A Forward Calculus for Countermodel Construction in IPL

Camillo Fiorentini[1], Mauro Ferrari[2]

[1] DI, Univ. degli Studi di Milano, Via Comelico, 39, 20135 Milano, Italy
[2] DiSTA, Univ. degli Studi dell'Insubria, Via Mazzini, 5, 21100, Varese, Italy

The inverse method, introduced in the 1960s by Maslov [10], is a saturation based theorem proving technique closely related to (hyper)resolution [5]; it relies on a forward proof-search strategy and can be applied to cut-free calculi enjoying the subformula property. Given a goal, a set of instances of the rules of the calculus at hand is selected; such specialized rules are repeatedly applied in the forward direction, starting from the axioms (i.e., the rules without premises). Proof-search terminates if either the goal is obtained or the database of proved facts saturates (no new fact can be added). The inverse method has been originally applied to Classical Logic and successively extended to some non-classical logics, see, e.g., [1,5,6,9]. A significant example are the (focused) forward calculi for Intuitionistic Logic presented in [3,4] and implemented in the prover Imogen [11].

In all the mentioned papers, the inverse method has been exploited to prove the validity of a goal in a specific logic. Here we follow the dual approach, namely: we design a forward calculus to derive the unprovability of a goal formula in Intuitionistic Propositional Logic (IPL). Our motivation is twofold. Firstly, we aim to define a calculus which is prone to constructively ascertain the unprovability of a formula by providing a concise countermodel for it. Differently from backward proof-search methods, where rules are applied bottom-up, in forward proof-search the proved sequents need not be duplicated; accordingly, the obtained derivations contain few redundancies and the extracted countermodel are in general small. We show that we can build a derivation of an unprovable formula so that the extracted countermodel has minimal height.

The second motivation is to clarify the role of the saturated database obtained as a result of a failed proof-search. In the case of the usual forward calculi for Intuitionistic provability, if proof-search fails, a saturated database is generated which *"may be considered a kind of countermodel for the goal sequent"* [11]. However, as far as we know, no method has been proposed to effectively extract it. Actually, the main problem comes from the high level of non-determinism involved in the construction of countermodels. Here, assuming the dual approach, the saturated database generated by a failed proof-search can be considered as *a kind of proof of the goal*; we give evidence of this by showing how to extract from such a database a derivation witnessing the Intuitionistic validity of the goal.

The preliminary results of this research have been introduced in [7], where the forward calculus is presented and the correctness of the proof-search procedure is proved. Here we discuss the new results concerning the minimality of generated countermodels and the role of saturated databases.

Here we only account for the main ideas and outcomes; for an in-depth presentation we refer to [8]. To evaluate the potential of our approach, we have implemented the proof-search procedures in the prover `frj` [3].

1 The calculus FRJ(G)

Let $\mathcal{L}$ be the propositional language based on a denumerable set of propositional variables $\mathcal{V}$, the connectives $\wedge$, $\vee$, $\supset$ and the logical constant $\bot$; $\neg A$ is a shorthand for $A \supset \bot$. By $\mathcal{V}^{\bot}$ we denote the set $\mathcal{V} \cup \{\bot\}$ and by $\mathcal{L}^{\supset}$ the set of the implicative formulas $A \supset B$ of $\mathcal{L}$. Capital Greek letters Γ, Σ, ... denote sets of formulas; we use notations like Γ^{At} and $\Gamma^{\supset}$ to mean that $\Gamma^{\mathrm{At}} \subseteq \mathcal{V}$ and $\Gamma^{\supset} \subseteq \mathcal{L}^{\supset}$. $\mathrm{Sf}(G)$ is the set of all subformulas of G (including G itself). By $\mathrm{S{\scriptsize L}}(G)$ (*left subformulas of* G) and $\mathrm{S{\scriptsize R}}(G)$ (*right subformulas* of G) we denote the smallest subsets of $\mathrm{Sf}(G)$ such that $G \in \mathrm{S{\scriptsize R}}(G)$ and, given $\mathrm{S{\scriptsize X}} \in \{\mathrm{S{\scriptsize L}}, \mathrm{S{\scriptsize R}}\}$ ($\overline{\mathrm{S{\scriptsize L}}} = \mathrm{S{\scriptsize R}}$ and $\overline{\mathrm{S{\scriptsize R}}} = \mathrm{S{\scriptsize L}}$):

- $A \odot B \in \mathrm{S{\scriptsize X}}(G)$ implies $\{A, B\} \subseteq \mathrm{S{\scriptsize X}}(G)$, where $\odot \in \{\wedge, \vee\}$;
- $A \supset B \in \mathrm{S{\scriptsize X}}(G)$ implies $B \in \mathrm{S{\scriptsize X}}(G)$ and $A \in \overline{\mathrm{S{\scriptsize X}}}(G)$.

The Forward Refutation calculus $\mathbf{FRJ}(G)$ is a calculus to infer the unprovability of a formula G (the *goal formula*) in IPL and it is designed to support forward proof-search. The calculus acts on $\mathbf{FRJ}(G)$-sequents that depend on the subformulas of G. We set: $\overline{\Gamma}^{\mathrm{At}} = \mathrm{S{\scriptsize L}}(G) \cap \mathcal{V}$, $\overline{\Gamma}^{\supset} = \mathrm{S{\scriptsize L}}(G) \cap \mathcal{L}^{\supset}$, $\overline{\Gamma} = \overline{\Gamma}^{\mathrm{At}} \cup \overline{\Gamma}^{\supset}$. There are two types of $\mathbf{FRJ}(G)$-sequents, we call *regular* (arrow $\Rightarrow$) and *irregular* (arrow $\rightarrow$), defined as follows:

- *regular sequents* have the form $\Gamma \Rightarrow C$, where $\Gamma \subseteq \overline{\Gamma}$ and $C \in \mathrm{S{\scriptsize R}}(G)$;
- *irregular sequents* have the form $\Sigma\,;\,\Theta \rightarrow C$, where $\Sigma \cup \Theta \subseteq \overline{\Gamma}$ and $C \in \mathrm{S{\scriptsize R}}(G)$.

The *left hand side* of a sequent σ is $\mathrm{Lhs}(\sigma) = \Gamma$ if σ is regular and $\mathrm{Lhs}(\sigma) = \Sigma \cup \Theta$ if σ is irregular; the *right hand side* of σ is $\mathrm{Rhs}(\sigma) = C$.

The rules of $\mathbf{FRJ}(G)$ are collected in Fig. 1 and discussed below. The forward proof-search procedure to build an $\mathbf{FRJ}(G)$-derivation of a goal formula G, namely an $\mathbf{FRJ}(G)$-derivation of a regular sequent of the form $\Gamma \Rightarrow G$, is a standard saturation procedure where the provable sequents of $\mathbf{FRJ}(G)$ are collected step-by-step in a database $\mathtt{D}_G$. Initially $\mathtt{D}_G$ contains all possible instances of axiom sequents; then a loop is entered where the rules of the calculus are repeatedly applied (in the forward direction) to the sequents in $\mathtt{D}_G$. The loop ends when either G is proved or no new sequent can be added to $\mathtt{D}_G$; since the number of sequents of $\mathbf{FRJ}(G)$ is bounded, the process eventually ends.

The calculus $\mathbf{FRJ}(G)$ consists of two axiom rules, some right introduction rules for the connectives $\wedge$, $\vee$, $\supset$ and the rules $\bowtie^{\mathrm{At}}$ and $\bowtie^{\vee}$ to join sequents; note that there are no left rules. Now, we provide an in-depth presentation of the rules.

[3] `frj` is available at `http://github.com/ferram/jtabwb_provers/`.

$\overline{\Gamma}^{\mathrm{At}} = \textsc{Sl}(G) \cap \mathcal{V}$, $\overline{\Gamma}^{\supset} = \textsc{Sl}(G) \cap \mathcal{L}^{\supset}$, $\overline{\Gamma} = \overline{\Gamma}^{\mathrm{At}} \cup \overline{\Gamma}^{\supset}$.

In the conclusion σ of each rule, $\mathrm{Rhs}(\sigma) \in \textsc{Sr}(G)$

$$\frac{}{\overline{\Gamma}^{\mathrm{At}} \setminus \{F\} \Rightarrow F}\ \mathrm{Ax}_{\Rightarrow} \qquad \frac{}{\cdot\,;\, \overline{\Gamma}^{\mathrm{At}} \setminus \{F\}, \overline{\Gamma}^{\supset} \to F}\ \mathrm{Ax}_{\to} \qquad F \in \mathcal{V}^{\perp}$$

$$\frac{\Gamma \Rightarrow A_k}{\Gamma \Rightarrow A_1 \wedge A_2}\ \wedge \qquad \frac{\Sigma\,;\,\Theta \to A_k}{\Sigma\,;\,\Theta \to A_1 \wedge A_2}\ \wedge \quad k \in \{1,2\}$$

$$\frac{\Sigma_1\,;\,\Theta_1 \to C_1 \qquad \Sigma_2\,;\,\Theta_2 \to C_2}{\Sigma_1, \Sigma_2\,;\,\Theta_1 \cap \Theta_2 \to C_1 \vee C_2}\ \vee \qquad \begin{array}{l} \Sigma_1 \subseteq \Sigma_2 \cup \Theta_2 \\ \Sigma_2 \subseteq \Sigma_1 \cup \Theta_1 \end{array}$$

$$\frac{\Gamma \Rightarrow B}{\Gamma \Rightarrow A \supset B}\ \supset_{\in} \quad A \in \mathcal{C}l(\Gamma) \qquad \frac{\Sigma\,;\,\Theta, \Lambda \to B}{\Sigma, \Lambda\,;\,\Theta \to A \supset B}\ \supset_{\in} \quad \begin{array}{l} \Theta \cap \Lambda = \emptyset \\ A \in \mathcal{C}l(\Sigma \cup \Lambda) \end{array}$$

$$\frac{\Gamma \Rightarrow B}{\cdot\,;\,\Theta \to A \supset B}\ \supset_{\not\in} \qquad \begin{array}{l} \Theta \ \subseteq\ \mathcal{C}l(\Gamma) \cap \overline{\Gamma} \\ A \ \in\ \mathcal{C}l(\Gamma) \setminus \mathcal{C}l(\Theta) \end{array}$$

Let, for $1 \leq j \leq n$, $\sigma_j = \underbrace{\Sigma_j^{\mathrm{At}}, \Sigma_j^{\supset}}_{\Sigma_j}\,;\,\underbrace{\Theta_j^{\mathrm{At}}, \Theta_j^{\supset}}_{\Theta_j} \to A_j$ and $\Upsilon = \{A_1, \ldots, A_n\}$

$$\Sigma^{\mathrm{At}} = \textstyle\bigcup_{1 \leq j \leq n} \Sigma_j^{\mathrm{At}} \qquad \Theta^{\mathrm{At}} = \bigcap_{1 \leq j \leq n} \Theta_j^{\mathrm{At}} \qquad \Sigma^{\supset} = \bigcup_{1 \leq j \leq n} \Sigma_j^{\supset} \qquad \Theta^{\supset} = (\bigcap_{1 \leq j \leq n} \Theta_j^{\supset})/\Upsilon$$

$$\frac{\sigma_1 \qquad \cdots \qquad \sigma_n}{\Sigma^{\mathrm{At}},\, \Theta^{\mathrm{At}} \setminus \{F\},\, \Sigma^{\supset},\, \Theta^{\supset} \Rightarrow F}\ \bowtie^{\mathrm{At}} \qquad \begin{array}{l} \Sigma_i \subseteq \Sigma_j \cup \Theta_j, \text{ for every } i \neq j \\ Y \supset Z \in \Sigma^{\supset} \text{ implies } Y \in \Upsilon \\ F \in \mathcal{V}^{\perp} \setminus \Sigma^{\mathrm{At}} \end{array}$$

$$\frac{\sigma_1 \qquad \cdots \qquad \sigma_n}{\Sigma^{\mathrm{At}},\, \Theta^{\mathrm{At}},\, \Sigma^{\supset},\, \Theta^{\supset} \Rightarrow C_1 \vee C_2}\ \bowtie^{\vee} \qquad \begin{array}{l} \Sigma_i \subseteq \Sigma_j \cup \Theta_j, \text{ for every } i \neq j \\ Y \supset Z \in \Sigma^{\supset} \text{ implies } Y \in \Upsilon \\ \{C_1, C_2\} \ \subseteq\ \Upsilon \end{array}$$

Fig. 1. The calculus **FRJ**(G).

Axiom rules. We have two axiom rules. Rule $\mathrm{Ax}_{\Rightarrow}$ introduces a regular axiom, rule $\mathrm{Ax}_{\to}$ an irregular one. Both axiom sequents have in the right a formula $F \in \mathcal{V}^{\perp}$; in irregular axioms the set Σ is empty (denoted by $\cdot$).

Rules for $\wedge$. We have two rules to introduce $\wedge$ in the right. In both rules the type of the sequents (regular or irregular) does not change.

Rules for $\supset$ In standard refutation calculi, the rule for right implication is

$$\frac{\Gamma \Rightarrow B}{\Gamma \Rightarrow A \supset B}\ R \supset \qquad A \in \Gamma$$

The side condition $A \in \Gamma$ is needed to guarantee that, assuming that B is not provable from Γ, then $A \supset B$ is not provable from Γ as well. With such a rule alone, the calculus **FRJ**(G) would be incomplete. For instance, we are not able to derive the goal formula $G = p_1 \wedge p_2 \supset q$, since the condition $p_1 \wedge p_2 \in \Gamma$ is never matched (Γ only contains propositional variables and $\supset$-formulas). To compensate for this, we have to relax the side condition. This leads to the rule $\supset_\in$ of $\mathbf{FRJ}(G)$ where we introduce the more general side condition $A \in \mathcal{C}l(\Gamma)$ (whence the $\in$ in the rule name), where $\mathcal{C}l(\Gamma)$ (the *closure* of Γ) is the smallest set containing the formulas X defined by the following grammar:

$$X \ ::= \ C \mid X \wedge X \mid A \vee X \mid X \vee A \mid A \supset X \qquad C \in \Gamma,\ A \text{ any formula}$$

Using this rule, we can prove the goal $G = p_1 \wedge p_2 \supset q$ as follows:

$$\dfrac{\dfrac{}{p_1, p_2 \Rightarrow q}\,\mathrm{Ax}_{\Rightarrow}}{p_1, p_2 \Rightarrow p_1 \wedge p_2 \supset q}\,\supset_\in \qquad p_1 \wedge p_2 \in \mathcal{C}l(\,\{p_1, p_2\}\,)$$

We can also apply rule $\supset_\in$ to an irregular sequent $\sigma_1 = \Sigma\,;\,\Theta' \to B$ to get an irregular sequent $\sigma = \Sigma'\,;\,\Theta \to A \supset B$. In this case the antecedent A must belong to $\mathcal{C}l(\Sigma')$ and we are allowed to transfer formulas from Θ' to Σ' so to satisfy such a condition. In rule $\supset_\in$ for irregular sequents the set Θ' is partitioned as $\Theta \cup \Lambda$, where the (possibly empty) set Λ is shifted to the left of semicolon. We also need a further rule to introduce $A \supset B$ in the right, having premise $\sigma_1 = \Gamma \Rightarrow B$ and conclusion $\sigma = \cdot\,;\,\Theta \to A \supset B$; this is the only rule of $\mathbf{FRJ}(G)$ turning a regular sequent into an irregular one. We require that $A \in \mathcal{C}l(\Gamma)$ and Θ is any subset of $\mathcal{C}l(\Gamma) \cap \overline{\Gamma}$ such that $A \notin \mathcal{C}l(\Theta)$.

Rule $\vee$. Apparently, the rule for $R\vee$ applied to $\Gamma_1 \Rightarrow A_1$ and $\Gamma_2 \Rightarrow A_2$ should generate $\Gamma_1 \cap \Gamma_2 \Rightarrow A_1 \vee A_2$, but such a rule is not sound. For instance, let $H = p \supset q_1 \vee q_2$, and let us take the unprovable sequents $q_2, p, H \Rightarrow q_1$ and $q_1, p, H \Rightarrow q_2$; the alleged $R\vee$ rule yields $p, H \Rightarrow q_1 \vee q_2$, which is provable. The drawback is that we cannot retain both p and H in the conclusion. This is the reason we introduce irregular sequents which provide a clever strategy to join sequents. The rule $\vee$ has as premises the irregular sequents $\sigma_1 = \Sigma_1\,;\,\Theta_1 \to C_1$ and $\sigma_2 = \Sigma_2\,;\,\Theta_2 \to C_2$. In the conclusion $\sigma = \Sigma_1, \Sigma_2\,;\,\Theta_1 \cap \Theta_2 \to C_1 \vee C_2$, introducing a disjunction in the right, the Σ-sets of the premises are preserved, while the Θ-sets are intersected; note that $\mathrm{Lhs}(\sigma) \subseteq \mathrm{Lhs}(\sigma_1) \cap \mathrm{Lhs}(\sigma_2)$.

Join rules. The *join* rules $\bowtie^{\mathrm{At}}$ and $\bowtie^{\vee}$ apply to $n \geq 1$ irregular sequents $\sigma_1 = \Sigma_1\,;\,\Theta_1 \to A_1, \ldots, \sigma_n = \Sigma_n\,;\,\Theta_n \to A_n$ and yield a regular sequent $\sigma = \Gamma \Rightarrow C$; this is the only way to obtain a regular sequent from irregular ones. These two rules have a similar structure and only differ in the form of C: in rule $\bowtie^{\mathrm{At}}$, $C \in \mathcal{V}^{\perp}$ while in $\bowtie^{\vee}$, C is an $\vee$-formula. For every $1 \leq j \leq n$, we write the premise σ_j as follows:

$$\sigma_j \ = \ \underbrace{\Sigma_j^{\mathrm{At}}, \Sigma_j^{\supset}}_{\Sigma_j}\ ;\ \underbrace{\Theta_j^{\mathrm{At}}, \Theta_j^{\supset}}_{\Theta_j} \to A_j \qquad \Sigma_j^{\mathrm{At}} \cup \Theta_j^{\mathrm{At}} \subseteq \mathcal{V} \text{ and } \Sigma_j^{\supset} \cup \Theta_j^{\supset} \subseteq \mathcal{L}^{\supset}$$

Similarly to rule $\vee$, formulas in Σ_j must be preserved in the conclusion and we need $\mathrm{Lhs}(\sigma) \subseteq \mathrm{Lhs}(\sigma_1) \cap \cdots \cap \mathrm{Lhs}(\sigma_n)$. Thus, we require the side condition of rule $\vee$ for every pair of distinct premises, namely:

(J1) $\Sigma_i \subseteq \Sigma_j \cup \Theta_j$ for every $i \neq j$.

From a semantic perspective, the role of join rules is to downward expand a countermodel under construction (see Sec. 2 for the formal definitions of Kripke model and countermodel). To properly perform this, we need a further side condition (J2) on the sets Σ_j. Let

$$\Sigma^{\mathrm{At}} \;=\; \bigcup_{1\leq j\leq n} \Sigma_j^{\mathrm{At}} \qquad\qquad \Sigma^{\supset} \;=\; \bigcup_{1\leq j\leq n} \Sigma_j^{\supset}$$

Since both Σ^{At} and $\Sigma^{\supset}$ must be kept in the conclusion $\sigma = \Gamma \Rightarrow C$ (namely, $\Sigma^{\mathrm{At}} \cup \Sigma^{\supset} \subseteq \Gamma$), we need that all the formulas in $\Sigma^{\mathrm{At}} \cup \Sigma^{\supset}$ are forced in the new world α. A formula $Y \supset Z$ is *supported* if there exists a premise σ_k $(1 \leq k \leq n)$ such that $Y = A_k$ (A_k is the right formula of σ_k). We require that all the $\supset$-formulas in the sets Σ_j are supported, and this is formalized by the following side condition:

(J2) $Y \supset Z \in \Sigma^{\supset}$ implies $Y \in \{A_1, \dots, A_n\}$.

Side conditions do not concern the sets Θ_j and in the conclusion we keep some of the formulas in the intersection of all the Θ_j's. More precisely, given a set of $\supset$-formulas $\Gamma^{\supset}$ and a set of formulas Υ, let

$$\Gamma^{\supset}/\Upsilon \;=\; \{\; Y \supset Z \in \Gamma^{\supset} \mid Y \in \Upsilon \;\}$$

We call $\Gamma^{\supset}/\Upsilon$ the *restriction* of $\Gamma^{\supset}$ to Υ. Then, the formulas Θ^{At} and $\Theta^{\supset}$ to be kept in the conclusion are defined as follows:

$$\Theta^{\mathrm{At}} \;=\; \bigcap_{1\leq j\leq n} \Theta_j^{\mathrm{At}} \qquad\qquad \Theta^{\supset} \;=\; \left(\bigcap_{1\leq j\leq n} \Theta_j^{\supset} \right) / \, \{A_1, \dots, A_n\}$$

We can now define the conclusion σ of a join rule having premises $\sigma_1, \dots, \sigma_n$ matching the side conditions (J1) and (J2). In rule $\bowtie^{\mathrm{At}}$, we can choose as right formula of σ any formula $F \in \mathcal{V}^{\perp} \cap \mathrm{Rhs}(G)$ such that $F \notin \Sigma^{\mathrm{At}}$. We set:

$$\sigma \;=\; \Sigma^{\mathrm{At}},\, \Theta^{\mathrm{At}} \setminus \{F\},\, \Sigma^{\supset},\, \Theta^{\supset} \Rightarrow F$$

In rule $\bowtie^{\vee}$, we can choose as right formula of σ any $\vee$-formula $C_1 \vee C_2 \in \mathrm{Rhs}(G)$ such that both C_1 and C_2 are among $A_1, \dots, A_n$. We set:

$$\sigma \;=\; \Sigma^{\mathrm{At}},\, \Theta^{\mathrm{At}},\, \Sigma^{\supset},\, \Theta^{\supset} \Rightarrow C_1 \vee C_2 \qquad \{C_1, C_2\} \;\subseteq\; \{A_1, \dots, A_n\}$$

To reduce the proof-search space, we can introduce further restrictions on rule applications.

We now provide a significant example of derivation.

Example 1. Let us consider the following instance T of *Anti-Scott* principle, which is equivalent to Nishimura formula N_9 [2]:

$$S \;=\; ((\neg\neg p \supset p) \supset \neg p \vee p) \;\supset\; \neg\neg p \vee \neg p \qquad T \;=\; S \;\supset\; (\neg\neg p \supset p) \vee \neg\neg p$$

Anti-Scott principle is valid in Classical Logic but not in IPL. Fig. 2 show an $\mathbf{FRJ}(T)$-derivation $\mathcal{D}_T$ of T in linear representation. We populate the database of proved sequents according with the naive recipe of [5] outlined above. For the sake of conciseness, we only show the sequents needed to get the goal. We denote with $\sigma_{(k)}$ the sequent derived at line (k). The tree-like structure of derivation $\mathcal{D}_T$ is displayed in Fig. 3 respectively. We point out that $\mathcal{D}_T$ contains an application of $\bowtie^\vee$ having four premises, namely:

$$\frac{\sigma_{(2)} \qquad \sigma_{(7)} \qquad \sigma_{(8)} \qquad \sigma_{(11)}}{\sigma_{(13)}} \bowtie^\vee$$

The sequent $\sigma_{(13)}$ is essential to build the countermodel since it corresponds to a world where both $\neg\neg p \supset p$ and S are forced, while $\neg p \vee p$ is not; to get $\Gamma = \{\neg\neg p \supset p, S\}$ in the left of $\sigma_{(13)}$, we have to use premises σ' such that $\Gamma \subseteq \mathrm{Lhs}(\sigma')$. Sequents $\sigma_{(8)}$ and $\sigma_{(2)}$ are needed to obtain $\neg p \vee p$ in the right of $\sigma_{(13)}$, while sequents $\sigma_{(7)}$ and $\sigma_{(11)}$ are needed to support $\neg\neg p \supset p$ and S respectively. One can easily check that the side conditions (J1) and (J2) hold, thus the displayed application of rule $\bowtie^\vee$ is sound. Finally, we point out that it is not possible to obtain $\sigma_{(13)}$ using less than four premises. $\Diamond$

2 Countermodels, soundness and minimality

First of all, let us introduce the Kripke semantics for IPL. A *Kripke model* is a structure $\mathcal{K} = \langle P, \leq, \rho, V \rangle$, where $\langle P, \leq \rangle$ is a finite poset with minimum ρ and $V : P \to 2^{\mathcal{V}}$ is a function such that $\alpha \leq \beta$ implies $V(\alpha) \subseteq V(\beta)$. The *forcing relation* $\Vdash \subseteq P \times \mathcal{L}$ is defined as follows:

- $\mathcal{K}, \alpha \nVdash \bot$ and, for every $p \in \mathcal{V}$, $\mathcal{K}, \alpha \Vdash p$ iff $p \in V(\alpha)$;
- $\mathcal{K}, \alpha \Vdash A \wedge B$ iff $\mathcal{K}, \alpha \Vdash A$ and $\mathcal{K}, \alpha \Vdash B$;
- $\mathcal{K}, \alpha \Vdash A \vee B$ iff $\mathcal{K}, \alpha \Vdash A$ or $\mathcal{K}, \alpha \Vdash B$;
- $\mathcal{K}, \alpha \Vdash A \supset B$ iff, for every $\beta \in P$ such that $\alpha \leq \beta$, $\mathcal{K}, \beta \nVdash A$ or $\mathcal{K}, \beta \Vdash B$.

Monotonicity property holds for arbitrary formulas, i.e.: $\mathcal{K}, \alpha \Vdash A$ and $\alpha \leq \beta$ imply $\mathcal{K}, \beta \Vdash A$. A formula A is *valid* in $\mathcal{K}$ iff $\mathcal{K}, \rho \Vdash A$. Intuitionistic Propositional Logic IPL coincides with the set of the formulas valid in all Kripke models [2]. If $\mathcal{K}, \rho \nVdash A$, we say that $\mathcal{K}$ is a *countermodel* for A. A *final* world γ of $\mathcal{K}$ is a maximal world in $\langle P, \leq \rangle$; for every classically valid formula A, we have $\mathcal{K}, \gamma \Vdash A$.

We show that we can extract from an $\mathbf{FRJ}(G)$-derivation $\mathcal{D}$ of G a countermodel for G, namely a model $\mathcal{K}$ such that G is not forced at the root of $\mathcal{K}$; this implies the soundness of $\mathbf{FRJ}(G)$. Moreover, we provides bounds on the height of $\mathbf{FRJ}(G)$-derivations and on the extracted countermodels.

Countermodels Given two sequents σ_1 and σ_2 of $\mathbf{FRJ}(G)$, $\sigma_1 \mapsto_0 \sigma_2$ iff σ_2 is the conclusion of a rule of $\mathbf{FRJ}(G)$ having σ_1 among the premises; thus, in forward computation, σ_2 is obtained from σ_1. By $\mapsto_*$ we denote the reflexive and transitive closure of $\mapsto_0$. We call *p-sequent* (*prime sequent*) of $\mathcal{D}$ any regular

$T \;=\; S \supset (\neg\neg p \supset p) \vee \neg\neg p \qquad S \;=\; H \supset \neg\neg p \vee \neg p \qquad H \;=\; (\neg\neg p \supset p) \supset \neg p \vee p$

$\mathrm{S\scriptsize{L}}(T) \;=\; \{\, S,\; \neg\neg p \supset p,\; \neg\neg p \vee \neg p,\; \neg\neg p,\; \neg p,\; p \,\}$

$\mathrm{S\scriptsize{R}}(T) \;=\; \{\, T,\; H,\; (\neg\neg p \supset p) \vee \neg\neg p,\; \neg\neg p \supset p,\; \neg\neg p,\; \neg p \vee p,\; \neg p,\; p,\; \bot \,\}$

(1)	$\cdot\,;\, p,\, S,\, \neg\neg p \supset p,\, \neg\neg p,\, \neg p \;\rightarrow\; \bot$	$\mathrm{Ax}_{\rightarrow}$
(2)	$\cdot\,;\, S,\, \neg\neg p \supset p,\, \neg\neg p,\, \neg p \;\rightarrow\; p$	$\mathrm{Ax}_{\rightarrow}$
(3)	$p\,;\, S,\, \neg\neg p \supset p,\, \neg\neg p,\, \neg p \;\rightarrow\; \neg p$	$\supset_{\in}$ (1)
(4)	$\neg p\,;\, p,\, S,\, \neg\neg p \supset p,\, \neg\neg p \;\rightarrow\; \neg\neg p$	$\supset_{\in}$ (1)
(5)	$\neg p,\, \neg\neg p \supset p \;\Rightarrow\; \bot$	$\bowtie^{\mathrm{At}}$ (2) (4)
(6)	$p,\, \neg\neg p \;\Rightarrow\; \bot$	$\bowtie^{\mathrm{At}}$ (3)
(7)	$\cdot\,;\, S,\, \neg\neg p \supset p \;\rightarrow\; \neg\neg p$	$\supset_{\notin}$ (5)
(8)	$\cdot\,;\, S,\, \neg\neg p \supset p,\, \neg\neg p \;\rightarrow\; \neg p$	$\supset_{\notin}$ (6)
(9)	$\cdot\,;\, S,\, \neg\neg p \supset p,\, \neg\neg p \;\rightarrow\; \neg p \vee p$	$\vee$ (2) (8)
(10)	$\neg\neg p \;\Rightarrow\; p$	$\bowtie^{\mathrm{At}}$ (8)
(11)	$\neg\neg p \supset p\,;\, S,\, \neg\neg p \;\rightarrow\; H$	$\supset_{\in}$ (9)
(12)	$\cdot\,;\, S \;\rightarrow\; \neg\neg p \supset p$	$\supset_{\notin}$ (10)
(13)	$\neg\neg p \supset p,\, S \;\Rightarrow\; \neg p \vee p$	$\bowtie^{\vee}$ (2) (7) (8) (11)
(14)	$\cdot\,;\, S \;\rightarrow\; H$	$\supset_{\notin}$ (13)
(15)	$S \;\Rightarrow\; (\neg\neg p \supset p) \vee \neg\neg p$	$\bowtie^{\vee}$ (7) (12) (14)
(16)	$S \;\Rightarrow\; T$	$\supset_{\in}$ (15)

Fig. 2. The **FRJ**(T)-derivation $\mathcal{D}_T$ of T.

sequent occurring in $\mathcal{D}$ which is either an axiom or the conclusion of a join rule. Given an **FRJ**(G)-derivation $\mathcal{D}$ of G let $\mathrm{Mod}(\mathcal{D}) \;=\; \langle\, \mathrm{P}(\mathcal{D}),\, \leq,\, \rho, V\,\rangle$ be the structure where:

– $\mathrm{P}(\mathcal{D})$ is the set of all p-sequents occurring in $\mathcal{D}$;
– for every $\sigma_1, \sigma_2 \in \mathrm{P}(\mathcal{D})$, $\sigma_1 \leq \sigma_2$ iff $\sigma_2 \mapsto_* \sigma_1$;
– ρ is the minimum of $\mathrm{P}(\mathcal{D})$ w.r.t. $\leq$;
– V maps $\sigma \in \mathrm{P}(\mathcal{D})$ to the set $V(\sigma) \;=\; \mathrm{Lhs}(\sigma) \cap \mathcal{V}$.

One can check that $\mathrm{Mod}(\mathcal{D})$ is a (Kripke) model. In particular, since the root sequent of $\mathcal{D}$ is regular, there exists $\rho \in \mathrm{P}(\mathcal{D})$ such that $\sigma_p \mapsto_* \rho$, for every $\sigma_p \in \mathrm{P}(\mathcal{D})$, hence ρ is the minimum of $\mathrm{P}(\mathcal{D})$ w.r.t. $\leq$. Moreover, by the structure of the rules, $\sigma_1 \leq \sigma_2$ implies $V(\sigma_1) \subseteq V(\sigma_2)$, hence the definition of V is sound. We call $\mathrm{Mod}(\mathcal{D})$ the *model extracted from* $\mathcal{D}$.

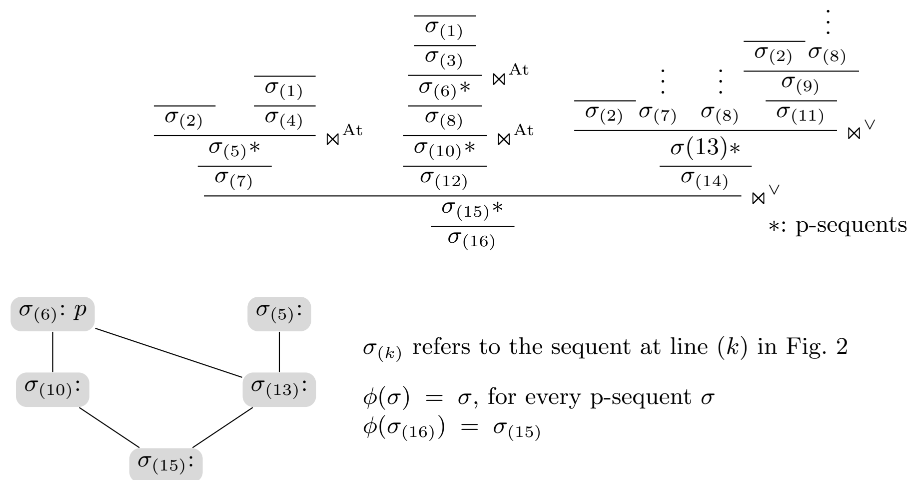

Fig. 3. The model $\mathrm{Mod}(\mathcal{D}_T)$ (see Fig. 2).

For every regular sequent σ occurring in $\mathcal{D}$, let $\phi(\sigma)$ be the p-sequent immediately above σ, namely:

$$\phi(\sigma) \;=\; \sigma_p \quad \text{iff} \quad \begin{array}{l} \sigma_p \in \mathrm{P}(\mathcal{D}) \text{ and } \sigma_p \mapsto_* \sigma \text{ and} \\ \text{for every } \sigma'_p \in \mathrm{P}(\mathcal{D}),\ \sigma_p \mapsto_* \sigma'_p \mapsto_* \sigma \text{ implies } \sigma'_p = \sigma_p. \end{array}$$

We call ϕ the *map associated with* $\mathcal{D}$. Soundness of $\mathbf{FRJ}(G)$ immediately follows by the following theorem:

Theorem 1 (Soundness). *Let $\mathcal{D}$ be an $\mathbf{FRJ}(G)$-derivation of G. Then,* $\mathrm{Mod}(\mathcal{D})$ *is a countermodel for G.* □

Example 2. The model $\mathrm{Mod}(\mathcal{D}_T)$ and the related map ϕ are shown in Fig. 3. The bottom world is the root and $\sigma < \sigma'$ iff the world σ is drawn below σ'. For each σ, we display the set $V(\sigma)$. In particular, $V(\sigma_{(6)}) = \{p\}$, while $V(\sigma_{(5)}) = V(\sigma_{(10)}) = V(\sigma_{(13)}) = V(\sigma_{(15)}) = \emptyset$. ◊

Bounds on derivations and countermodels Given a formula A, by $|A|$ we denote the number of symbols occurring in A. Let $\mathcal{D}$ be an $\mathbf{FRJ}(G)$-derivation of σ; the *height of* $\mathcal{D}$, denoted by $\mathrm{h}(\mathcal{D})$, is the length of the longest branch from σ to an axiom sequent of $\mathcal{D}$. Similarly, let $\mathcal{K}$ be a Kripke model with root ρ; the *height of* $\mathcal{K}$, denoted by $\mathrm{h}(\mathcal{K})$, is the length of the longest path from ρ to a final world of $\mathcal{K}$. The following theorem settles an upper bound on the height of $\mathbf{FRJ}(G)$-derivations and of extracted countermodels:

Theorem 2. *Let $\mathcal{D}$ be an $\mathbf{FRJ}(G)$-derivation and $N = |G|$. Then: (i)* $\mathrm{h}(\mathcal{D}) = O(N^2)$*; (ii)* $\mathrm{h}(\mathrm{Mod}(\mathcal{D})) \leq N$*.* □

Minimality Let $G \notin$ IPL; the *height* of G, denoted by $\mathrm{h}(G)$, is the minimum height of a countermodel for G; for instance, the formula T of Ex. 1 has height 2 ($\mathrm{h}(G) = 0$ iff G is not classically valid). We can build an $\mathbf{FRJ}(G)$-derivation $\mathcal{D}$ of G such that $\mathrm{Mod}(\mathcal{D})$ is a countermodel of G having minimal height. Formally:

Theorem 3. *Let $G \notin$ IPL. Then, there exists an $\mathbf{FRJ}(G)$-derivation $\tilde{\mathcal{D}}$ of G such that $\mathrm{h}(\mathrm{Mod}(\tilde{\mathcal{D}})) = \mathrm{h}(G)$.* □

Given two sequents σ_1 and σ_2, we say that σ_2 *subsumes* σ_1, and we write $\sigma_1 \sqsubseteq \sigma_2$, iff one of the following conditions hold:

(1) $\sigma_1 = \Gamma_1 \Rightarrow C$ and $\sigma_2 = \Gamma_2 \Rightarrow C$ and $\Gamma_1 \subseteq \Gamma_2$;
(2) $\sigma_1 = \Sigma\,;\, \Theta_1 \to C$ and $\sigma_2 = \Sigma\,;\, \Theta_2 \to C$ and $\Theta_1 \subseteq \Theta_2$.

We can modify the forward proof-search procedure presented above so to generate $\mathbf{FRJ}(G)$-derivations of minimal height. This is obtained by implementing forward subsumption (if at some step σ is proved and σ is subsumed by a sequent already in $\mathtt{D}_G$, then σ is discarded and not added to $\mathtt{D}_G$) and delaying the application of join rules as much as possible. We remark that the above theorem provides a semantic proof of completeness of $\mathbf{FRJ}(G)$-derivations; an alternative proof is discussed in the next section.

3 Saturated databases and completeness of FRJ(G)

Now, we introduce the key notion of saturated database. Let G be a formula and let $\mathtt{D}_G$ be a set of $\mathbf{FRJ}(G)$-sequents: $\mathtt{D}_G$ is a *database for* G iff, for every $\sigma \in \mathtt{D}_G$, σ is provable in $\mathbf{FRJ}(G)$. A database $\mathtt{D}_G$ for G is *saturated* iff, for every sequent σ provable in $\mathbf{FRJ}(G)$, there exists $\sigma' \in \mathtt{D}_G$ such that $\sigma \sqsubseteq \sigma'$.

Let **G3i** be the well-known sequent calculus for IPL described in [12]. We can prove the following result:

Theorem 4. *Let $G \notin$ IPL. Then, the forward proof-search procedure for $\mathbf{FRJ}(G)$ returns a saturated database $\mathtt{D}_G$ from which a **G3i**-derivation of G can be extracted.* □

The crucial point about the proof of the above result is that the derivation of G in **G3i** can be constructed by means of a backward proof search procedure that does not require backtracking: in presence of multiple non-deterministic choices, we exploit $\mathtt{D}_G$ to select the right way so to successfully continue proof-search.

As a future work we plan to investigate the applicability of our method to other logics, in particular to modal logics such as **S4** and intermediate logics such as Gödel-Dummett logic characterized by linear Kripke models.

References

1. T. Brock-Nannestad and K. Chaudhuri. Disproving using the inverse method by iterative refinement of finite approximations. In H. De Nivelle, editor, *TABLEAUX 2015*, volume 9323 of *LNCS*, pages 153–168. Springer, 2015.
2. A. Chagrov and M. Zakharyaschev. *Modal Logic*. Oxford University Press, 1997.
3. K. Chaudhuri and F. Pfenning. A focusing inverse method theorem prover for first-order linear logic. In R. Nieuwenhuis, editor, *CADE-20*, volume 3632 of *LNCS*, pages 69–83. Springer, 2005.
4. K. Chaudhuri, F. Pfenning, and G. Price. A logical characterization of forward and backward chaining in the inverse method. In U. Furbach et al., editor, *IJCAR 2006*, volume 4130 of *LNCS*, pages 97–111. Springer, 2006.
5. A. Degtyarev and A. Voronkov. The inverse method. In J.A. Robinson et al., editor, *Handbook of Automated Reasoning*, pages 179–272. Elsevier and MIT Press, 2001.
6. K. Donnelly, T. Gibson, N. Krishnaswami, S. Magill, and S. Park. The inverse method for the logic of bunched implications. In F. Baader et al., editor, *LPAR 2004*, volume 3452 of *LNCS*, pages 466–480. Springer, 2004.
7. C. Fiorentini and M. Ferrari. A forward unprovability calculus for intuitionistic propositional logic. In R. A. Schmidt and C. Nalon, editors, *TABLEAUX 2017*, volume 10501 of *LNCS*, pages 114–130. Springer, 2017.
8. C. Fiorentini and M. Ferrari. Duality between unprovability and provability in forward proof-search for intuitionistic propositional logic. *CoRR*, arXiv:1804.06689, 2018.
9. L. Kovács, A. Mantsivoda, and A. Voronkov. The inverse method for many-valued logics. In F. Castro-Espinoza et al., editor, *MICAI 2013*, volume 8265 of *LNCS*, pages 12–23. Springer, 2013.
10. S. Ju. Maslov. An invertible sequential version of the constructive predicate calculus. *Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)*, 4:96–111, 1967.
11. S. McLaughlin and F. Pfenning. Imogen: Focusing the polarized inverse method for intuitionistic propositional logic. In I. Cervesato et al., editor, *LPAR 2008*, volume 5330 of *LNCS*, pages 174–181. Springer, 2008.
12. A.S. Troelstra and H. Schwichtenberg. *Basic Proof Theory*, volume 43 of *Cambridge Tracts in Theoretical Computer Science*. Camb. Univ. Press, 2ed edition, 2000.