Intuitionistic Multi-Agent Subatomic Natural Deduction for Belief and Knowledge

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1 Introduction

In proof theory, the study of intensional operators is mainly focused on the adaptation of proof systems for classical (intuitionistic, minimal) logic to some given modal logic. In a natural deduction setting, a basic element of such adaptations is a suitably chosen side condition on the introduction rule for formulae in which the intensional operator is principal. An early example of this component is the introduction rule for $\Box$ in [3]. In labelled deductive proof systems (e.g., [2], [6], [8]), this component is combined with an enrichment of the modal language with labelled and relational formulae which serve to internalize the notions of truth at an index and of the accessibility between indices, both of which are explicit in the relational models of the Kripke-semantics for the modal logic in question.

Our aim in this paper is to outline a proof-theoretic approach to the semantics of intensional operators for intuitionistic belief and knowledge, which does neither intend to capture a given target modal logic, nor to import ideas from relational model-theoretic semantics into the language. On this approach, the meaning of intensional operators is explained exclusively by appeal to the structure of proofs which are unaided by labelled and relational formulae.

The proof-theoretic framework chosen for this purpose is subatomic natural deduction (cf. [9], [10], [11]). This framework refines natural deduction by extending it with subatomic systems. The main ingredients of subatomic systems are term assumptions for non-logical constants and rules for the introduction and elimination of atomic sentences. In contrast to natural deduction systems extended with atomic systems, used to represent the proof-theoretic meaning of atomic sentences (e.g., [4]), derivations in subatomic systems normalize.

Essentially, the idea is to use subatomic systems to represent an agent's beliefs about basic facts. More generally, an agent's beliefs about such facts, or about his beliefs about his beliefs, or his beliefs about other agent's beliefs are represented by means of an intuitionistic subatomic natural deduction system for first-order logic which contains agent-labelled introduction and elimination rules. The meaning of an agent's belief operator is explained in terms of an agent-labelled introduction rule which can be applied to any premiss derivable in the agent's belief system. By contrast, the meaning of an agent's knowledge operator is explained in terms of an agent-labelled introduction rule which can be applied only to premisses which have been derived in the agent's belief system as theorems. In a multi-agent setting, derivations in distinct belief systems can be combined. Such systems can be used for the logical analysis of reasoning patterns which involve complex multi-agent belief constructions (e.g., reciprocating or universal beliefs; see Examples 3.1 below).

As main technical results, we obtain normalization and the subexpression property (a refinement of the subformula property) for the systems. The first result allows us to formulate a proof-theoretic semantics (see [5] for an overview) for belief and knowledge, the second guarantees full analyticity.

The full paper compares the present perspective on intuitionistic belief and knowledge with conceptions underlying other intuitionistic approaches to these notions (e.g., [1], [13]).

2 Language

The language $\mathcal{L}$ is the standard language of first-order logic extended with constants, variables, and quantifiers for agents, and with two intensional operators.

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Definition 2.1 Alphabet of $\mathcal{L}$:

1. *Non-logical terms:*

   (a) Predicate constants (n-ary): φ^n, φ_i^n, ...

   (b) Individual (or nominal) constants: α, α_i, ...

   (c) Nominal variables: x, x_i, y, ...

   (d) Nominal terms: *Any nominal constant and any nominal variable is a* nominal term. *We use the notation o, o_i, ... for nominal terms.*

   (e) Agent constants: $\underline{a}$, $\underline{a}_i$, ...

   (f) Agent variables: $\underline{x}$, $\underline{x}_i$, $\underline{y}$, ...

   (g) Agent terms: *Any agent constant and any agent variable is an* agent term. *We use the notation $\underline{o}$, $\underline{o}_i$, ... for agent terms.*

   We take the supply of non-logical terms to be denumerable for each category.

2. Logical symbols:

   (a) Sentential constant: $\perp$ (absurdity).

   (b) Sentential connectives: & *('and')*, $\vee$ *('or')*, $\supset$ *('implies')*,

   (c) Quantifiers: $\forall$ *('for all')*, $\exists$ *('for some')*,

   (d) Agent quantifiers: $\underline{\forall}$ *('for all agents')*, $\underline{\exists}$ *('for some agent')*.

   (e) Intensional operators: $\mathsf{B}_{\underline{o}}$ *('$\underline{o}$ believes')*, $\mathsf{K}_{\underline{o}}$ *('$\underline{o}$ knows')*,

3. Auxiliary symbols: (,) (parentheses).

$\mathcal{P}$ is the set of predicate constants. $\mathcal{C}$ is the set of nominal constants. $\mathcal{C} \cup \mathcal{P}$ is the set of non-logical constants $\tau, \tau_i, \ldots$. *$\underline{\mathcal{C}}$ is the set of agent constants.*

Definition 2.2 *An* atomic formula *of $\mathcal{L}$ is a string of the form $\varphi^n o_1 \ldots o_n$. An atomic sentence is an atomic formula of the form $\varphi^n \alpha_1 \ldots \alpha_n$. We distinguish two classes of atomic sentences:*

1. *For any $\varphi^n \in \mathcal{P}$: $Atm(\varphi^n) =_{def} \{A \in Atm$: A contains an occurrence of $\varphi^n\}$.*

2. *For any $\alpha \in \mathcal{C}$: $Atm(\alpha) =_{def} \{A \in Atm$: A contains at least one occurrence of $\alpha\}$.*

We call the elements of $Atm(\varphi^n)$ φ^n-atoms *and the elements of $Atm(\alpha)$* α-atoms.

Definition 2.3 Formula of $\mathcal{L}$: *This notion is defined inductively by:*

1. *Any atomic formula is a formula (of $\mathcal{L}$).*

2. *If A is a formula, then $\mathsf{B}_{\underline{o}}A$, $\mathsf{K}_{\underline{o}}A$ are formulae.*

3. *If A and B are formulae, then $A\&B$, $A \vee B$, $A \supset B$ are formulae.*

4. *If A is a formula, o [$\underline{o}$] is either a (free) nominal variable y [agent variable $\underline{y}$] or a nominal constant α [agent constant $\underline{\alpha}$], and x [$\underline{x}$] a (bound) nominal variable [agent variable] not occurring in A, then $\forall x A(o/x)$, $\underline{\forall}\, \underline{x} A(\underline{o}/\underline{x})$, $\exists x A(o/x)$, $\underline{\exists}\, \underline{x} A(\underline{o}/\underline{x})$ are formulae, where $A(o/x)$ [$A(\underline{o}/\underline{x})$] is obtained from A by writing x [$\underline{x}$] in place of o [$\underline{o}$] at each occurrence of o [$\underline{o}$] in A.*

Definition 2.4 Defined symbols of $\mathcal{L}$: $\neg$ *('it is not the case that')*, $\leftrightarrow$ *('if and only if')*, *and* $\ddot{=}$ *('is identical with')*, *and* $\underline{\ddot{=}}$ *(agent-identity).*

1. $\neg A =_{def} A \supset \perp$

2. $A \leftrightarrow B =_{def} (A \supset B)\&(B \supset A)$

3. *Let φ^n be an n-ary predicate constant.*

$$\begin{aligned}
&K^n_{\varphi^n}(o_1, o_2) =_{def} \\
&\quad \forall z_1 \ldots \forall z_{n-1} \forall z_n \; ((\varphi^n o_1 z_2 \ldots z_n \leftrightarrow \varphi^n o_2 z_2 \ldots z_n) \\
&\quad \& \; (\varphi^n z_1 o_1 \ldots z_n \leftrightarrow \varphi^n z_1 o_2 \ldots z_n) \\
&\quad \& \; \ldots \; \& \; (\varphi^n z_1 \ldots z_{n-1} o_1 \leftrightarrow \varphi^n z_1 \ldots z_{n-1} o_2)).
\end{aligned}$$

Let $\varphi_1^{k_1}, ..., \varphi_m^{k_m}$ be all the atomic predicates in $\mathcal{P}$, where φ_i is k_i-ary.

$$o_1 \ddot{=} o_2 =_{def} K^{k_1}_{\varphi_1}(o_1, o_2) \ \& \ ... \ \& \ K^{k_m}_{\varphi_m}(o_1, o_2).$$

4. Let A be an atomic formula of $\mathcal{L}$.

$$\begin{aligned} &\underline{K}^n_A(\underline{o}_1, \underline{o}_2) =_{def} \\ &\quad \underline{\forall}\ \underline{z}_1 ... \underline{\forall}\ \underline{z}_{n-1} \underline{\forall}\ \underline{z}_n\ ((\mathsf{B}_{\underline{o}_1} \mathsf{B}_{\underline{z}_2} ... \mathsf{B}_{\underline{z}_n} A \leftrightarrow \mathsf{B}_{\underline{o}_2} \mathsf{B}_{\underline{z}_2} ... \mathsf{B}_{\underline{z}_n} A) \\ &\quad \& \ (\mathsf{B}_{\underline{z}_1} \mathsf{B}_{\underline{o}_1} ... \mathsf{B}_{\underline{z}_n} A \leftrightarrow \mathsf{B}_{\underline{z}_1} \mathsf{B}_{\underline{o}_2} ... \mathsf{B}_{\underline{z}_n} A) \\ &\quad \& \ ... \ \& \ (\mathsf{B}_{\underline{z}_1} ... \mathsf{B}_{\underline{z}_{n-1}} ... \mathsf{B}_{\underline{o}_1} A \leftrightarrow \mathsf{B}_{\underline{z}_1} ... \mathsf{B}_{\underline{z}_{n-1}} ... \mathsf{B}_{\underline{o}_2} A)). \end{aligned}$$

Let $A_1, ..., A_m$ be a finite list of atomic formulae.

$$\underline{o}_1 \ \underline{\ddot{=}} \ \underline{o}_2 =_{def} \underline{K}_{A_1}(\underline{o}_1, \underline{o}_2) \ \& \ ... \ \& \ \underline{K}_{A_m}(\underline{o}_1, \underline{o}_2).$$

Remarks 2.1 *1. It is apparent from the definition of* $\ddot{=}$ *that only a finite number of predicate constants may figure in the definiens. We, therefore, have to make a finiteness stipulation, when we take the language to contain* $\ddot{=}$*. For simplicity, in such a case, we assume that* $\mathcal{P}$ *is finite.*

2. Intuitively, $\underline{\ddot{=}}$ *is defined in terms of the indistinguishability of two agents with respect to their (possibly) finitely nested and/or finitely embedded beliefs concerning a finite list of atomic formulae.*

Definition 2.5 Subformula*:*

1. *A is a positive and strictly positive subformula of itself.*
2. *If* $\mathsf{B}_{\underline{o}}B$ *or* $\mathsf{K}_{\underline{o}}B$ *is a positive [negative, strictly positive] subformula of* A*, then so is* B*.*
3. *If* $B \& C$ *or* $B \vee C$ *is a positive [negative, strictly positive] subformula of* A*, then so are* B, C*.*
4. *If* $\forall x B$ *or* $\exists x B$ *is a positive [negative, strictly positive] subformula of* A*, then so is* $B(x/o)$ *for any* o *free for* x *in* B*.*
5. *If* $\underline{\forall}\ \underline{x} B$ *or* $\underline{\exists}\ \underline{x} B$ *is a positive [negative, strictly positive] subformula of* A*, then so is* $B(\underline{x}/\underline{o})$ *for any* $\underline{o}$ *free for* $\underline{x}$ *in* B*.*
6. *If* $B \supset C$ *is a positive [negative] subformula of* A*, then* B *is a negative [positive] subformula of* A *and then* C *is a positive [negative] subformula of* A*.*
7. *If* $B \supset C$ *is a strictly positive subformula of* A*, then so is* C*.*
8. *If* $o_1 \ddot{=} o_2$ *is a positive [negative, strictly positive] subformula of* A*, then so is* $K^{k_i}_{\varphi_i}(o_1, o_2)$ *where* $i \in \{1, ..., m\}$*.*
9. *If* $\underline{o}_1 \ \underline{\ddot{=}} \ \underline{o}_2$ *is a positive [negative, strictly positive] subformula of* A*, then so is* $\underline{K}_i(\underline{o}_1, \underline{o}_2)$ *where* $i \in \{1, ..., m\}$*.*

Definition 2.6 Subexpression*:*

1. *Any formula* A*, nominal term* o*, and agent term* $\underline{o}$ *is a positive and strictly positive subexpression of itself.*
2. *If formula* B *is a subexpression of* A*, then so is any subformula of* B*.*
3. *Any nominal term* o *[agent term* $\underline{o}$*] occurring in formula* A *is a subexpression of* A*.*

In clauses 2 and 3 the positivity, negativity, strict positivity of subexpressions is determined by the positivity, negativity, strict positivity of the subformulae in which they occur.

3 Systems

Subatomic natural deduction systems can be seen as alternatives to natural deduction systems extended with atomic systems containing simple rules (e.g., [4], [7]: sect. 6.4). Unlike the latter, subatomic natural deduction systems do not undermine the subformula property. Ideed, they enjoy the subexpression property. Moreover, whereas standard natural deduction systems allow only for a proof-theoretic semantics for the usual superatomic logical operators of first-order logic, subatomic natural deduction systems also admit a proof-theoretic account of the semantics of atomic sentences, their components (see, in particular, [11]), and subatomic operators which operate on non-logical terms [12].

Intuitionistic multi-agent subatomic natural deduction systems can be defined in a stepwise manner.

Definition 3.1 *An* agent-relative subatomic base $\mathcal{I}_{\boldsymbol{a}}$ *is a pair* $\langle \mathcal{C}, \mathcal{P}, v_{\boldsymbol{a}} \rangle$*, where* $\boldsymbol{a}$ *is an agent and* $v_{\boldsymbol{a}}$ *is such that:*

1. *for any* $\alpha \in \mathcal{C}$*,* $v_{\boldsymbol{a}} : \mathcal{C} \to \wp(Atm)$*, where* $v_{\boldsymbol{a}}(\alpha) \subseteq Atm(\alpha)$*;*
2. *for any* $\varphi^n \in \mathcal{P}$*,* $v_{\boldsymbol{a}} : \mathcal{P} \to \wp(Atm)$*, where* $v_{\boldsymbol{a}}(\varphi^n) \subseteq Atm(\varphi^n)$*.*

For any $\tau \in \mathcal{C} \cup \mathcal{P}$*, we define:* $\tau\Gamma^a =_{def} v_{\boldsymbol{a}}(\tau)$*.* $\tau\Gamma^a$ *is the set of* agent-relative term assumptions *for* τ*.*

Definition 3.2 *An* agent-relative subatomic system $\mathcal{S}_{\boldsymbol{a}}$ *is a pair* $\langle \mathcal{I}_{\boldsymbol{a}}, \mathcal{R}_{\boldsymbol{a}} \rangle$*, where:* $\mathcal{I}_{\boldsymbol{a}}$ *is an agent-relative subatomic base and* $\mathcal{R}_{\boldsymbol{a}}$ *is a set of agent-labelled I/E-rules for atomic sentences:*

$$\boldsymbol{a}\,\dfrac{\begin{array}{c}\mathcal{D}_0\\ \varphi_0^n\Gamma^{\boldsymbol{a}}\end{array}\quad \begin{array}{c}\mathcal{D}_1\\ \alpha_1\Gamma^{\boldsymbol{a}}\end{array}\quad \dots \quad \begin{array}{c}\mathcal{D}_n\\ \alpha_n\Gamma^{\boldsymbol{a}}\end{array}}{\varphi_0^n\alpha_1\dots\alpha_n}\ (asI)$$

where $\varphi_0^n\alpha_1\dots\alpha_n \in \varphi_0^n\Gamma^{\boldsymbol{a}} \cap \alpha_1\Gamma^{\boldsymbol{a}} \cap \dots \cap \alpha_n\Gamma^{\boldsymbol{a}}$

$$\boldsymbol{a}\,\dfrac{\begin{array}{c}\mathcal{D}'\\ \varphi_0^n\alpha_1\dots\alpha_n\end{array}}{\tau_i\Gamma^{\boldsymbol{a}}}\ (asE_i)$$

where $i \in \{0, \dots, n\}$ *and* $\tau_i \in \{\varphi_0^n, \alpha_1, \dots, \alpha_n\}$

Definition 3.3 Derivations in $\mathcal{S}_{\boldsymbol{a}}$*:*

Basic step. *Let* $\mathcal{S}_{\boldsymbol{a}} = \langle \mathcal{I}_{\boldsymbol{a}}, \mathcal{R}_{\boldsymbol{a}} \rangle$ *be an agent-relative subatomic system. Every term assumption determined by* $\mathcal{I}_{\boldsymbol{a}}$ *and every solitary occurrence of an atomic sentence* $\varphi_0^n\alpha_1\dots\alpha_n$ *enclosed in* $\langle\rangle_{\boldsymbol{a}}$ *(i.e., an agent-relative derivation of* $\varphi_0^n\alpha_1\dots\alpha_n$ *from the agent-relative open assumption of* $\varphi_0^n\alpha_1\dots\alpha_n$*), is a derivation (in* $\mathcal{S}_{\boldsymbol{a}}$*).*

Induction step. *When* $\mathcal{D}_0$*,* $\mathcal{D}_1$*, ...,* $\mathcal{D}_n$*, and* $\mathcal{D}'$ *are derivations, a derivation* $\mathcal{D}$ *can be constructed by means of the rules asI and* asE_i *contained in* $\mathcal{R}_{\boldsymbol{a}}$ *(see Definition 3.2).*

Definition 3.4 *Let* $\varphi(\alpha_1)$ *and* $\varphi(\alpha_2)$ *be atomic sentences which are exactly alike except that the former contains occurrences of* α_1 *where the latter contains occurrences of* α_2*. If this is the case,* $\varphi(\alpha_1)$ *and* $\varphi(\alpha_2)$ *are* mirror atomic sentences.

Definition 3.5 *An* agent-relative subatomic identity system $\mathcal{S}^{\ddot{=}}_{\boldsymbol{a}}$ *is a pair* $\langle \mathcal{I}_{\boldsymbol{a}}, \mathcal{R}^{\ddot{=}}_{\boldsymbol{a}} \rangle$*, where:* $\mathcal{I}_{\boldsymbol{a}}$ *is an agent-relative subatomic base and* $\mathcal{R}^{\ddot{=}}_{\boldsymbol{a}}$ *is* $\mathcal{R}_{\boldsymbol{a}}$ *extended with agent-relative I/E-rules for* $\ddot{=}$*:*

$$\boldsymbol{a}\,\dfrac{\begin{array}{c}[\langle\varphi_1(\alpha_1)\rangle_{\boldsymbol{a}}]^{(1_1)}_{\boldsymbol{a}}\\ \mathcal{D}_{1_1}\\ \varphi_1(\alpha_2)\end{array}\quad \begin{array}{c}[\langle\varphi_1(\alpha_2)\rangle_{\boldsymbol{a}}]^{(1_2)}_{\boldsymbol{a}}\\ \mathcal{D}_{1_2}\\ \varphi_1(\alpha_1)\end{array}\quad \dots \quad \begin{array}{c}[\langle\varphi_k(\alpha_1)\rangle_{\boldsymbol{a}}]^{(k_1)}_{\boldsymbol{a}}\\ \mathcal{D}_{k_1}\\ \varphi_k(\alpha_2)\end{array}\quad \begin{array}{c}[\langle\varphi_k(\alpha_2)\rangle_{\boldsymbol{a}}]^{(k_2)}_{\boldsymbol{a}}\\ \mathcal{D}_{k_2}\\ \varphi_k(\alpha_1)\end{array}}{\alpha_1 \ddot{=} \alpha_2}\ {\scriptstyle (\ddot{=}I),\ 1_1, 1_2, \dots, k_1, k_2}$$

$$\boldsymbol{a}\,\dfrac{\begin{array}{c}\mathcal{D}_1\\ \alpha_1 \ddot{=} \alpha_2\end{array}\quad \begin{array}{c}\mathcal{D}_{i_2}\\ \varphi_i(\alpha_1)\end{array}}{\varphi_i(\alpha_2)}\ (\ddot{=}E_i1) \qquad \boldsymbol{a}\,\dfrac{\begin{array}{c}\mathcal{D}_1\\ \alpha_1 \ddot{=} \alpha_2\end{array}\quad \begin{array}{c}\mathcal{D}_{i_1}\\ \varphi_i(\alpha_2)\end{array}}{\varphi_i(\alpha_1)}\ (\ddot{=}E_i2)$$

where $i \in \{1, \dots, k\}$

These rules do not exclude the case in which α_1 and α_2 are literally identical (i.e., $\alpha_1 \equiv \alpha_2$; cf. [7]: 2). (Recall that we take $\mathcal{P}$ to be finite in case $\ddot{=}$ is in the language.)

Definition 3.6 Derivations in $\mathcal{S}^{\ddot{=}}_{\boldsymbol{a}}$:

Basic step. *Let $\mathcal{S}^{\ddot{=}}_{\boldsymbol{a}} = \langle \mathcal{I}_{\boldsymbol{a}}, \mathcal{R}^{\ddot{=}}_{\boldsymbol{a}} \rangle$ be an agent-relative subatomic identity system based on $\mathcal{S}_{\boldsymbol{a}} = \langle \mathcal{I}_{\boldsymbol{a}}, \mathcal{R}_{\boldsymbol{a}} \rangle$. Any derivation in $\mathcal{S}_{\boldsymbol{a}}$ and any solitary occurrence of an identity sentence enclosed in $\langle\rangle_{\boldsymbol{a}}$ (i.e., an agent-relative derivation of $\alpha_1 \ddot{=} \alpha_2$ from the agent-relative open assumption of itself), is a derivation (in $\mathcal{S}^{\ddot{=}}_{\boldsymbol{a}}$).*

Induction step. *When $\mathcal{D}_1$, $\mathcal{D}_{i_j}$ for any i with $1 \leq i \leq k$ and $j \in \{1, 2\}$ are derivations in $\mathcal{S}^{\ddot{=}}_{\boldsymbol{a}}$, a derivation $\mathcal{D}$ can be generated by means of the rules $\ddot{=}I$ and $\ddot{=}E_ij$ (see Definition 3.5).*

Definition 3.7 *Let $\mathcal{A} = \{\boldsymbol{a}_1, \dots, \boldsymbol{a}_n\}$ be a finite set of agents, let $\mathcal{S}^{\ddot{=}}_{\mathcal{A}} = \{\mathcal{S}^{\ddot{=}}_{\boldsymbol{a}_1}, \dots, \mathcal{S}^{\ddot{=}}_{\boldsymbol{a}_n}\}$, and let $\underline{\mathcal{C}} = \{\underline{a}_1, \dots, \underline{a}_n\}$. A* multi-agent belief base *$\mathcal{I}_{\mathcal{A}}$ is a tuple $\langle \mathcal{A}, \mathcal{S}^{\ddot{=}}_{\mathcal{A}}, \underline{\mathcal{C}}, f, g \rangle$, where for each $i \in \{1, \dots, n\}$:*

$f : \mathcal{A} \to \mathcal{S}^{\ddot{=}}_{\mathcal{A}}$ such that $f(\boldsymbol{a}_i) = \mathcal{S}^{\ddot{=}}_{\boldsymbol{a}_i}$;

$g : \mathcal{A} \to \underline{\mathcal{C}}$ such that $g(\boldsymbol{a}_i) = \underline{a}_i$.

Definition 3.8 *An* intuitionsitic multi-agent natural deduction belief system *$\mathcal{B}_{\boldsymbol{I}(\mathcal{S}^{\ddot{=}}_{\mathcal{A}})}$ is a pair $\langle \mathcal{I}_{\mathcal{A}}, \mathcal{R}_{\mathcal{A}} \rangle$, where $\mathcal{I}_{\mathcal{A}}$ is a multi-agent belief base and $\mathcal{R}_{\mathcal{A}}$ a set which contains for each $\boldsymbol{a}_i \in \mathcal{A}$:*

1. *Agent-relative I/E-rules for the standard logical operators:*

$$\boldsymbol{a}_i\ \dfrac{\begin{matrix}\mathcal{D}_1 & \mathcal{D}_2\\ A & B\end{matrix}}{A\&B}\ (\&I) \qquad \boldsymbol{a}_i\ \dfrac{\begin{matrix}\mathcal{D}_1\\ A\&B\end{matrix}}{A}\ (\&E1) \qquad \boldsymbol{a}_i\ \dfrac{\begin{matrix}\mathcal{D}_1\\ A\&B\end{matrix}}{B}\ (\&E2)$$

$$\boldsymbol{a}_i\ \dfrac{\begin{matrix}\mathcal{D}_1\\ A\end{matrix}}{A \vee B}\ (\vee I1) \qquad \boldsymbol{a}_i\ \dfrac{\begin{matrix}\mathcal{D}_1\\ B\end{matrix}}{A \vee B}\ (\vee I2) \qquad \boldsymbol{a}_i\ \dfrac{\begin{matrix} & [\langle A\rangle_{\boldsymbol{a}_i}]^{(u)}_{\boldsymbol{a}_i} & [\langle B\rangle_{\boldsymbol{a}_i}]^{(v)}_{\boldsymbol{a}_i}\\ \mathcal{D}_1 & \mathcal{D}_2 & \mathcal{D}_3\\ A \vee B & C & C\end{matrix}}{C}\ (\vee E),\ u, v$$

$$\boldsymbol{a}_i\ \dfrac{\begin{matrix}[\langle A\rangle_{\boldsymbol{a}_i}]^{(u)}_{\boldsymbol{a}_i}\\ \mathcal{D}_1\\ B\end{matrix}}{A \supset B}\ (\supset I),\ u \qquad \boldsymbol{a}_i\ \dfrac{\begin{matrix}\mathcal{D}_1 & \mathcal{D}_2\\ A \supset B & A\end{matrix}}{B}\ (\supset E)$$

$$\boldsymbol{a}_i\ \dfrac{\begin{matrix}\mathcal{D}_1\\ A(x/o)\end{matrix}}{\forall xA}\ (\forall I) \qquad \boldsymbol{a}_i\ \dfrac{\begin{matrix}\mathcal{D}_1\\ \forall xA\end{matrix}}{A(x/o)}\ (\forall E) \qquad \boldsymbol{a}_i\ \dfrac{\begin{matrix}\mathcal{D}_1\\ A(x/o)\end{matrix}}{\exists xA}\ (\exists I) \qquad \boldsymbol{a}_i\ \dfrac{\begin{matrix} & [\langle A(x/o)\rangle_{\boldsymbol{a}_i}]^{(u)}_{\boldsymbol{a}_i}\\ \mathcal{D}_1 & \mathcal{D}_2\\ \exists xA & C\end{matrix}}{C}\ (\exists E),\ u$$

$$\boldsymbol{a}_i\ \dfrac{\begin{matrix}\mathcal{D}_1\\ \bot\end{matrix}}{A}\ (\bot i)$$

Side conditions*:*

(a) In $\forall I$: (i) if o is a proper variable y, then $o \equiv x$ or o is not free in A, and o is not free in any assumption of a formula which is open in the derivation of $A(x/o)$; (ii) if o is a nominal constant, then o does neither occur in an undischarged assumption of a formula, nor in $\forall xA$, nor in a term assumption leaf $o\Gamma^{\boldsymbol{a}_i}$; (iii) o is a nominal constant and $\begin{matrix}\mathcal{D}_1\\ A(x/o)\end{matrix}$ for all $o \in \mathcal{C}$.

(b) In $\forall E$: o is free for x in A.

(c) In $\exists E$: (i) if o is a proper variable y, then $o \equiv x$ or o is not free in A, and o is not free in C nor in any assumption of a formula which is open in the derivation of the upper occurrence of C other than $[\langle A(x/o)\rangle_{\boldsymbol{a}_i}]^{(u)}_{\boldsymbol{a}_i}$; (ii) if o is a nominal constant, then o does neither occur in an undischarged assumption of a formula, nor in $\exists xA$, nor in C, nor in a term assumption leaf $o\Gamma^{\boldsymbol{a}_i}$.

(d) *In $\exists I$: o is free for x in A.*

(e) *In $\bot i$: $A \in Atm$.*

We write $\forall I.i$, $\forall I.ii$, $\forall I.iii$ and $\exists E.i$, $\exists E.ii$ when we use the rules $\forall I$ and $\exists E$ according to the conditions given in (i), (ii), and (iii).

In the remaining rules, $i, j \in \{1, ..., n\}$ and possibly $i = j$.

2. *I/E-rules for the belief operator:*

$$\boldsymbol{a}_i \; \frac{\begin{matrix}\mathcal{D}_1\\ A\end{matrix}}{\mathsf{B}_{\underline{a}_j}(A)} \; (\mathsf{B}_{\underline{a}_j} I) \qquad\qquad \boldsymbol{a}_i \; \frac{\begin{matrix}\mathcal{D}_1\\ \mathsf{B}_{\underline{a}_j}(A)\end{matrix}}{A} \; (\mathsf{B}_{\underline{a}_j} E)$$

3. *I/E-rules for the knowledge operator:*

$$\boldsymbol{a}_i \; \frac{\begin{matrix}\mathcal{D}_1\\ A\end{matrix}}{\mathsf{K}_{\underline{a}_j}(A)} \; (\mathsf{K}_{\underline{a}_j} I) \qquad\qquad \boldsymbol{a}_i \; \frac{\begin{matrix}\mathcal{D}_1\\ \mathsf{K}_{\underline{a}_j}(A)\end{matrix}}{A} \; (\mathsf{K}_{\underline{a}_j} E)$$

Side condition on $\mathsf{K}_{\underline{a}_j} I$: A does not depend on any open assumption.

4. *I/E-rules for the universal agent quantifier:*

$$\boldsymbol{a}_i \; \frac{\begin{matrix}\mathcal{D}_1\\ \mathsf{B}_{\underline{a}_j}(A)\end{matrix}}{\underline{\forall}\; \underline{x}\mathsf{B}_{\underline{x}}(A)} \; (\underline{\forall} I) \qquad\qquad \boldsymbol{a}_i \; \frac{\begin{matrix}\mathcal{D}_1\\ \underline{\forall}\; \underline{x}\mathsf{B}_{\underline{x}}(A)\end{matrix}}{\mathsf{B}_{\underline{a}_j}(A)} \; (\underline{\forall} E)$$

Side condition on $\underline{\forall} I$: $\mathsf{B}_{\underline{a}_j} A$ in $\mathcal{D}_1$ for each $\underline{a}_j \in \underline{\mathcal{C}}$.

5. *I/E-rules for the existential agent quantifier:*

$$\boldsymbol{a}_i \; \frac{\begin{matrix}\mathcal{D}_1\\ \mathsf{B}_{\underline{a}_j}(A)\end{matrix}}{\underline{\exists}\; \underline{x}\mathsf{B}_{\underline{x}}(A)} \; (\underline{\exists} I) \qquad\qquad \boldsymbol{a}_i \; \frac{\begin{matrix}\\ \mathcal{D}_1\\ \underline{\exists}\; \underline{x}\mathsf{B}_{\underline{x}}(A)\end{matrix} \qquad \begin{matrix}[\langle \mathsf{B}_{\underline{a}_j}(A)\rangle_{\boldsymbol{a}_i}]^{(u)}_{\boldsymbol{a}_i}\\ \mathcal{D}_2\\ C\end{matrix}}{C} \; (\underline{\exists} E),\, u$$

Side condition on $\underline{\exists} E$: $\underline{a}$ does neither occur in an undischarged assumption, nor in C.

Definition 3.9 Derivations in $\mathcal{B}_{\boldsymbol{I}(\mathcal{S}^{\ddot{=}}_{\mathcal{A}})}$*-systems:*

Basic step. *Let $\mathcal{B}_{\boldsymbol{I}(\mathcal{S}^{\ddot{=}}_{\mathcal{A}})}$ be an intuitionistic multi-agent belief system based on $\mathcal{S}^{\ddot{=}}_{\mathcal{A}}$. Any derivation in any agent-relative identity system in $\mathcal{S}^{\ddot{=}}_{\mathcal{A}}$ and any single occurrence of a logically compound formula A enclosed in $\langle\rangle_{\boldsymbol{a}_i}$, for any $\boldsymbol{a}_i \in \mathcal{A}$ (i.e., an agent-relative derivation of A from the agent-relative open assumption of A), is a derivation (in $\mathcal{B}_{\boldsymbol{I}(\mathcal{S}^{\ddot{=}}_{\mathcal{A}})}$).*

Induction step. *When $\mathcal{D}_1$, $\mathcal{D}_2$, and $\mathcal{D}_3$ are derivations, a derivation $\mathcal{D}$ can be generated by means of the rules listed in Definition 3.8.*

Remarks 3.1 1. *A single-agent natural deduction belief system is a special case of a $\mathcal{B}_{\boldsymbol{I}(\mathcal{S}^{\ddot{=}}_{\mathcal{A}})}$-system in which the sets in $\mathcal{I}_{\mathcal{A}}$ are singletons.*

2. *We take the I-rules of the system to be meaning determining. Note that the I-rule for the belief operator can be applied to any premiss. By contrast, the I-rule for the knowledge operator can be applied only to theorems. So, intuitively, knowledge is a special case of belief, and only theorems can be known.*

3. *The present systems aim at simplicity and liberty. Of course, they can be modified in various ways. For instance, conditions on the embedding of subderivations can be imposed.*

Examples 3.1

$$\boldsymbol{a}_1\,\dfrac{\boldsymbol{a}_1\,\dfrac{\boldsymbol{a}_1\,\dfrac{\boldsymbol{a}_1\,\dfrac{[\langle A\rangle_{\boldsymbol{a}_1}]^{(1)}_{\boldsymbol{a}_1}}{B_{\underline{a}_1}(A)}\,(B_{\underline{a}_1}I)}{A\supset B_{\underline{a}_1}(A)}\,(\supset I),\ 1 \qquad \boldsymbol{a}_1\,\dfrac{\boldsymbol{a}_1\,\dfrac{[\langle B_{\underline{a}_1}(A)\rangle_{\boldsymbol{a}_1}]^{(2)}_{\boldsymbol{a}_1}}{A}\,(B_{\underline{a}_1}E)}{B_{\underline{a}_1}(A)\supset A}\,(\supset I),\ 2}{A\leftrightarrow B_{\underline{a}_1}(A)}\,(\&I)}{K_{\underline{a}_1}(A\leftrightarrow B_{\underline{a}_1}(A))}\,(K_{\underline{a}_1}I) \tag{1}$$

$$\boldsymbol{a}_1\,\dfrac{\boldsymbol{a}_1\,\dfrac{\boldsymbol{a}_1\,\dfrac{\boldsymbol{a}_1\,\dfrac{[\langle A\rangle_{\boldsymbol{a}_1}]^{(1)}_{\boldsymbol{a}_1}}{A\supset A}\,(\supset I),\ 1}{K_{\underline{a}_1}(A\supset A)}\,(K_{\underline{a}_1}I)}{A\supset K_{\underline{a}_1}(A\supset A)}\,(\supset I)}{B_{\underline{a}_1}(A\supset K_{\underline{a}_1}(A\supset A))}\,(B_{\underline{a}_1}I) \tag{2}$$

$$\boldsymbol{a}_1\,\dfrac{\boldsymbol{a}_1\,\dfrac{\boldsymbol{a}_1\,\dfrac{\boldsymbol{a}_1\,\dfrac{[\langle K_{\underline{a}_1}(A)\rangle_{\boldsymbol{a}_1}]^{(1)}_{\boldsymbol{a}_1}}{A}\,(K_{\underline{a}_1}E)}{B_{\underline{a}_1}(A)}\,(B_{\underline{a}_1}I)}{K_{\underline{a}_1}(A)\supset B_{\underline{a}_1}(A)}\,(\supset I),\ 1 \qquad \boldsymbol{a}_1\,\dfrac{\boldsymbol{a}_1\,\dfrac{\boldsymbol{a}_1\,\dfrac{[\langle B_{\underline{a}_1}(A)\rangle_{\boldsymbol{a}_1}]^{(2)}_{\boldsymbol{a}_1}}{A}\,(B_{\underline{a}_1}E)}{K_{\underline{a}_1}(A)}\,(K_{\underline{a}_1}I)\ \text{illegal!}}{B_{\underline{a}_1}(A)\supset K_{\underline{a}_1}(A)}\,(\supset I),\ 2}{K_{\underline{a}_1}(A)\leftrightarrow B_{\underline{a}_1}(A)}\,(\&I) \tag{3}$$

$$\boldsymbol{a}_1\,\dfrac{\boldsymbol{a}_1\,\dfrac{\boldsymbol{a}_1\,\dfrac{\boldsymbol{a}_1\,\dfrac{\boldsymbol{a}_1\,\dfrac{\mathcal{D}\qquad \boldsymbol{a}_1\,\dfrac{\boldsymbol{a}_1\,\dfrac{\boldsymbol{a}_1\,\dfrac{[\langle B_{\underline{a}_1}(\neg(\alpha_1\ddot{=}\alpha_2))\rangle_{\boldsymbol{a}_1}]^{(1)}_{\boldsymbol{a}_1}}{\neg(\alpha_1\ddot{=}\alpha_2)}\,(B_{\underline{a}_1}E)\qquad \boldsymbol{a}_1\,\dfrac{[\langle B_{\underline{a}_1}(\alpha_1\ddot{=}\alpha_2)\rangle_{\boldsymbol{a}_1}]^{(2)}_{\boldsymbol{a}_1}}{\alpha_1\ddot{=}\alpha_2}\,(B_{\underline{a}_1}E)}{\bot}\,(\supset E)}{\varphi^n_k(\alpha_3)}\,(\bot i)}{\varphi^n_k(\alpha_3)}\,(\ddot{=}E_k 2)}{B_{\underline{a}_1}(\varphi^n_k(\alpha_3))}\,(B_{\underline{a}_1}I)}{B_{\underline{a}_1}(\alpha_1\ddot{=}\alpha_2)\supset B_{\underline{a}_1}(\varphi^n_k(\alpha_3))}\,(\supset I),\ 2}{B_{\underline{a}_1}(\neg(\alpha_1\ddot{=}\alpha_2))\supset(B_{\underline{a}_1}(\alpha_1\ddot{=}\alpha_2)\supset B_{\underline{a}_1}(\varphi^n_k(\alpha_3)))}\,(\supset I),\ 1}{B_{\underline{a}_1}(B_{\underline{a}_1}(\neg(\alpha_1\ddot{=}\alpha_2))\supset(B_{\underline{a}_1}(\alpha_1\ddot{=}\alpha_2)\supset B_{\underline{a}_1}(\varphi^n_k(\alpha_3))))}\,(B_{\underline{a}_1}I)$$

$\mathcal{D}$, with $i\neq 0$, is:

$$\boldsymbol{a}_1\,\dfrac{\boldsymbol{a}_1\,\dfrac{\boldsymbol{a}_1\,\dfrac{[\langle\varphi^n_1(\alpha_3)\rangle_{\boldsymbol{a}_1}]^{(1_1)}_{\boldsymbol{a}_1}}{\varphi^n_1\Gamma}\,(asE_0)\ \ldots\qquad \boldsymbol{a}_1\,\dfrac{[\langle\varphi^n_1(\alpha_3)\rangle_{\boldsymbol{a}_1}]^{(1_1)}_{\boldsymbol{a}_1}}{\alpha_3\Gamma}\,(asE_i)}{\varphi^n_1(\alpha_3)}\,(asI)\quad\ldots\qquad \boldsymbol{a}_1\,\dfrac{\boldsymbol{a}_1\,\dfrac{[\langle\varphi^n_k(\alpha_3)\rangle_{\boldsymbol{a}_1}]^{(k_2)}_{\boldsymbol{a}_1}}{\varphi^n_k\Gamma}\ \ldots\qquad \boldsymbol{a}_1\,\dfrac{[\langle\varphi^n_k(\alpha_3)\rangle_{\boldsymbol{a}_1}]^{(k_2)}_{\boldsymbol{a}_1}}{\alpha_3\Gamma}}{\varphi^n_k(\alpha_3)}\,(asI)}{\alpha_3\ddot{=}\alpha_3}\,(\ddot{=}I),\ 1_1,1_2,\ldots,k_1,k_2 \tag{4}$$

$$\boldsymbol{a}_1\,\dfrac{\boldsymbol{a}_4\,\dfrac{\boldsymbol{a}_4\,\dfrac{\boldsymbol{a}_3\,\dfrac{\boldsymbol{a}_3\,\dfrac{\boldsymbol{a}_2\,\dfrac{\boldsymbol{a}_1\,\dfrac{\boldsymbol{a}_1\,\dfrac{\boldsymbol{a}_1\,\dfrac{\varphi^2\Gamma^{\boldsymbol{a}}\quad \alpha_1\Gamma^{\boldsymbol{a}}\quad \alpha_2\Gamma^{\boldsymbol{a}}}{\varphi^2\alpha_1\alpha_2}\,(asI)}{B_{\underline{a}_1}(\varphi^2\alpha_1\alpha_2)}\,(B_{\underline{a}_1}I)}{\exists x(B_{\underline{a}_1}(\varphi^2 x\alpha_2))}\,(\exists I)}{B_{\underline{a}_2}(\exists x(B_{\underline{a}_1}(\varphi^2 x\alpha_2)))}\,(B_{\underline{a}_2}I)}{\underline{\exists}\ \underline{x}(B_{\underline{x}}(\exists x(B_{\underline{a}_1}(\varphi^2 x\alpha_2))))}\,(\underline{\exists}I)}{B_{\underline{a}_3}(\underline{\exists}\ \underline{x}(B_{\underline{x}}(\exists x(B_{\underline{a}_1}(\varphi^2 x\alpha_2)))))}\,(B_{\underline{a}_3}I)}{B_{\underline{a}_4}(B_{\underline{a}_3}(\underline{\exists}\ \underline{x}(B_{\underline{x}}(\exists x(B_{\underline{a}_1}(\varphi^2 x\alpha_2))))))}\,(B_{\underline{a}_4}I)}{\underline{\forall}\ \underline{y}(B_{\underline{y}}(B_{\underline{a}_3}(\underline{\exists}\ \underline{x}(B_{\underline{x}}(\exists x(B_{\underline{a}_1}(\varphi^2 x\alpha_2)))))))}\,(\underline{\forall}I)}{B_{\underline{a}_1}(\underline{\forall}\ \underline{y}(B_{\underline{y}}(B_{\underline{a}_3}(\underline{\exists}\ \underline{x}(B_{\underline{x}}(\exists x(B_{\underline{a}_1}(\varphi^2 x\alpha_2))))))))}\,(B_{\underline{a}_1}I) \tag{5}$$

$$\boldsymbol{a}_1\,\dfrac{\boldsymbol{a}_2\,\dfrac{\boldsymbol{a}_3\,\dfrac{\boldsymbol{a}_1\,\dfrac{\boldsymbol{a}_1\,\dfrac{[\langle \underline{\forall}\ \underline{x} K_{\underline{x}}(\neg A)\rangle_{\boldsymbol{a}_1}]^{(1)}_{\boldsymbol{a}_1}}{K_{\underline{a}_1}(\neg A)}\,(\underline{\forall}E)}{\neg A}\,(K_{\underline{a}_1}E) \qquad \boldsymbol{a}_2\,\dfrac{[\langle B_{\underline{a}_2}(A)\rangle_{\boldsymbol{a}_2}]^{(2)}_{\boldsymbol{a}_2}}{A}\,(B_{\underline{a}_2}E)}{\bot}\,(\supset E)}{B_{\underline{a}_2}(A)\supset\bot}\,(\supset I),\ 2}{\underline{\forall}\ \underline{x}K_{\underline{x}}(\neg A)\supset\neg B_{\underline{a}_2}(A)}\,(\supset I),\ 1 \tag{6}$$

(3) is not a derivation in the system. (4) contains a detour and is, therefore, not a normal $\mathcal{B}_{\mathbf{I}(\mathcal{S}^{\ddot{=}}_{\mathcal{A}})}$-derivation. All conclusions of the derivations are theorems, except for the conclusion of (5).

4 Results

Theorem 4.1 *Any derivation $\mathcal{D}$ in a $\mathcal{B}_{\mathbf{I}(\mathcal{S}^{\ddot{=}}_{\mathcal{A}})}$-system can be transformed into a normal derivation.*

Theorem 4.2 *If $\mathcal{D}$ is a normal $\mathcal{B}_{\mathbf{I}(\mathcal{S}^{\ddot{=}}_{\mathcal{A}})}$-derivation of a unit U (i.e., a term assumption or a formula) from a set of units Γ, then each expression in a unit in $\mathcal{D}$ is a subexpression of an expression in $\Gamma \cup \{U\}$.*

The proofs build on definitions and methods due to Prawitz (see [3], [7]). They are not part of this extended abstract.

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