Using the Simplex Method in Mixed Integer Linear Programming

Carlos Castro

UTFSM

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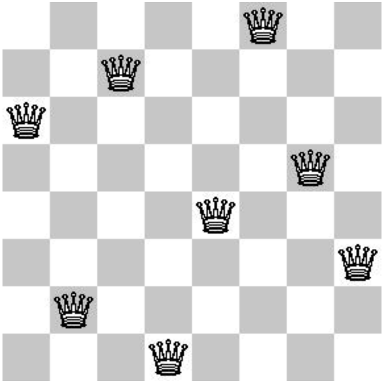
Outline

Constraint Satisfaction Problems

A Constraint Satisfaction Problem (CSP) consists of:

Variables:	The unknown, what we can modify
Parameters:	The known, what we cannot modify
Constraints:	Relationships among variables and parameters

N-Queens Problem



Map Colouring Problem

Constraint Satisfaction Problems

Given a CSP we can define:

Solution:	An assignment of values to variables
Feasible solution:	A solution satisfying all constraints

A CSP can be

Satisfiable:	If there exists at least one feasible solution
Unsatisfiable:	If there does not exist a feasible solution

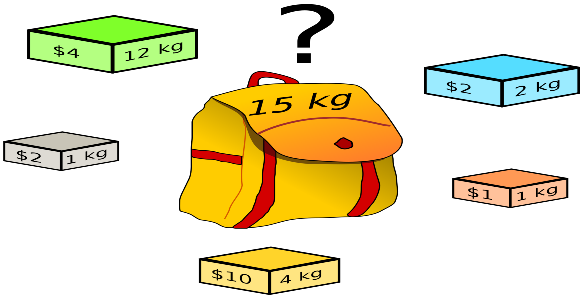
Solving a CSP means to search for one or all feasible solutions, or to prove that the CSP is unsatisfiable.

Constraint Satisfaction Optimisation Problems

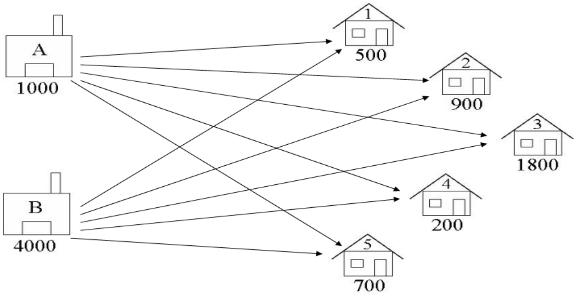
A Constraint Satisfaction Optimisation Problem (CSOP) consists of:

CSP:	Variables, parameters, and constraints
Objective function(s):	Function(s) to be optimised satisfying all constraints

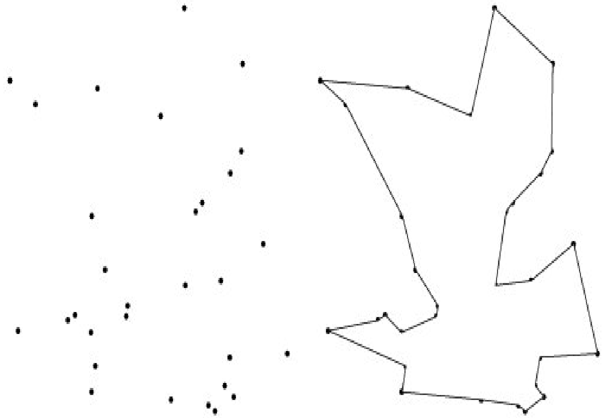
Knapsack Problem



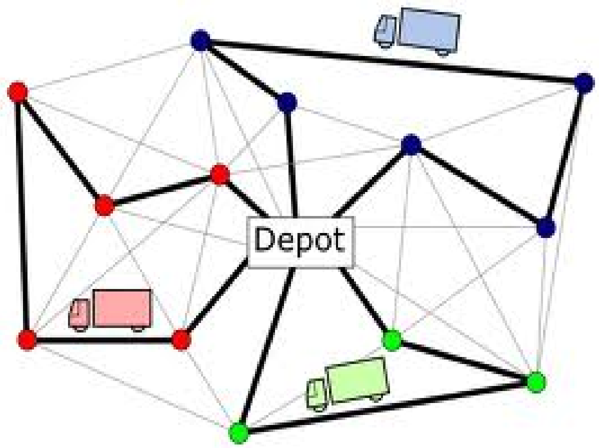
Transportation Problems



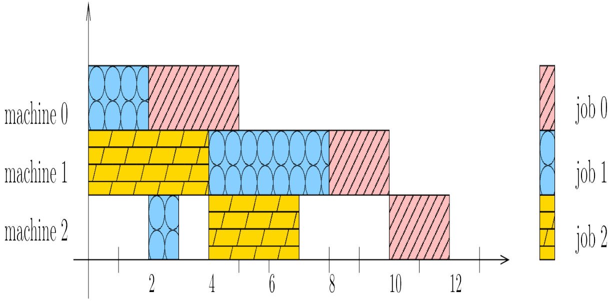
Traveling Salesperson Problem (TSP)



Vehicle Routing Problems (VRP)



Scheduling Problem



Given a CSOP we can define:

Optimal solution: A feasible solution better than or equal to all other feasible solutions

Depending on the set of constraints, a CSOP can be

Feasible: If there exists at least one feasible solution

Unfeasible: If there does not exist a feasible solution

Depending on the set of constraints and the objective function, a feasible CSOP can have

Unbounded Feasible Solutions: Feasible solutions can grow indefinitely

Unique Optimal Solution: Just one optimal solution

Multiple Optimal Solutions: Several optimal solutions

Solving a CSOP means to search for optimal solution(s) or to prove that the CSOP is unfeasible.

Solving Techniques

Both CSPs and CSOPs can be tackled using two approaches:

- Global or Complete Techniques: Search the global optimal solution, the best one among all feasible solutions
- Local or Incomplete Techniques: Search for a local optimal solution, the best one among the considered feasible solutions

Linear Programming

- Linear: All mathematical functions are linear
- Programming: In the sense of planning activities
- Industrial Applications: Blending, transportation, etc.
- Origin: George B. Dantzig, 1947

Graphical method

1. Assign variables x_1 and x_2 to axes x and y of plane, respectively.
2. Identify the feasible area:
   - Applying non-negativity constraints, select positive side of both axes.
   - Draw functional constraints.
3. Identify the feasible solution in the feasible area that optimise the value of the objective function.

Example

Considering the following Linear Programming problem:

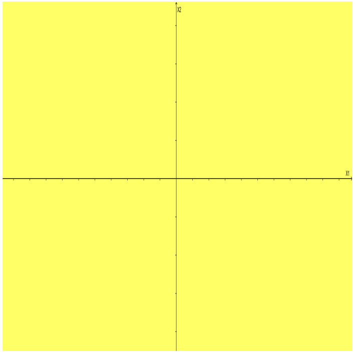
$$Max \quad z = 11x_1 + 14x_2$$

Subject to

$$\begin{aligned} 1x_1 + 1x_2 &\leq 17 \\ 3x_1 + 7x_2 &\leq 63 \\ 3x_1 + 5x_2 &\leq 48 \\ 3x_1 + 1x_2 &\leq 30 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Variable representation

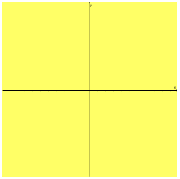
- x_1 and x_2 are assigned to axes x and y of a plane, respectively.



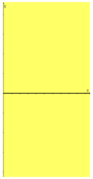
- Each point (x_1, x_2) represents a solution.
- The solution where $x_1 = 0 \ \wedge \ x_2 = 0$ is called the origin.

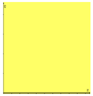
Set of feasible solutions

- Non-negativity constraints, $x_1 \geq 0$ y $x_2 \geq 0$, impose (x_1, x_2) will be in the positive side of both axes.



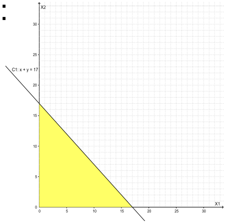
$$\Downarrow$$



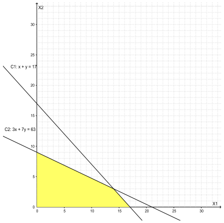


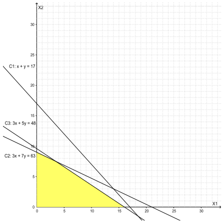
Set of feasible solutions

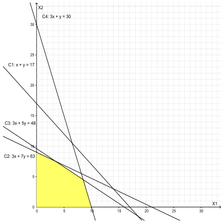
- Constraint $1x_1 + 1x_2 \leq 17$ allows solutions (x_1, x_2) under line $1x_1 + 1x_2 = 170$:



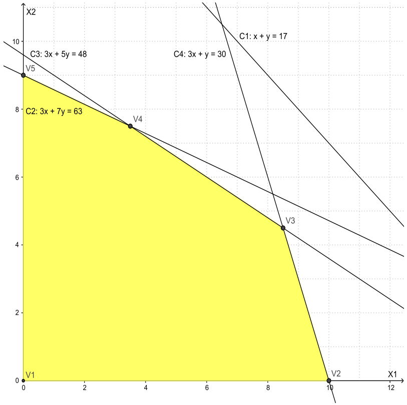
- Doing the same for all constraints:







Set of feasible solutions

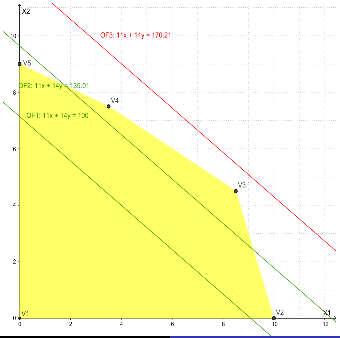


- The set of feasible solutions is called feasible area.

Optimising

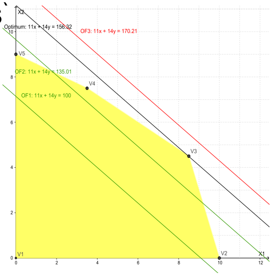
Assigning arbitrary values to the objective function $11x_1 + 14x_2$:

- $11x_1 + 14x_2 = 100$: there exist some feasible solutions
- $11x_1 + 14x_2 = 135$: there exist some feasible solutions
- $11x_1 + 14x_2 = 170$: there does not exist any feasible solution



Optimising

Different values of the objective function generate different parallel lines, the higher the value, the farther the line from the origin. Feasible solution(s) on the farthest line correspond to the optimal solution(s`



Optimal solution is in the intersection of constraints c3 and c4:

$$3x_1 + 5x_2 = 48 \quad \wedge \quad 3x_1 + 1x_2 = 30$$

solving the system of two equations we get:

$$(x_1^*, x_2^*) = (17/2, 9/2); \quad z^* = 11 \times 17/2 + 14 \times 9/2 = 313/2$$

Properties of feasible solutions at vertices

1. Location of optimal solutions:
   - If there exists an optimal solution, it has to be a feasible solution at a vertex.
   - If there exist multiple optimal solutions, at least two of them have to be feasible solutions at adjacent vertices.
2. There exists a finite number of feasible solutions at the vertices.
3. If a feasible solution at a vertex is equal or better than all the feasible solutions at adjacent vertices, then it is equal or better than all the feasible solutions at the vertices, that is, it is optimal.

Standard form

$$Min\ \ z = 1x_1 + 1x_2$$

Subject to

$$\begin{aligned} 1x_1 + 2x_2 &\leq 30 \\ 2x_1 + 1x_2 &\geq 28 \\ 1x_1 - 2x_2 &= 10 \\ x_1, x_2 &\geq 0 \end{aligned}$$

The standard form of this problem is the following:

$$Min\ \ z = 1x_1 + 1x_2 + 0s_1 + 0s_2$$

Subject to

$$\begin{aligned} 1x_1 + 2x_2 + 1s_1 &= 30 \\ 2x_1 + 1x_2 - 1s_2 &= 28 \\ 1x_1 - 2x_2 &= 10 \\ x_1, x_2, s_1, s_2 &\geq 0 \end{aligned}$$

Iterative Optimisation

- Algorithm Structure
  - Initialisation procedure
  - Iterative process
  - Stopping criterion

- Optimisation Algorithm Structure
  - Search for an initial solution
  - Iteratively search for a better solution
  - Optimality test

Overview of the Simplex Method

1. Initial step:
   - Start in a feasible basic solution at a vertex.
2. Iterative step:
   - Move to a better feasible basic solution at an adjacent vertex
3. Optimality test:
   - A feasible basic solution at a vertex is optimal when it is equal or better than feasible basic solutions at all adjacent vertices.

Creating the initial simplex tableau

Given the following linear programming problem in standard form:

$$Max \quad z = 11x_1 + 14x_2 + 0s_1 + 0s_2 + 0s_3 + 0s_4$$

Subject to

$$\begin{aligned}
1x_1 + 1x_2 \quad + \quad 1s_1 \quad\quad\quad\quad &= 17 \\
3x_1 + 7x_2 \quad + \quad\quad 1s_2 \quad\quad\quad &= 63 \\
3x_1 + 5x_2 \quad + \quad\quad\quad 1s_3 \quad\quad &= 48 \\
3x_1 + 1x_2 \quad + \quad\quad\quad\quad 1s_4 &= 30 \\
x_1, x_2, s_1, s_2, s_3, s_4 &\geq 0
\end{aligned}$$

Let:

c_j: coefficient for variable j in the objective function

a_{ij}: coefficient for variable j in constraint i

b_i: right-hand side value of constraint i

A simplex tableau is the following:

$$\begin{array}{ccc|c} c_1 & \dots & c_n & \\ \hline a_{11} & \dots & a_{1n} & b_1 \\ \vdots & \ddots & \vdots & \vdots \\ a_{m1} & \dots & a_{mn} & b_m \end{array}$$

$$\begin{array}{c|c} Row\ \ c & \\ \hline Matrix\ \ A & Column\ \ b \end{array}$$

Creating the initial simplex tableau

$$Max \quad z = 11x_1 + 14x_2 + 0s_1 + 0s_2 + 0s_3 + 0s_4$$

Subject to

$$\begin{array}{rcl}
1x_1 + 1x_2 + 1s_1 & = & 17 \\
3x_1 + 7x_2 + 1s_2 & = & 63 \\
3x_1 + 5x_2 + 1s_3 & = & 48 \\
3x_1 + 1x_2 + 1s_4 & = & 30
\end{array}$$

$$x_1, x_2, s_1, s_2, s_3, s_4 \geq 0$$

x_1	x_2	s_1	s_2	s_3	s_4	
11	14	0	0	0	0	b_i
1	1	1	0	0	0	17
3	7	0	1	0	0	63
3	5	0	0	1	0	48
3	1	0	0	0	1	30

To read the solutions we add two columns:

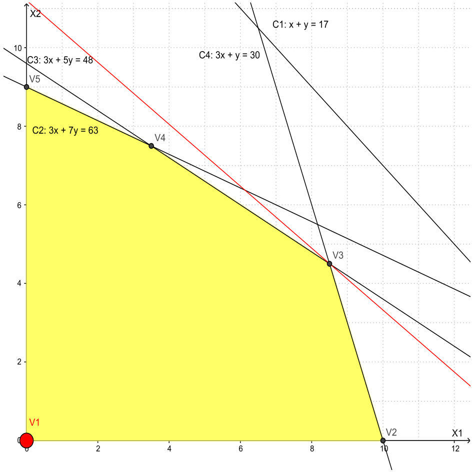
Basis: Current basic variables

c_j: Coefficients in the objective function of current basic variables

		x_1	x_2	s_1	s_2	s_3	s_4	
Basis	c_j	11	14	0	0	0	0	b_i
s_1	0	1	1	1	0	0	0	17
s_2	0	3	7	0	1	0	0	63
s_3	0	3	5	0	0	1	0	48
s_4	0	3	1	0	0	0	1	30
								0

- As all no-basic variables are assigned value zero, the value of basic variables is obtained from column b_i at the corresponding row:
$$(x_1, x_2, s_1, s_2, s_3, s_4) = (0, 0, 17, 63, 48, 30)$$
- The product of the column c_j and the column b_i turns in the value of the objective function (indicated under column b_i)

Graphically



Evaluating adjacent feasible basic solutions

How about the quality of this basic solution with respect to adjacent basic solutions?

Changing the value of a variable:

- Impacts directly the objective function through its coefficient in the objective function (c_j)
- Impacts indirectly the objective function through variations in other variables
- We need to take into account both impacts

Evaluating adjacent feasible basic solutions

The direct impact in the objective function is easily computed by c_j, but for the decrease and the net change we will add two rows into the tableau:

z_j:	Decrease in the objective function when increasing variable j by one.
$c_j - z_j$:	Net change in the objective function when increasing variable j by one.

Computing z_j

$$
\begin{aligned}
z_1 &= 0 \times 1 + 0 \times 3 + 0 \times 3 + 0 \times 3 = 0 \\
z_2 &= 0 \times 1 + 0 \times 7 + 0 \times 5 + 0 \times 1 = 0 \\
z_3 &= 0 \times 1 + 0 \times 0 + 0 \times 0 + 0 \times 0 = 0 \\
z_4 &= 0 \times 0 + 0 \times 1 + 0 \times 0 + 0 \times 0 = 0 \\
z_5 &= 0 \times 0 + 0 \times 0 + 0 \times 1 + 0 \times 0 = 0 \\
z_6 &= 0 \times 0 + 0 \times 0 + 0 \times 0 + 0 \times 1 = 0
\end{aligned}
$$

Adding this information into the tableau ...

		x_1	x_2	s_1	s_2	s_3	s_4	
Basis	c_j	11	14	0	0	0	0	b_i
s_1	0	1	1	1	0	0	0	17
s_2	0	3	7	0	1	0	0	63
s_3	0	3	5	0	0	1	0	48
s_4	0	3	1	0	0	0	1	30
z_j		0	0	0	0	0	0	0

Entering x_1 or x_2 does not decrease the obj func

Computing $c_j - z_j$

$$
\begin{aligned}
c_1 - z_1 &= 11 - 0 = 11 \\
c_2 - z_2 &= 14 - 0 = 14 \\
c_3 - z_3 &= 0 - 0 = 0 \\
c_4 - z_4 &= 0 - 0 = 0 \\
c_5 - z_5 &= 0 - 0 = 0 \\
c_6 - z_6 &= 0 - 0 = 0
\end{aligned}
$$

Adding this information into the tableau ...

		x_1	x_2	s_1	s_2	s_3	s_4	
Basis	c_j	11	14	0	0	0	0	b_i
s_1	0	1	1	1	0	0	0	17
s_2	0	3	7	0	1	0	0	63
s_3	0	3	5	0	0	1	0	48
s_4	0	3	1	0	0	0	1	30
z_j		0	0	0	0	0	0	0
$c_j - z_j$		11	14	0	0	0	0	

It is not optimal, entering x_1 or x_2 allows to improve the obj func

Criterion to enter a non-basic variable into the basis

The first criterion proposed to select the entering non-basic variable was to choose the variable with highest $c_j - z_j$, i.e, with highest unit increase in the objective function.

		x_1	x_2	s_1	s_2	s_3	s_4	
Basis	c_j	11	14	0	0	0	0	b_i
s_1	0	1	1	1	0	0	0	17
s_2	0	3	7	0	1	0	0	63
s_3	0	3	5	0	0	1	0	48
s_4	0	3	1	0	0	0	1	30
z_j		0	0	0	0	0	0	0
$c_j - z_j$		11	14	0	0	0	0	
			$\uparrow$					

Of course, we would like to increase the value of x_2 as much as possible, but we have to take into account all constraints!

Criterion to remove a basic variable from the basis

The maximum value for x_2 is giving by the first current basic variable getting value zero when x_1 is increasing.
Considering column j corresponding to the non-basic variable entering into the basis, we compute rate $\frac{b_i}{a_{ij}}$ (only for $a_{ij} > 0$) for each row i, and we select the basic variable giving the minimum $\frac{b_i}{a_{ij}}$.

		x_1	x_2	s_1	s_2	s_3	s_4			
Basis	c_j	11	14	0	0	0	0	b_i	$\frac{b_i}{a_{ij}}$	
s_1	0	1	1	1	0	0	0	17	$\frac{17}{1} = 17$	
s_2	0	3	7	0	1	0	0	63	$\frac{63}{7} = 9$	$\rightarrow$
s_3	0	3	5	0	0	1	0	48	$\frac{48}{5} = 9\frac{3}{5}$	
s_4	0	3	1	0	0	0	1	30	$\frac{30}{1} = 30$	
z_j		0	0	0	0	0	0	0		
$c_j - z_j$		11	14	0	0	0	0			
			$\uparrow$							

Pivoting

Basis	c_j	x_1 11	x_2 14	s_1 0	s_2 0	s_3 0	s_4 0	b_i	$\frac{b_i}{a_{ij}}$	
s_1	0	1	1	1	0	0	0	17	17	
s_2	0	3	7	0	1	0	0	63	9	$\rightarrow$
s_3	0	3	5	0	0	1	0	48	$9\frac{3}{5}$	
s_4	0	3	1	0	0	0	1	30	30	
z_j		0	0	0	0	0	0	0		
$c_j - z_j$		11	14	0	0	0	0			
			$\uparrow$							

The adjacent basic solution is obtained replacing s_2 by x_2 in the basis, and operating rows to obtain in column x_2 the following vector:

$$\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Pivoting

Using the Gauss Elimination Method, we can

- Multiply a constraint by a constant k ($k \neq 0$)
- Add constraints

In our case:

- Multiplying the second row by $\frac{1}{7}$ we get the new second row.
- Multiplying the new second row by -1 and adding it to the first row we get the new first row.
- Multiplying the new second row by -5 and adding it to the third row we get the new third row.
- Multiplying the new second row by -1 and adding it to the fourth row we get the new fourth row.

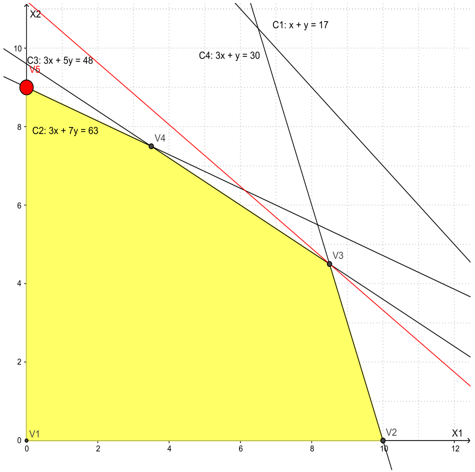
Pivoting

		x_1	x_2	s_1	s_2	s_3	s_4			
Basis	c_j	11	14	0	0	0	0	b_i	$\frac{b_i}{a_{ij}}$	
s_1	0	1	1	1	0	0	0	17	17	
s_2	0	3	7	0	1	0	0	63	9	$\rightarrow$
s_3	0	3	5	0	0	1	0	48	$9\frac{3}{5}$	
s_4	0	3	1	0	0	0	1	30	30	
z_j		0	0	0	0	0	0	0		
$c_j - z_j$		11	14	0	0	0	0			
			$\uparrow$							

		x_1	x_2	s_1	s_2	s_3	s_4	
Basis	c_j	11	14	0	0	0	0	b_i
s_1	0	$\frac{4}{7}$	0	1	$\frac{-1}{7}$	0	0	8
x_2	14	$\frac{3}{7}$	1	0	$\frac{1}{7}$	0	0	9
s_3	0	$\frac{6}{7}$	0	0	$\frac{-5}{7}$	1	0	3
s_4	0	$\frac{18}{7}$	0	0	$\frac{-1}{7}$	0	1	21
								126

New feasible basic solution: $(x_1, x_2, s_1, s_2, s_3, s_4) = (0, 9, 8, 0, 3, 21)$; $z = 126$

Graphically



Evaluating adjacent feasible basic solutions

		x_1	x_2	s_1	s_2	s_3	s_4	
Basis	c_j	11	14	0	0	0	0	b_i
s_1	0	$\frac{4}{7}$	0	1	$\frac{-1}{7}$	0	0	8
x_2	14	$\frac{3}{7}$	1	0	$\frac{1}{7}$	0	0	9
s_3	0	$\frac{6}{7}$	0	0	$\frac{-5}{7}$	1	0	3
s_4	0	$\frac{18}{7}$	0	0	$\frac{-1}{7}$	0	1	21

Computing z_j and $c_j - z_j$ ...

		x_1	x_2	s_1	s_2	s_3	s_4	
Basis	c_j	11	14	0	0	0	0	b_i
s_1	0	$\frac{4}{7}$	0	1	$\frac{-1}{7}$	0	0	8
x_2	14	$\frac{3}{7}$	1	0	$\frac{1}{7}$	0	0	9
s_3	0	$\frac{6}{7}$	0	0	$\frac{-5}{7}$	1	0	3
s_4	0	$\frac{18}{7}$	0	0	$\frac{-1}{7}$	0	1	21
z_j		6	14	0	2	0	0	126
$c_j - z_j$		5	0	0	-2	0	0	

$\exists \; c_j - z_j > 0$ so there exists a better adjacent basic solution and the current one is not optimal

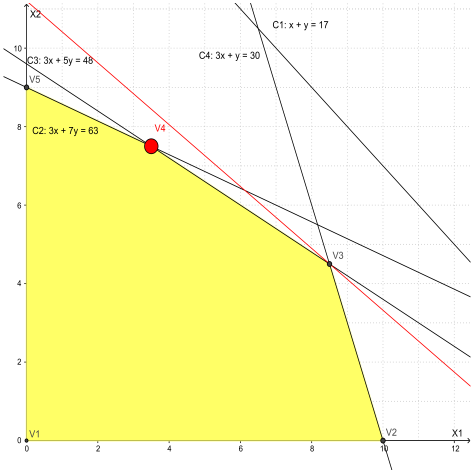
Pivoting

		x_1	x_2	s_1	s_2	s_3	s_4			
Basis	c_j	11	14	0	0	0	0	b_i	$\frac{b_i}{a_{ij}}$	
s_1	0	$\frac{4}{7}$	0	1	$\frac{-1}{7}$	0	0	8	$\frac{8}{\frac{4}{7}} = 14$	
x_2	14	$\frac{3}{7}$	1	0	$\frac{1}{7}$	0	0	9	$\frac{9}{\frac{3}{7}} = 21$	
s_3	0	$\frac{6}{7}$	0	0	$\frac{-5}{7}$	1	0	3	$\frac{3}{\frac{6}{7}} = 3\frac{1}{2}$	$\rightarrow$
s_4	0	$\frac{18}{7}$	0	0	$\frac{-1}{7}$	0	1	21	$\frac{21}{\frac{18}{7}} = 8\frac{1}{6}$	
z_j		6	14	0	2	0	0	126		
$c_j - z_j$		5	0	0	-2	0	0			
		$\uparrow$								

		x_1	x_2	s_1	s_2	s_3	s_4	
Basis	c_j	11	14	0	0	0	0	b_i
s_1	0	0	0	1	$\frac{1}{3}$	$\frac{-2}{3}$	0	6
x_2	14	0	1	0	$\frac{1}{2}$	$\frac{-1}{2}$	0	$\frac{15}{2}$
x_1	11	1	0	0	$\frac{-5}{6}$	$\frac{7}{6}$	0	$\frac{7}{2}$
s_4	0	0	0	0	2	-3	1	12
								$\frac{287}{2}$

New feasible basic solution: $(x_1, x_2, s_1, s_2, s_3, s_4) = (\frac{7}{2}, \frac{15}{2}, 6, 0, 0, 12)$; $z = \frac{287}{2}$

Graphically



Evaluating adjacent feasible basic solutions

		x_1	x_2	s_1	s_2	s_3	s_4	
Basis	c_j	11	14	0	0	0	0	b_i
s_1	0	0	0	1	$\frac{1}{3}$	$\frac{-2}{3}$	0	6
x_2	14	0	1	0	$\frac{1}{2}$	$\frac{-1}{2}$	0	$\frac{15}{2}$
x_1	11	1	0	0	$\frac{-5}{6}$	$\frac{7}{6}$	0	$\frac{7}{2}$
s_4	0	0	0	0	2	-3	1	12

Computing z_j and $c_j - z_j$ ...

		x_1	x_2	s_1	s_2	s_3	s_4	
Basis	c_j	11	14	0	0	0	0	b_i
s_1	0	0	0	1	$\frac{1}{3}$	$\frac{-2}{3}$	0	6
x_2	14	0	1	0	$\frac{1}{2}$	$\frac{-1}{2}$	0	$\frac{15}{2}$
x_1	11	1	0	0	$\frac{-5}{6}$	$\frac{7}{6}$	0	$\frac{7}{2}$
s_4	0	0	0	0	2	-3	1	12
z_j		11	14	0	$\frac{-13}{6}$	$\frac{35}{6}$	0	$\frac{287}{2}$
$c_j - z_j$		0	0	0	$\frac{13}{6}$	$\frac{-35}{6}$	0	

$\exists\ c_j - z_j > 0$ so there exists a better adjacent basic solution and the current one is not optimal

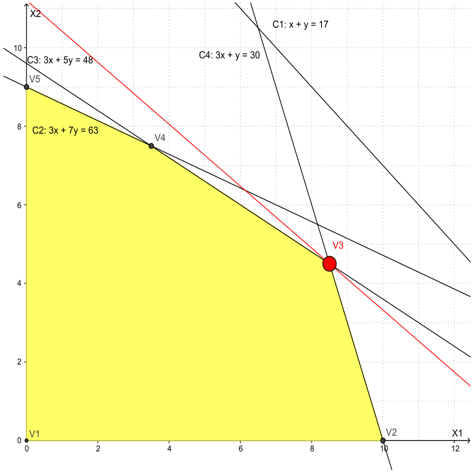
Pivoting

		x_1	x_2	s_1	s_2	s_3	s_4			
Basis	c_j	11	14	0	0	0	0	b_i	$\frac{b_i}{a_{ij}}$	
s_1	0	0	0	1	$\frac{1}{3}$	$\frac{-2}{3}$	0	6	$\frac{6}{\frac{1}{3}} = 18$	
x_2	14	0	1	0	$\frac{1}{2}$	$\frac{-1}{2}$	0	$\frac{15}{2}$	$\frac{\frac{15}{2}}{\frac{1}{2}} = 15$	
x_1	11	1	0	0	$\frac{-5}{6}$	$\frac{7}{6}$	0	$\frac{7}{2}$	—	
s_4	0	0	0	0	2	-3	1	12	$\frac{12}{2} = 6$	$\rightarrow$
z_j		11	14	0	$\frac{-13}{6}$	$\frac{35}{6}$	0	$\frac{287}{2}$		
$c_j - z_j$		0	0	0	$\frac{13}{6}$	$\frac{-35}{6}$	0			
					$\uparrow$					

		x_1	x_2	s_1	s_2	s_3	s_4	
Basis	c_j	11	14	0	0	0	0	b_i
s_1	0	0	0	1	0	$\frac{-1}{6}$	$\frac{-1}{6}$	4
x_2	14	0	1	0	0	$\frac{1}{4}$	$\frac{-1}{4}$	$\frac{9}{2}$
x_1	11	1	0	0	0	$\frac{-1}{12}$	$\frac{5}{12}$	$\frac{17}{2}$
s_2	0	0	0	0	1	$\frac{-3}{2}$	$\frac{1}{2}$	6
								$\frac{313}{2}$

New feasible basic solution: $(x_1, x_2, s_1, s_2, s_3, s_4) = (\frac{17}{2}, \frac{9}{2}, 4, 6, 0, 0)$; $z = \frac{313}{2}$

Graphically



Evaluating adjacent feasible basic solutions

Basis	c_j	x_1 11	x_2 14	s_1 0	s_2 0	s_3 0	s_4 0	b_i
s_1	0	0	0	1	0	$\frac{-1}{6}$	$\frac{-1}{6}$	4
x_2	14	0	1	0	0	$\frac{1}{4}$	$\frac{-1}{4}$	$\frac{9}{2}$
x_1	11	1	0	0	0	$\frac{-1}{12}$	$\frac{5}{12}$	$\frac{17}{2}$
s_2	0	0	0	0	1	$\frac{-3}{2}$	$\frac{1}{2}$	6

Computing z_j and $c_j - z_j$ ...

Basis	c_j	x_1 11	x_2 14	s_1 0	s_2 0	s_3 0	s_4 0	b_i
s_1	0	0	0	1	0	$\frac{-1}{6}$	$\frac{-1}{6}$	4
x_2	14	0	1	0	0	$\frac{1}{4}$	$\frac{-1}{4}$	$\frac{9}{2}$
x_1	11	1	0	0	0	$\frac{-1}{12}$	$\frac{5}{12}$	$\frac{17}{2}$
s_2	0	0	0	0	1	$\frac{-3}{2}$	$\frac{1}{2}$	6
z_j		11	14	0	0	$\frac{31}{12}$	$\frac{13}{12}$	$\frac{313}{2}$
$c_j - z_j$		0	0	0	0	$\frac{-31}{12}$	$\frac{-13}{12}$	

All values in row $c_j - z_j$ are zero or negative so the Stopping Criterion is verified!

Integer Linear Programming

Why?

- Many practical problems need variables to take integer values

How to solve it?

- Exhaustive enumeration is only useful when dealing with few combinations of values
- In general, we deal with a combinatorial problem
- Solving techniques are based on divide and conquer approach

Branch and Bound for General IP

- Solve linear programming relaxations to bound the objective function
- Create branches by adding constraints that eliminate non-integer values

Example

Considering the following Integer Programming problem:

$$Max \quad z = 11x_1 + 14x_2$$

Subject to

$$\begin{aligned}
1x_1 + 1x_2 &\leq 17 \\
3x_1 + 7x_2 &\leq 63 \\
3x_1 + 5x_2 &\leq 48 \\
3x_1 + 1x_2 &\leq 30 \\
x_1, x_2 &\geq 0 \\
x_1, x_2 &\in \mathbb{Z}
\end{aligned}$$

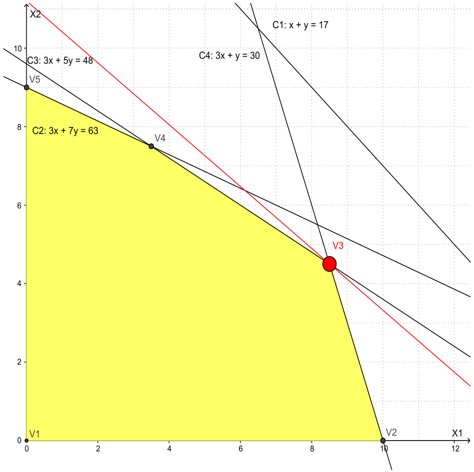
Solving the Linear Programming Relaxation

$$Max \;\; z = 11x_1 + 14x_2$$

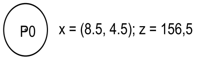
Subject to

$$\begin{aligned} 1x_1 + 1x_2 &\leq 17 \\ 3x_1 + 7x_2 &\leq 63 \\ 3x_1 + 5x_2 &\leq 48 \\ 3x_1 + 1x_2 &\leq 30 \\ x_1, x_2 &\geq 0 \quad x^* = (8.5, 4.5) \;\; z^* = 156.5 \end{aligned}$$

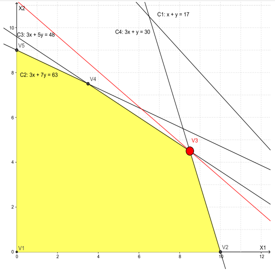
Graphically



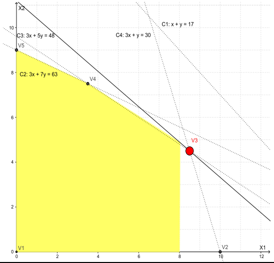
Enumeration tree

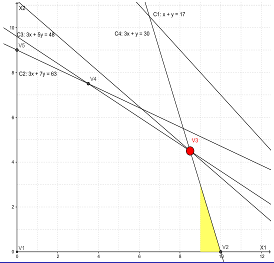


Branching



Branching on x_1:





Solving the Linear Programming Relaxation

$$Max \;\; z = 11x_1 + 14x_2$$

Subject to

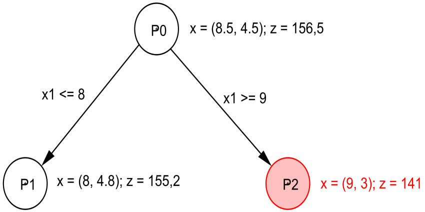
$$\begin{aligned}
1x_1 + 1x_2 &\leq 17 \\
3x_1 + 7x_2 &\leq 63 \\
3x_1 + 5x_2 &\leq 48 \\
3x_1 + 1x_2 &\leq 30 \\
x_1 &\leq 8 \\
x_1, x_2 &\geq 0 \quad x^* = (8, 4.8) \;\; z^* = 155.2
\end{aligned}$$

$$Max \;\; z = 11x_1 + 14x_2$$

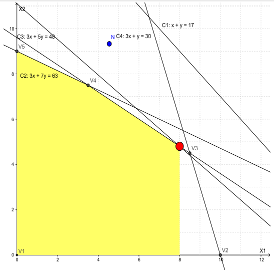
Subject to

$$\begin{aligned}
1x_1 + 1x_2 &\leq 17 \\
3x_1 + 7x_2 &\leq 63 \\
3x_1 + 5x_2 &\leq 48 \\
3x_1 + 1x_2 &\leq 30 \\
x_1 &\geq 9 \\
x_1, x_2 &\geq 0 \quad x^* = (9, 3) \;\; z^* = 141
\end{aligned}$$

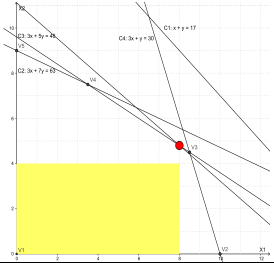
Enumeration tree

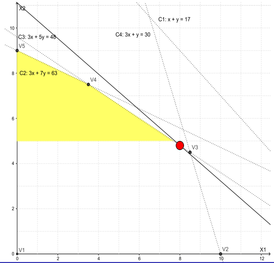


Branching



Branching on x_2:





Solving the Linear Programming Relaxation

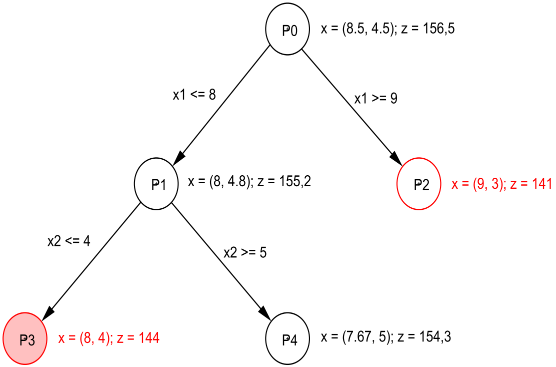
$$Max \quad z = 11x_1 + 14x_2$$

Subject to
$$\begin{aligned} 1x_1 + 1x_2 &\leq 17 \\ 3x_1 + 7x_2 &\leq 63 \\ 3x_1 + 5x_2 &\leq 48 \\ 3x_1 + 1x_2 &\leq 30 \\ x_1 \leq 8 \quad &\& \quad x_2 \leq 4 \\ x_1, x_2 &\geq 0 \quad x^* = (8,4) \quad z^* = 144 \end{aligned}$$

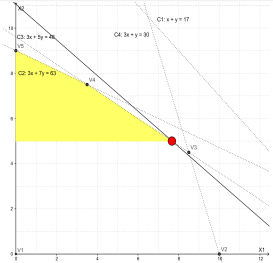
$$Max \quad z = 11x_1 + 14x_2$$

Subject to
$$\begin{aligned} 1x_1 + 1x_2 &\leq 17 \\ 3x_1 + 7x_2 &\leq 63 \\ 3x_1 + 5x_2 &\leq 48 \\ 3x_1 + 1x_2 &\leq 30 \\ x_1 \leq 8 \quad &\& \quad x_2 \geq 5 \\ x_1, x_2 &\geq 0 \quad x^* = (7.67,5) \quad z^* = 154.3 \end{aligned}$$

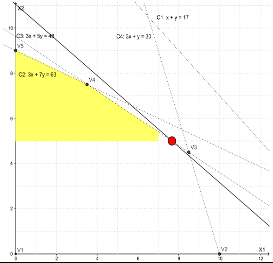
Enumeration tree

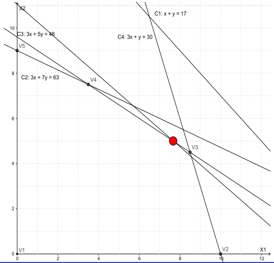


Branching



Branching again on x_1:





Solving the Linear Programming Relaxation

$$Max \quad z = 11x_1 + 14x_2$$

Subject to

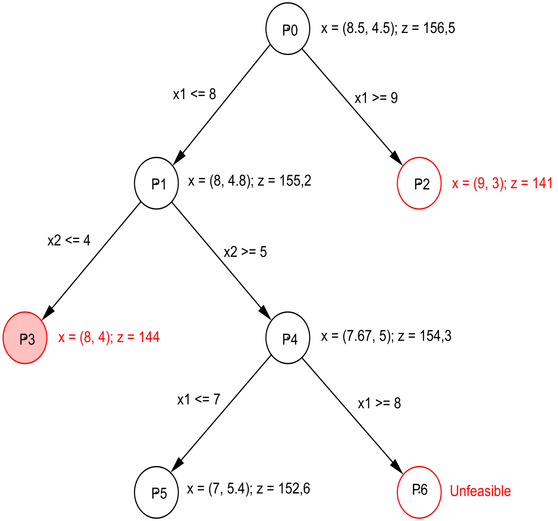
$$\begin{aligned} 1x_1 + 1x_2 &\leq 17 \\ 3x_1 + 7x_2 &\leq 63 \\ 3x_1 + 5x_2 &\leq 48 \\ 3x_1 + 1x_2 &\leq 30 \\ x_1 \leq 8 \ \& \ x_2 &\geq 5 \ \& \ x_1 \leq 7 \\ x_1, x_2 &\geq 0 \quad x^* = (7, 5.4) \quad z^* = 152.6 \end{aligned}$$

$$Max \quad z = 11x_1 + 14x_2$$

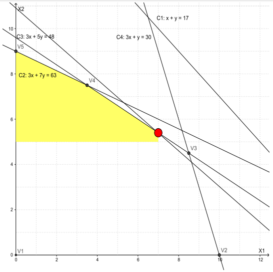
Subject to

$$\begin{aligned} 1x_1 + 1x_2 &\leq 17 \\ 3x_1 + 7x_2 &\leq 63 \\ 3x_1 + 5x_2 &\leq 48 \\ 3x_1 + 1x_2 &\leq 30 \\ x_1 \leq 8 \ \& \ x_2 &\geq 5 \ \& \ x_1 \geq 8 \\ x_1, x_2 &\geq 0 \quad \textit{Unfeasible} \end{aligned}$$

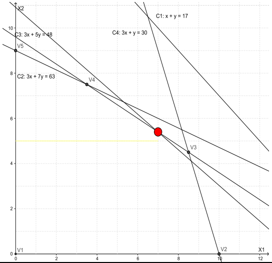
Enumeration tree

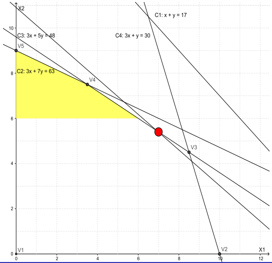


Branching



Branching again on x_2:





Solving the Linear Programming Relaxation

$$Max \quad z = 11x_1 + 14x_2$$

Subject to

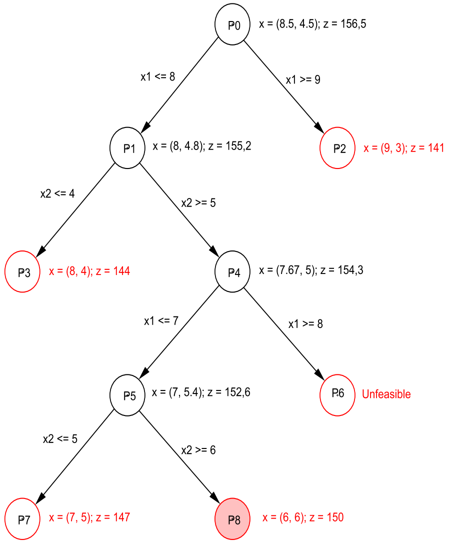
$$\begin{aligned} 1x_1 + 1x_2 &\leq 17 \\ 3x_1 + 7x_2 &\leq 63 \\ 3x_1 + 5x_2 &\leq 48 \\ 3x_1 + 1x_2 &\leq 30 \\ x_1 \leq 8 \;\&\; x_2 \geq 5 \quad &\& \quad x_1 \leq 7 \;\&\; x_2 \leq 5 \\ x_1, x_2 &\geq 0 \quad x^* = (7,5) \;\; z^* = 147 \end{aligned}$$

$$Max \quad z = 11x_1 + 14x_2$$

Subject to

$$\begin{aligned} 1x_1 + 1x_2 &\leq 17 \\ 3x_1 + 7x_2 &\leq 63 \\ 3x_1 + 5x_2 &\leq 48 \\ 3x_1 + 1x_2 &\leq 30 \\ x_1 \leq 8 \;\&\; x_2 \geq 5 \quad &\& \quad x_1 \leq 7 \;\&\; x_2 \geq 6 \\ x_1, x_2 &\geq 0 \quad x^* = (6,6) \;\; z^* = 150 \end{aligned}$$

Enumeration tree



Branch and Bound for Binary IP

- Solve linear programming relaxations to bound the objective function
- Create branches by adding constraints that fix variables

Example

$$Max \;\; z = 9x_1 + 5x_2 + 6x_3 + 4x_4$$

Subject to

$$\begin{aligned} 6x_1 + 3x_2 + 5x_3 + 2x_4 &\leq 10 \\ x_3 + x_4 &\leq 1 \\ -x_1 + x_3 &\leq 0 \\ -x_2 + x_4 &\leq 0 \\ x_i &\in \{0,1\} \;\; \forall i = 1,\ldots,4 \end{aligned}$$

Constraints allowing values

A constraint allowing values for variables:

$$x_i \in \{0, 1\}$$

is equivalent to the following constraints:

$$\begin{aligned} x_i &\geq 0 \\ x_i &\leq 1 \\ x_i &\in \mathbb{Z} \end{aligned}$$

This correspond to an IP problem with upper bounds for all variables.

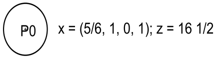
Solving the Linear Programming Relaxation

$$Max \quad z = 9x_1 + 5x_2 + 6x_3 + 4x_4$$

Subject to

$$\begin{aligned}
6x_1 + 3x_2 + 5x_3 + 2x_4 &\leq 10 \\
x_3 + x_4 &\leq 1 \\
-x_1 + x_3 &\leq 0 \\
-x_2 + x_4 &\leq 0 \\
x_i &\leq 1 \;\; \forall i = 1, \ldots, 4 \\
x_i &\geq 0 \;\; \forall i = 1, \ldots, 4 \;\; x^* = (\frac{5}{6}, 1, 0, 1); \; z^* = 16\frac{1}{2}
\end{aligned}$$

Enumeration tree



Branching

Remember that $x_i \in \{0, 1\}$ is equivalent to the following constraints:

$$\begin{aligned} x_i &\geq 0 \\ x_i &\leq 1 \\ x_i &\in \mathbb{Z} \end{aligned}$$

When branching we will add constraints $x_i \leq 0$ and $x_i \geq 1$:

$$\begin{array}{cc} x_i \geq 0 & x_i \geq 0 \\ x_i \leq 1 & x_i \leq 1 \\ x_i \leq 0 & x_i \geq 1 \\ x_i \in \mathbb{Z} & x_i \in \mathbb{Z} \\ \Downarrow & \Downarrow \\ x_i = 0 & x_i = 1 \end{array}$$

Branching

Subproblem 1 (original problem $\wedge$ $x_1 = 0$):

$$\begin{array}{lrcl} & \multicolumn{3}{c}{Max\ \ z = 9x_1 + 5x_2 + 6x_3 + 4x_4} \\ \text{Subject to} & 6x_1 + 3x_2 + 5x_3 + 2x_4 & \leq & 10 \\ & x_3 + x_4 & \leq & 1 \\ & -x_1 + x_3 & \leq & 0 \\ & -x_2 + x_4 & \leq & 0 \\ & x_1 & = & 0 \\ & x_i & \in & \{0,1\}\ \ \forall i = 1,\dots,4 \end{array}$$

Subproblem 2 (original problem $\wedge$ $x_1 = 1$):

$$\begin{array}{lrcl} & \multicolumn{3}{c}{Max\ \ z = 9x_1 + 5x_2 + 6x_3 + 4x_4} \\ \text{Subject to} & 6x_1 + 3x_2 + 5x_3 + 2x_4 & \leq & 10 \\ & x_3 + x_4 & \leq & 1 \\ & -x_1 + x_3 & \leq & 0 \\ & -x_2 + x_4 & \leq & 0 \\ & x_1 & = & 1 \\ & x_i & \in & \{0,1\}\ \ \forall i = 1,\dots,4 \end{array}$$

Solving the Linear Programming Relaxation

Subproblem 1: $Max\ \ z = 5x_2 + 6x_3 + 4x_4$

Subject to

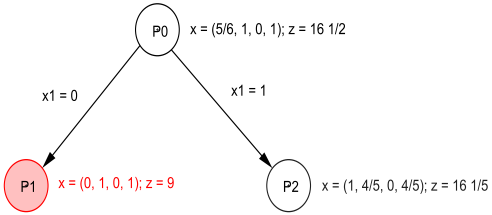
$$\begin{aligned} 3x_2 + 5x_3 + 2x_4 &\leq 10 \\ x_3 + x_4 &\leq 1 \\ x_3 &\leq 0 \\ -x_2 + x_4 &\leq 0 \\ x_i &\leq 1\ \ \forall i = 2, \dots, 4 \\ x_i &\geq 0\ \ \forall i = 2, \dots, 4\ \ x^* = (0, 1, 0, 1);\ \ z^* = 9 \end{aligned}$$

Subproblem 2: $Max\ \ z = 9 + 5x_2 + 6x_3 + 4x_4$

Subject to

$$\begin{aligned} 3x_2 + 5x_3 + 2x_4 &\leq 4 \\ x_3 + x_4 &\leq 1 \\ x_3 &\leq 1 \\ -x_2 + x_4 &\leq 0 \\ x_i &\leq 1\ \ \forall i = 2, \dots, 4 \\ x_i &\geq 0\ \ \forall i = 2, \dots, 4\ \ x^* = (1, \frac{4}{5}, 0, \frac{4}{5});\ \ z^* = 16\frac{1}{5} \end{aligned}$$

Enumeration tree


P0
x = (5/6, 1, 0, 1); z = 16 1/2
x1 = 0
x1 = 1
P1
x = (0, 1, 0, 1); z = 9
P2
x = (1, 4/5, 0, 4/5); z = 16 1/5

Branching

Subproblem 3 (subproblem 2 $\wedge$ $x_2 = 0$):

Subject to

$$Max \;\; z = 9 + 5x_2 + 6x_3 + 4x_4$$

$$\begin{array}{rcl} 3x_2 + 5x_3 + 2x_4 & \leq & 4 \\ x_3 + x_4 & \leq & 1 \\ x_3 & \leq & 1 \\ -x_2 + x_4 & \leq & 0 \\ x_2 & = & 0 \\ x_i & \in & \{0,1\} \;\; \forall i = 2, \ldots, 4 \end{array}$$

Subproblem 4 (subproblem 2 $\wedge$ $x_2 = 1$):

$$Max \;\; z = 9 + 5x_2 + 6x_3 + 4x_4$$

Subject to

$$\begin{array}{rcl} 3x_2 + 5x_3 + 2x_4 & \leq & 4 \\ x_3 + x_4 & \leq & 1 \\ x_3 & \leq & 1 \\ -x_2 + x_4 & \leq & 0 \\ x_2 & = & 1 \\ x_i & \in & \{0,1\} \;\; \forall i = 2, \ldots, 4 \end{array}$$

Solving the Linear Programming Relaxation

Subproblem 3: $Max \;\; z = 9 + 6x_3 + 4x_4$

Subject to

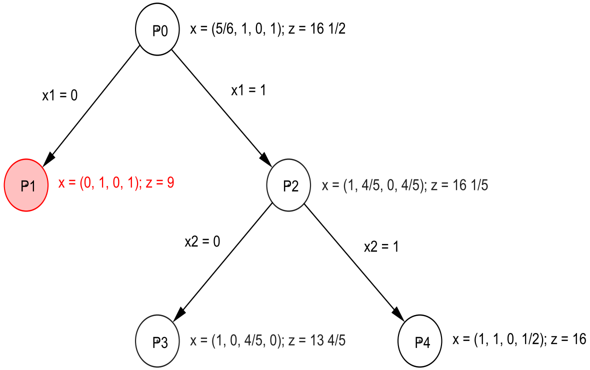
$$\begin{aligned} 5x_3 + 2x_4 &\leq 4 \\ x_3 + x_4 &\leq 1 \\ x_3 &\leq 1 \\ x_4 &\leq 0 \\ x_i &\leq 1 \;\; \forall i = 3,4 \\ x_i &\geq 0 \;\; \forall i = 3,4 \;\; x^* = (1, 0, \frac{4}{5}, 0); \;\; z = 13\frac{4}{5} \end{aligned}$$

Subproblem 4: $Max \;\; z = 14 + 6x_3 + 4x_4$

Subject to

$$\begin{aligned} 5x_3 + 2x_4 &\leq 1 \\ x_3 + x_4 &\leq 1 \\ x_3 &\leq 1 \\ x_4 &\leq 1 \\ x_i &\leq 1 \;\; \forall i = 3,4 \\ x_i &\geq 0 \;\; \forall i = 3,4 \;\; x^* = (1, 1, 0, \frac{1}{2}); \;\; z^* = 16 \end{aligned}$$

Enumeration tree


P0
x = (5/6, 1, 0, 1); z = 16 1/2
x1 = 0
x1 = 1
P1
x = (0, 1, 0, 1); z = 9
P2
x = (1, 4/5, 0, 4/5); z = 16 1/5
x2 = 0
x2 = 1
P3
x = (1, 0, 4/5, 0); z = 13 4/5
P4
x = (1, 1, 0, 1/2); z = 16

Branching

Subproblem 5 (subproblem 4 $\wedge$ $x_4 = 0$):

Subject to

$$
\begin{aligned}
Max \;\; z = 14 + 6x_3 + 4x_4 \\
5x_3 + 2x_4 &\leq 1 \\
x_3 + x_4 &\leq 1 \\
x_3 &\leq 1 \\
x_4 &\leq 1 \\
x_4 &= 0 \\
x_i &\in \{0, 1\} \;\; \forall i = 3, 4
\end{aligned}
$$

Subproblem 6 (subproblem 4 $\wedge$ $x_4 = 1$):

Subject to

$$
\begin{aligned}
Max \;\; z = 14 + 6x_3 + 4x_4 \\
5x_3 + 2x_4 &\leq 1 \\
x_3 + x_4 &\leq 1 \\
x_3 &\leq 1 \\
x_4 &\leq 1 \\
x_4 &= 1 \\
x_i &\in \{0, 1\} \;\; \forall i = 3, 4
\end{aligned}
$$

Solving the Linear Programming Relaxation

Subproblem 5:

Subject to

$$Max \;\; z = 14 + 6x_3$$

$$\begin{aligned} 5x_3 &\leq 1 \\ x_3 &\leq 1 \\ x_3 &\leq 1 \\ x_3 &\geq 0 \;\; x^* = (1, 1, \frac{1}{5}, 0);\; z^* = 15\frac{1}{5} \end{aligned}$$

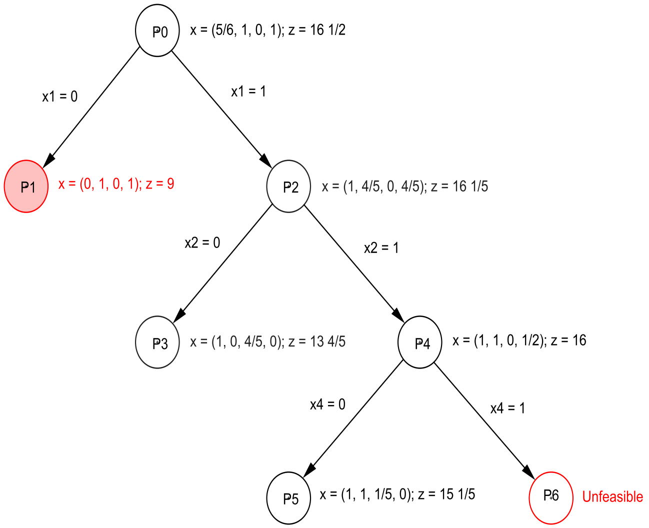
Subproblem 6:

Subject to

$$Max \;\; z = 18 + 6x_3$$

$$\begin{aligned} 5x_3 &\leq -1 \\ x_3 &\leq 0 \\ x_3 &\leq 1 \\ x_3 &\leq 1 \\ x_3 &\geq 0 \;\; \textit{Unfeasible} \end{aligned}$$

Enumeration tree


P0
x = (5/6, 1, 0, 1); z = 16 1/2
x1 = 0
x1 = 1
P1
x = (0, 1, 0, 1); z = 9
P2
x = (1, 4/5, 0, 4/5); z = 16 1/5
x2 = 0
x2 = 1
P3
x = (1, 0, 4/5, 0); z = 13 4/5
P4
x = (1, 1, 0, 1/2); z = 16
x4 = 0
x4 = 1
P5
x = (1, 1, 1/5, 0); z = 15 1/5
P6
Unfeasible

Branching

Subproblem 7 (subproblem 5 $\wedge$ $x_3 = 0$):

Subject to

$$\begin{aligned} Max \quad z &= 14 + 6x_3 \\ 5x_3 &\le 1 \\ x_3 &\le 1 \\ x_3 &\le 1 \\ x_3 &\ge 0 \\ x_3 &= 0 \\ x_3 &\in \{0, 1\} \end{aligned}$$

Subproblem 8 (subproblem 5 $\wedge$ $x_3 = 1$):

Subject to

$$\begin{aligned} Max \quad z &= 14 + 6x_3 \\ 5x_3 &\le 1 \\ x_3 &\le 1 \\ x_3 &\le 1 \\ x_3 &\ge 0 \\ x_3 &= 1 \\ x_3 &\in \{0, 1\} \end{aligned}$$

Solving the Linear Programming Relaxation

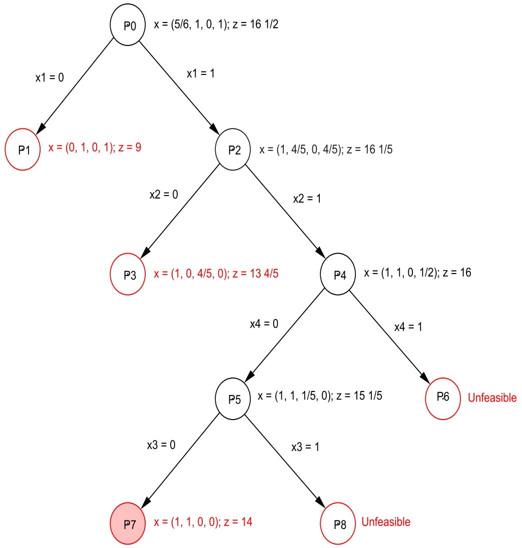
Subproblem 7:

- Variable enumeration: $x = (1, 1, 0, 0)$
- Feasible solution
- $z = 14$

Subproblem 8:

- Variable enumeration: $x = (1, 1, 1, 0)$
- Unfeasible solution

Enumeration tree


P0
x = (5/6, 1, 0, 1); z = 16 1/2
x1 = 0
x1 = 1
P1
x = (0, 1, 0, 1); z = 9
P2
x = (1, 4/5, 0, 4/5); z = 16 1/5
x2 = 0
x2 = 1
P3
x = (1, 0, 4/5, 0); z = 13 4/5
P4
x = (1, 1, 0, 1/2); z = 16
x4 = 0
x4 = 1
P5
x = (1, 1, 1/5, 0); z = 15 1/5
P6
Unfeasible
x3 = 0
x3 = 1
P7
x = (1, 1, 0, 0); z = 14
P8
Unfeasible

Solving Mixed Integer Programming Problems

Mixed Integer Programming Problems involve real, integer, and binary variables:

- Real variables: Simplex Method
- Integer variables: Simplex Method + constraints $\leq$ and $\geq$
- Binary variables: Simplex Method + constraints $=$