |A_2|$

Combination Lemma: Proof Sketch

($\Leftarrow$) Consider a model $\mathcal{A}_1$ of the Σ_1-theory T_1 and a model $\mathcal{A}_2$ of the Σ_2-theory T_2 such that $\mathcal{A}_1[\Phi_1 \cup \Phi_0]$ and $\mathcal{A}_2[\Phi_2 \cup \Phi_0]$ are true.

Let $h : A_1 \to A_2$ be a **bijective** mapping.

- construct a model $\mathcal{A}$ of $T_1 \cup T_2$ such that $\mathcal{A}^{\Sigma_1} \simeq \mathcal{A}_1$ and $\mathcal{A}^{\Sigma_2} \simeq \mathcal{A}_2$:
  The domain of $\mathcal{A}$ is defined as the domain A_1 of $\mathcal{A}_1$ and the function symbols are interpreted in $\mathcal{A}$ as follows:
  $\mathcal{A}[f_1] = \mathcal{A}_1[f_1]$ if $f_1 \in \Sigma_1$
  $\forall a_1, \ldots, a_n \in A_1,\ \mathcal{A}[f_2](a_1, \ldots, a_n) = h^{-1}(\mathcal{A}_2[f_2](h(a_1), \ldots, h(a_n)))$ if $f_2 \in \Sigma_2$
- construct an interpretation $\mathcal{A}$ satisfying both $\Phi_1 \cup \Phi_0$ and $\Phi_2 \cup \Phi_0$.
  Thanks to the arrangement Φ_0, we have $\mathcal{A}_1[x] = \mathcal{A}_1[y]$ iff $\mathcal{A}_2[x] = \mathcal{A}_2[y]$ for any shared variables x, y
  ➻ h is chosen such that $\mathcal{A}_1[x] = h^{-1}(\mathcal{A}_2[x])$ for any shared variable x
  $\mathcal{A}[x] = \mathcal{A}_1[x]$ for $x \in Var(\Phi_1)$
  $\mathcal{A}[x] = h^{-1}(\mathcal{A}_2[x])$ for $x \in Var(\Phi_2)$

Eventually $\mathcal{A}$ is a model of $T_1 \cup T_2$ such that $\mathcal{A}[\Phi_1 \cup \Phi_0 \cup \Phi_2]$ is true.

Combination Lemma with Stably Infinite Theories

Theorem (Upward Lowenheim-Skolem Theorem)

If a formula is satisfiable in a model of infinite cardinality, then it is satisfiable in a model of greater cardinality

Consequence: the assumption $|A_1| = |A_2|$ can always be satisfied in stably infinite theories.

How to get rid of arrangements?

Problem: there is a lot of arrangements to consider...

Remark

Number of arrangements = Bell number

$B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k$ where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

$B_0 = 1, B_1 = 1, B_2 = 2, B_3 = 5, B_4 = 15, B_5 = 52, B_6 = 203, \ldots$

Solutions: use a more deductive approach to reduce the number of cases to consider

- Computation of entailed elementary equalities (for convex theories)
- Computation of entailed disjunctions of elementary equalities (for arbitrary theories)

Deterministic Nelson-Oppen Algorithm (1/2)

Assumption: T_1 and T_2 are **convex** (length of disjunctions is 1)

$$\text{Contradiction}_1 \quad \Phi_1 ; \Phi_2 \quad \vdash \quad \mathit{false}$$

if Φ_1 is T_1-unsatisfiable

$$\text{Deduction}_1 \quad \Phi_1 ; \Phi_2 \quad \vdash \quad \Phi_1 \cup \{x = y\} ; \Phi_2 \cup \{x = y\}$$

$$\text{if} \begin{cases} \Phi_1 \text{ is } T_1\text{-satisfiable,} \\ \Phi_2 \text{ is } T_2\text{-satisfiable,} \\ T_1 \models \Phi_1 \Rightarrow x = y \\ T_2 \not\models \Phi_2 \Rightarrow x = y \end{cases}$$

Contradiction$_2$ and Deduction$_2$ obtained by symmetry

Deterministic Nelson-Oppen Algorithm (2/2)

$$\text{Contradiction}_2 \quad \Phi_1 ; \Phi_2 \quad \vdash \quad \mathit{false}$$

if Φ_2 is T_2-unsatisfiable

$$\text{Deduction}_2 \quad \Phi_1 ; \Phi_2 \quad \vdash \quad \Phi_1 \cup \{x = y\} ; \Phi_2 \cup \{x = y\}$$

$$\text{if} \begin{cases} \Phi_1 \text{ is } T_1\text{-satisfiable,} \\ \Phi_2 \text{ is } T_2\text{-satisfiable,} \\ T_2 \models \Phi_2 \Rightarrow x = y \\ T_1 \not\models \Phi_1 \Rightarrow x = y \end{cases}$$

Soundness Proof (Deterministic Nelson-Oppen Algo)

Consider $\mathrm{R} = \{\mathrm{Contradiction_i}, \mathrm{Deduction_i}\}_{i=1,2}$
Termination: since there are finitely many shared variables
Soundness: $\varphi_{\downarrow_\mathrm{R}} \neq$ *false* iff φ is $T_1 \cup T_2$-satisfiable
($\Leftarrow$) because each rule of R transforms the premise into a $T_1 \cup T_2$-equivalent conclusion.
($\Rightarrow$) By contradiction: Assume that a normal form $\varphi'_1 \wedge \varphi'_2$ ($\neq$ *false*) is unsatisfiable
Apply Combination Lemma, with the arrangement Φ_0 defined by the shared equalities entailed by both φ'_1 and φ'_2: we have that $\varphi'_i \cup \Phi_0$ is T_i-unsatisfiable.

- If Φ_0 contains no disequality, then φ'_i is T_i-equivalent to $\varphi'_i \cup \Phi_0$. Hence, φ'_i is T_i-unsatisfiable and $\mathrm{Contradiction_i}$ applies: impossible.
- Otherwise, $\varphi'_i \cup \Phi_0$ is T_i-unsatisfiable and $\varphi'_j \cup \Phi_0$ is T_j-satisfiable for $j \neq i$. So φ'_i implies a new equality (by convexity) and $\mathrm{Deduction_i}$ applies: impossible.

Nelson-Oppen in Practice

How to deduce elementary equalities?
Default case: by refutation, with the satisfiability procedure

$$T \models \Phi \Rightarrow x = y \text{ iff } \Phi \wedge x \neq y \text{ is } T\text{-unsatisfiable}$$

other *cleverer* solutions?
➽ see the combination procedure proposed by Shostak

What is the Shostak Combination Procedure?

- a seminal paper published in '80, whith a strong impact in the Automated Deduction Community
- applied for combining the arithmetic and the equality theory in an efficient way
- but difficult to understand and to prove
- a lot of papers, that aims at correcting the description initially given by Shostak using pseudo-code

Shostak Theories

A *Shostak* theory T admits two specialized procedures:

- **Solver** for T: $solve(s = t)$
  - If $T \models s \neq t$ then return *false*
  - Else, return a most general **solution** (substitution) $\sigma = \{x_i \to t_i\}_{i \in I}$, where $\widehat{\sigma}$ denotes the related **solved form**

$$\widehat{\sigma} = \bigwedge_{i \in I} x_i = t_i$$

- **Canonizer** for T: an idempotent function *canon* such that

$$T \models s = t \text{ iff } \models canon(s) = canon(t)$$

- Motivation: **Solver + Canonizer** implies (decidability of) satisfiability

Shostak Theories: Example

Linear Arithmetic over the Rationals is a Shostak theory, where a solver can be implemented thanks to Gauss Elimination, and a canonizer is provided by the canonical forms of linear expressions (using a given ordering on variables).

For instance,

$z + 1 + y + x + z + y + 3 + y$ canonized into $4 + x + 3y + 2z$

A solved form of

$$\Gamma = \begin{cases} x + y + z = 10 \\ 2x + y + 3z = 20 \\ v + y + 5 = -z + 15 \end{cases}$$

is $\widehat{\sigma} = \{x = 10 - 2z, y = z, v = -z + 15 - z - 5\}$ and the corresponding most general solution is the substitution $\sigma = \{x \mapsto 10 - 2z, y \mapsto z, v \mapsto -z + 15 - z - 5\}$

Shostak Theories: Counter-Example

The theory of equality does not admit a solver, it is not a Shostak theory...

Satisfiability Procedure for Shostak Theories

Γ is a set equalities and Δ is a set of disequalities

- **Solve – fail**

$$\frac{\Gamma, \Delta}{\mathit{false}} \quad \text{if } \mathit{solve}(\Gamma) = \mathit{false}$$

- **Solve – success**

$$\frac{\Gamma, \Delta}{\widehat{\sigma}, \Delta} \quad \text{if} \begin{cases} \Gamma \text{ is not in solved form} \\ \sigma = \mathit{solve}(\Gamma) \neq \mathit{false} \end{cases}$$

- **Contradiction**

$$\frac{\widehat{\sigma}, \Delta}{\mathit{false}} \quad \text{if} \begin{cases} s \neq t \in \Delta \\ \mathit{canon}(s\sigma) = \mathit{canon}(t\sigma) \end{cases}$$

Entailement of Elementary Equalities

1. Disequalities are useless to entail elementary equalities
2. The canonizer can be used to check the equality of solved variables

Proposition

1. *(thanks to convexity)*

$$T \models (\Gamma \cup \Delta \Rightarrow \mathbf{x} = \mathbf{y}) \textit{ iff } T \models (\Gamma \Rightarrow \mathbf{x} = \mathbf{y})$$

where Γ *is set of equalities,* Δ *is a set of disequalities, and* $\Gamma \cup \Delta$ *is* T*-satisfiable*

2. $T \models (\widehat{\sigma} \Rightarrow \mathbf{x} = \mathbf{y})$ *iff* $canon(\mathbf{x}\sigma) = canon(\mathbf{y}\sigma)$ *where* σ *is the substitution associated to the solved form* $\widehat{\sigma}$

Entailement of Elementary Equalities: An Example

Let us consider the solved form

$$\widehat{\sigma} = \{x = 10 - 2z, y = z, v = -z + 15 - z - 5\}$$

We have $\widehat{\sigma} \Rightarrow y = z$ since $y\sigma = z$ and $z\sigma = z$ have the same canonical form z

We have $\widehat{\sigma} \Rightarrow x = v$
since $x\sigma = 10 - 2z$ and $v\sigma = -z + 15 - z - 5$
have the same canonical form, say $10 - 2z$

Combination Algorithms for Convex Theories

- Consider the following classes of theories:
  NOc = stably infinite + convex + satisfiability proc.
  SH = stably infinite + convex + canonizer + solver
- Study the combination of theories $T_1 \cup T_2$ such that ($T_1, T_2 \in$ **NOc**) or ($T_1 \in$ **NOc**, $T_2 \in$ **SH**)
  1. Develop combination rules for **NOc** $+$ **NOc**
  2. Refine these combination rules for the case $T_2 \in$ **SH**
     ➼ Important case in practice: $T_1 = UF$, $T_2 = LRA$

NOc + NOc Combination (1/2)

Transform $\Phi_1 ; \Phi_2$ such that Φ_i consists of T_i-pure literals
NO_1 rules:

- **Contradiction$_1$**

$$\frac{\Phi_1 ; \Phi_2}{\mathit{false}} \quad \text{if } \Phi_1 \text{ is } T_1\text{-unsatisfiable}$$

- **Deduction$_1$**

$$\frac{\Phi_1 ; \Phi_2}{\Phi_1 ; \Phi_2 \cup \{x = y\}} \quad \text{if } \begin{cases} \Phi_1 \text{ is } T_1\text{-satisfiable} \\ \Phi_1 \wedge x \neq y \text{ is } T_1\text{-unsatisfiable} \\ \Phi_2 \wedge x \neq y \text{ is } T_2\text{-satisfiable} \\ x, y \in Var(\Phi_1) \cap Var(\Phi_2) \end{cases}$$

NO_2 rules: obtained by symmetry

NOc + NOc Combination (2/2)

NO_2 rules:

- **Contradiction$_2$**

$$\frac{\Phi_1 ; \Phi_2}{\mathit{false}} \quad \text{if } \Phi_2 \text{ is } T_2\text{-unsatisfiable}$$

- **Deduction$_2$**

$$\frac{\Phi_1 ; \Phi_2}{\Phi_1 \cup \{x = y\} ; \Phi_2} \quad \text{if} \begin{cases} \Phi_2 \text{ is } T_2\text{-satisfiable} \\ \Phi_2 \wedge x \neq y \text{ is } T_2\text{-unsatisfiable} \\ \Phi_1 \wedge x \neq y \text{ is } T_1\text{-satisfiable} \\ x, y \in Var(\Phi_1) \cap Var(\Phi_2) \end{cases}$$

NOc $+$ SH Combination

Transform $\Phi_1; \Gamma_2, \Delta_2$ such that
Φ_1 consists of T_1-pure literals
Γ_2 (resp. Δ_2) consists of of T_2-pure equalities (resp. disequalities)

Combination rules: use of Shostak rules to check satisfiability and to perform the entailment of shared equalities

NOc + SH Combination: NO rules

- **Contradiction$_1$**

$$\frac{\Phi_1 ; \Gamma_2, \Delta_2}{\mathit{false}} \quad \text{if } \Phi_1 \text{ is } T_1\text{-unsatisfiable}$$

- **Deduction$_1$**

$$\frac{\Phi_1 ; \widehat{\sigma_2}, \Delta_2}{\Phi_1 ; \widehat{\sigma_2} \cup \{x = y\}, \Delta_2} \quad \text{if } \begin{cases} \Phi_1 \text{ is } T_1\text{-satisfiable} \\ \Phi_1 \wedge x \neq y \text{ is } T_1\text{-unsatisfiable} \\ \mathit{canon}_2(x\sigma_2) \neq \mathit{canon}_2(y\sigma_2) \\ x, y \in \mathit{Var}(\Phi_1) \cap \mathit{Var}(\widehat{\sigma_2}) \end{cases}$$

NOc + SH Combination: SH rules (using Solver)

- **Solve** – **fail$_2$**

$$\frac{\Phi_1 ; \Gamma_2, \Delta_2}{\mathit{false}} \quad \text{if } \mathit{solve}_2(\Gamma_2) = \mathit{false}$$

- **Solve** – **success$_2$**

$$\frac{\Phi_1 ; \Gamma_2, \Delta_2}{\Phi_1 ; \widehat{\sigma_2}, \Delta_2} \quad \text{if } \begin{cases} \Gamma_2 \text{ is not in solved form} \\ \sigma_2 = \mathit{solve}_2(\Gamma_2) \neq \mathit{false} \end{cases}$$

NOc + SH Combination: SH rules (using Canonizer)

- **Contradiction$_2$**

$$\frac{\Gamma_1; \widehat{\sigma_2}, \Delta_2}{false} \quad \text{if} \begin{cases} s \neq t \in \Delta_2 \\ canon_2(s\sigma_2) = canon_2(t\sigma_2) \end{cases}$$

- **Deduction$_2$**

$$\frac{\Phi_1; \widehat{\sigma_2}, \Delta_2}{\Phi_1 \cup \{x = y\}; \widehat{\sigma_2}, \Delta_2} \quad \text{if} \begin{cases} \Phi_1 \wedge x \neq y \text{ is } T_1\text{-satisfiable} \\ canon_2(x\sigma_2) = canon_2(y\sigma_2) \\ x, y \in Var(\Phi_1) \cap Var(\widehat{\sigma_2}) \end{cases}$$

Deductive Nelson-Oppen Algorithm (general case)

$$\text{Contradiction}_1 \quad \Phi_1 ; \Phi_2 \quad \vdash \quad \mathit{false}$$

if Φ_1 is T_1-unsatisfiable

$$\text{Deduction}_1 \quad \Phi_1 ; \Phi_2 \quad \vdash \quad \bigvee_{k \in K} (\Phi_1 \cup \{x_k = y_k\} ; \Phi_2 \cup \{x_k = y_k\})$$

$$\text{if} \begin{cases} \Phi_1 \text{ is } T_1\text{-satisfiable,} \\ \Phi_2 \text{ is } T_2\text{-satisfiable,} \\ T_1 \models \Phi_1 \Rightarrow \bigvee_{k \in K} x_k = y_k \\ T_2 \not\models \Phi_2 \Rightarrow \bigvee_{k \in K} x_k = y_k \end{cases}$$

Contradiction_2 and Deduction_2 obtained by symmetry

Combination Results à la Nelson-Oppen

Nelson-Oppen Deductive algorithm (1979)

Nelson's thesis introduction of stably infinite theories, combination lemma

Harandi & Tinelli correction of a flaw in the seminal Nelson-Oppen paper

Ringeissen & Tinelli a non-disjoint extension (using shared constructors)

Baader & Tinelli application to the word-problem, in a non-disjoint constructor-based case

Baader & Ghilardi & Tinelli non-disjoint extension, with applications to modal logics

Zarba combining lists/sets/multisets with integers

Tinelli & Zarba Beyond stably infinite theories (mono-sorted logic)

Combination Results à la Shostak

- **Shostak** combination algorithm (1980)
- **Ruess & Shankar** correction of flaws in the seminal Shostak paper, using the same pseudo-code presentation
- **Barrett, Kapur** first comparison with Nelson-Oppen
- **Conchon & Krstic** a rule-based description of the Shostak combination method implemented in ICS
- **Manna & Zarba** use of Shostak decision procedures in the Nelson-Oppen combination method
- **Ganzinger et al.** integration of a solver and a canonizer in the superposition calculus