Toward intuitionistic non-normal modal logic and its calculi*

Tiziano Dalmonte, Charles Grellois, Nicola Olivetti

Aix Marseille Univ, Université de Toulon, CNRS, LIS, Marseille, France

Abstract

We propose a minimal non-normal intuitionistic modal logic. We present it by Hilbert axiomatisation and an equivalent cut-free calculus. We then propose a semantic characterisation of this logic in terms of bi-neighbourhood models with respect to which the logic is sound. We finally present a cut-free labelled calculus matching with the bi-neighbourhood semantics.

1 Introduction

Both Intuitionistic modal logic and non-normal modal logic have a strong tradition. Concerning the former we can very schematically identify two traditions: so-called Intuitionistic modal logics and Constructive modal logics. Intuitionistic modal logics have been extensively investigated by Simpson [11], whose main goal is to define analogues of classical modalities justified from an intuitionistic point of view. Constructive modal logics are mainly motivated by their type-theoretic interpretations (Curry–Howard correspondence, typed lambda calculus) [9], but also by other applications to computer science, such as verification. A full account of the literature on the subject is beyond the scope of othe present work.

On the other hand Non-Normal modal logics have been strongly motivated on a philosophical and epistemic ground. They are called "Non-normal" as they do not satisfy all the axioms and rules of the minimal normal modal logic K. They have been studied since the seminal work of Scott, Lemmon, and Chellas and can be seen as generalisations of standard modal logics. They have found an interest in several areas such as epistemic and deontic reasoning, reasoning about games, and reasoning about probabilistic notions such as 'truth in most of the cases'.

It can be noticed that some intuitionistic or constructive modal logics investigated in the literature contain non-normal modalities. The prominent example is the logic proposed by Wijesekera [13], its propositional fragment has been recently investigated by Kojima [6]. This logic has a normal $\Box$ modality and a non-normal $\Diamond$ modality, where $\Diamond$ does not distribute on the $\vee$, that is $\Diamond(A \vee B)$ is not equivalent to $\Diamond A \vee \Diamond B$. The original motivation by Wijesekera comes

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from Constructive Concurrent Dynamic Logic, but the logic has also an interesting epistemic interpretation in terms of internal/external observers proposed by Kojima.

However, to the best of our knowledge and rather surprisingly, there is no investigation of non-normal modalities with an intuitionistic base. We wish to develop a theory of non-normal modal logics, analogous to the classical one, but with an intuitionistic, whence a constructive base. Here we begin to tackle this issue. The question we aim at answering is: what is the minimal extension of intuitionistic logic which defines a non-normal modal logic?

We propose a possible answer by defining the minimal logic EI. This logic is obtained by adding two independent modalities $\Box$ and $\Diamond$ to intuitionistic logic, with the weakest connection between the two. We first present the logic EI syntactically by its Hilbert axiomatisation, and then provide an equivalent cut-free internal calculus for it. We then tackle the problem of a semantic characterisation of this logic. To this purpose we introduce a particular kind of neighbourhood models, called intuitionistic bi-neighbourhood models (IBNM). Neighbourhood models are the standard semantics for classical non-normal modal logics. Moreover a neighbourhood semantics has been proposed by Kojima to characterise the propositional fragment of Wijesekera's logic. IBNM models are neighbourhood models based on intuitionistic frames, but as a difference with standard neighbourhood models (including Kojima's ones), in these models every world is equipped with a set of pairs of neighbourhoods: the two components of a pair give, so to say, independent "positive" and "negative" support for the truth of a modal formula. As a preliminary result, we prove that the logic EI is sound with respect to validity in IBNM models, and its mere $\Box$-fragment is also complete with respect to it. We conjecture that completeness holds for the full language including $\Diamond$, but we have not proved it yet. We then define a labelled sequent calculus for validity in IBNM models, we show that the calculus admits cut-elimination and is sound and complete with respect to IBNM models. We finally discuss several open issues and extensions of the proposed framework.

2 Axiomatization and internal calculus

In this section, we introduce a Hilbert axiomatization for the logic **EI**. After that we propose an equivalent cut-free internal calculus for **EI**.

We consider a propositional language $\mathcal{L}$ extendend with two unary operators $\Box$ and $\Diamond$ for the modalities. As usual $\neg A$ is defined as $A \to \bot$.

Definition 2.1 (Hilbert axiomatization $\mathbf{H_{EI}}$)**.** The system $\mathbf{H_{EI}}$ is based on the intuitionistic propositional calculus, extended with axioms and rules for the modalities.

Propositional axioms and rules: $\bot \to A$ $\quad A \to (B \to A)$

$(A \to (B \to C)) \to ((A \to B) \to (A \to C))$ $\quad A \land B \to A$ $\quad A \land B \to B$

$A \to A \lor B$ $\quad B \to A \lor B$ $\quad (A \to C) \to ((B \to C) \to (A \lor B \to C))$

$A \to (B \to (A \land B))$

$$\frac{A \qquad A \to B}{B}\ \text{MP}$$

Modal axioms:

$$\Box\top \to \neg\Diamond\bot \qquad\qquad \Diamond\top \to \neg\Box\bot$$

Modal rules:

$$\frac{A \leftrightarrow B}{\Box A \leftrightarrow \Box B}\ \mathrm{RE}_\Box \qquad\qquad \frac{A \leftrightarrow B}{\Diamond A \leftrightarrow \Diamond B}\ \mathrm{RE}_\Diamond$$

Let us then define the internal sequent calculus **S-Int$_{\mathbf{EI}}$**.

Definition 2.2 (Sequent calculus **S-Int$_{\mathbf{EI}}$**)**.** We base our internal calculus **S-Int$_{\mathbf{EI}}$** on the intuitionistic sequent calculus **G3i**, that we extend with rules for handling our modalities. The resulting calculus is the following:

Propositional rules of G3i:

$$\frac{}{\Gamma, A \Rightarrow A}\ \mathrm{Ax} \qquad \frac{}{\Gamma, \bot \Rightarrow A}\ \mathrm{L}\bot \qquad \frac{}{\Gamma \Rightarrow A, \top}\ \mathrm{R}\top$$

$$\frac{\Gamma, A, B \Rightarrow C}{\Gamma, A \wedge B \Rightarrow C}\ \mathrm{L}\wedge \qquad \frac{\Gamma \Rightarrow A \qquad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B}\ \mathrm{R}\wedge \qquad \frac{\Gamma, A \Rightarrow C \qquad \Gamma, B \Rightarrow C}{\Gamma, A \vee B \Rightarrow C}\ \mathrm{L}\vee$$

$$\frac{\Gamma \Rightarrow A_i}{\Gamma \Rightarrow A_1 \vee A_2}\ \mathrm{R}\vee \qquad \frac{\Gamma, A \to B \Rightarrow A \qquad \Gamma, B \Rightarrow C}{\Gamma, A \to B \Rightarrow C}\ \mathrm{L}\to \qquad \frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \to B}\ \mathrm{R}\to$$

Rules for the modalities:

$$\frac{A \Rightarrow B \qquad B \Rightarrow A}{\Gamma, \Box A \Rightarrow \Box B}\ \mathrm{C}_\Box \qquad\qquad \frac{A \Rightarrow B \qquad B \Rightarrow A}{\Gamma, \Diamond A \Rightarrow \Diamond B}\ \mathrm{C}_\Diamond$$

$$\frac{A \Rightarrow \qquad \Rightarrow B}{\Gamma, \Diamond A, \Box B \Rightarrow C}\ \Box\Diamond_1 \qquad\qquad \frac{A \Rightarrow \qquad \Rightarrow B}{\Gamma, \Box A, \Diamond B \Rightarrow C}\ \Box\Diamond_2$$

The Cut rule is the following: $$\frac{\Gamma \Rightarrow A \qquad \Gamma, A \Rightarrow B}{\Gamma \Rightarrow B}\ \mathrm{Cut}$$

We now show that Cut is admissible in the calculus.

Theorem 2.1. The Cut rule is admissible in **S-Int$_{\mathbf{EI}}$**.

Proof. The proof follows the usual inductive structure. We only treat the case of modal rules, since the proof of admissibility of Cut for the core intuitionistic calculus **G3i** is standard. As usual, we focus on the cases in which the cut formula is principal in the last rules of the derivation of both premisses.

1. $\mathrm{C}_\Box$ vs. $\mathrm{C}_\Box$: the situation is as follows:

$$\mathrm{Cut}\ \frac{\dfrac{A \Rightarrow B \qquad B \Rightarrow A}{\Gamma, \Box A \Rightarrow \Box B} \qquad \dfrac{B \Rightarrow C \qquad C \Rightarrow B}{\Gamma, \Box A, \Box B \Rightarrow \Box C}}{\Gamma, \Box A \Rightarrow \Box C}$$

   and is transformed into:

$$\frac{\mathrm{Cut}\ \dfrac{A \Rightarrow B \qquad B \Rightarrow C}{A \Rightarrow C} \qquad \dfrac{C \Rightarrow B \qquad B \Rightarrow A}{C \Rightarrow A}\ \mathrm{Cut}}{\Gamma, \Box A \Rightarrow \Box C}$$

2. $\mathrm{C}_\Box$ vs. $\Box\Diamond_1$: we have:

$$\mathrm{Cut}\ \frac{\dfrac{A \Rightarrow B \qquad B \Rightarrow A}{\Gamma, \Diamond C, \Box A \Rightarrow \Box B} \qquad \dfrac{C \Rightarrow \qquad \Rightarrow B}{\Gamma, \Box A, \Diamond C, \Box B \Rightarrow D}}{\Gamma, \Diamond C, \Box A \Rightarrow D}$$

   which becomes:

$$\frac{C \Rightarrow \qquad \dfrac{\Rightarrow B \qquad B \Rightarrow A}{\Rightarrow A}\ \mathrm{Cut}}{\Gamma, \Diamond C, \Box A \Rightarrow D}$$

3. $C_\Diamond$ vs. $\Box\Diamond_1$: we have:

$$\text{Cut}\ \dfrac{\dfrac{A \Rightarrow B \qquad B \Rightarrow A}{\Gamma, \Box C, \Diamond A \Rightarrow \Diamond B} \qquad \dfrac{B \Rightarrow \qquad \Rightarrow C}{\Gamma, \Diamond A, \Diamond B, \Box C \Rightarrow D}}{\Gamma, \Box C, \Diamond A \Rightarrow D}$$

which becomes:

$$\dfrac{\text{Cut}\ \dfrac{A \Rightarrow B \qquad B \Rightarrow}{A \Rightarrow} \qquad \Rightarrow C}{\Gamma, \Box C, \Diamond A \Rightarrow D}$$

□

Theorem 2.2. A is derivable in $\mathbf{H_{EI}}$ iff $\Rightarrow A$ is derivable in $\mathbf{S\text{-}Int_{EI}}$.

Proof. From left to right, we show that if $\Gamma \Rightarrow A$ is derivable in $\mathbf{S\text{-}Int_{EI}}$, then $\bigwedge \Gamma \rightarrow A$ is derivable in $\mathbf{H_{EI}}$. It suffices to show that the rules of the internal calculus are derivable in the axiomatization. The rules $C_\Box$ and $C_\Diamond$ are directly obtained from $RE_\Box$ and $RE_\Diamond$.

($\Box\Diamond_1$) Assume $\vdash \neg A$ and $\vdash B$. Then $\vdash A \leftrightarrow \bot$ and $\vdash B \leftrightarrow \top$. By $RE_\Diamond$ and $RE_\Box$, $\vdash \Diamond A \leftrightarrow \Diamond\bot$ and $\vdash \Box B \leftrightarrow \Box\top$. By the axiom $\Box\top \rightarrow \neg\Diamond\bot$, we obtain $\vdash \Box B \rightarrow \neg\Diamond\bot$. Thus $\vdash \Diamond A \wedge \Box B \rightarrow \Diamond\bot \wedge \neg\Diamond\bot$, and finally $\vdash \Diamond A \wedge \Box B \rightarrow \bot$.

($\Box\Diamond_2$) Suppose $\vdash \neg A$ and $\vdash B$. Then $\vdash A \leftrightarrow \bot$ and $\vdash B \leftrightarrow \top$. By $RE_\Box$ and $RE_\Diamond$, $\vdash \Box A \leftrightarrow \Box\bot$ and $\vdash \Diamond B \leftrightarrow \Diamond\top$. By axiom $\Diamond\top \rightarrow \neg\Box\bot$ we obtain $\vdash \Diamond B \rightarrow \neg\Box\bot$. Thus $\vdash \Box A \wedge \Diamond B \rightarrow \Box\bot \wedge \neg\Box\bot$, and finally $\vdash \Box A \wedge \Diamond B \rightarrow \bot$.

From right to left, we show that if A is derivable in $\mathbf{H_{EI}}$, then the sequent $\Rightarrow A$ is derivable in $\mathbf{S\text{-}Int_{EI}}$. It suffices to show that the axioms and rules of $\mathbf{H_{EI}}$ are derivable or admissible in $\mathbf{S\text{-}Int_{EI}}$. The axioms $\Box\top \rightarrow \neg\Diamond\bot$ and $\Box\bot \rightarrow \neg\Diamond\top$ are obtained through the following derivations:

$$\dfrac{\dfrac{\dfrac{\dfrac{\bot \Rightarrow \qquad \Rightarrow \top}{\Box\top, \Diamond\bot \Rightarrow \bot}}{\Box\top \Rightarrow \Diamond\bot \rightarrow \bot}}{\Box\top \Rightarrow \neg\Diamond\bot}}{\Rightarrow \Box\top \rightarrow \neg\Diamond\bot} \qquad\qquad \dfrac{\dfrac{\dfrac{\dfrac{\bot \Rightarrow \qquad \Rightarrow \top}{\Diamond\top, \Box\bot \Rightarrow \bot}}{\Diamond\top \Rightarrow \Box\bot \rightarrow \bot}}{\Diamond\top \Rightarrow \neg\Box\bot}}{\Rightarrow \Diamond\top \rightarrow \neg\Box\bot}$$

The rules $RE_\Box$ and $RE_\Diamond$ are obtained directly from $C_\Box$ and $C_\Diamond$. Finally, the Modus Ponens rule is treated as usual using Cut, whence it is admissible in $\mathbf{S\text{-}Int_{EI}}$. □

3 A possible semantics for $\mathbf{H_{EI}}$

In this section we introduce a possible semantic interpretation of system $\mathbf{H_{EI}}$. This semantics is defined in terms of *intuitionistic bi-neighbourhood models* (IBNMs). IBNMs are neighbourhood models, in which to each world is associated a set of pairs of neighbourhood, each neighbourhood being a set of worlds itself. This kind of model is similar to the models for classical non-normal modal logics introduced in [2].

Definition 3.1. An *intuitionistic bi-neighbourhood model* is a tuple $\mathcal{M} = \langle W, \leq, \mathcal{N}, V \rangle$ where W is a non-empty set; $\leq$ is a preorder over W; $\mathcal{N}$ is a bi-neighbourhood function, *i.e.* for all $w \in W$, $\mathcal{N}(w) \subseteq \mathcal{P}(W) \times \mathcal{P}(W)$ s.t.

$(\alpha, \beta) \in \mathcal{N}(w)$ implies $\alpha \cap \beta = \emptyset$; and V is a valuation function $W \longrightarrow \mathcal{P}(At)$. Moreover we assume the following properties: for all $w, v \in W$,

$$w \leq v \text{ implies } V(w) \subseteq V(v) \qquad (V\text{-property})$$
$$w \leq v \text{ implies } \mathcal{N}(v) \subseteq \mathcal{N}(w) \qquad (\mathcal{N}\text{-property})$$

The forcing relation $\mathcal{M}, w \Vdash A$ associated with this semantics is the following:

$\mathcal{M}, w \Vdash p$	iff	$p \in V(w)$;
$\mathcal{M}, w \nVdash \bot$;		
$\mathcal{M}, w \Vdash A \wedge B$	iff	$\mathcal{M}, w \Vdash A$ and $\mathcal{M}, w \Vdash B$;
$\mathcal{M}, w \Vdash A \vee B$	iff	$\mathcal{M}, w \Vdash A$ or $\mathcal{M}, w \Vdash B$;
$\mathcal{M}, w \Vdash A \to B$	iff	for all $v \in W$, $w \leq v$ and $\mathcal{M}, v \Vdash A$ implies $\mathcal{M}, v \Vdash B$;
$\mathcal{M}, w \Vdash \Box A$	iff	for all $v \in W$, if $w \leq v$, then there is $(\alpha, \beta) \in \mathcal{N}(v)$ s.t. for all $u \in \alpha$, $\mathcal{M}, u \Vdash A$ and for all $z \in \beta$, $\mathcal{M}, z \nVdash A$.
$\mathcal{M}, w \Vdash \Diamond A$	iff	for all $(\alpha, \beta) \in \mathcal{N}(w)$, there is $u \in \alpha$ s.t. $\mathcal{M}, u \Vdash A$ or there is $v \in \beta$ s.t. $\mathcal{M}, v \nVdash A$.

Proposition 3.1. The relation $\Vdash$ of Def. 3.1 enjoys the monotonicity property: for all $w, v \in W$ and any $A \in \mathcal{L}$, $w \leq v$ and $\mathcal{M}, w \Vdash A$ implies $\mathcal{M}, v \Vdash A$.

Proof. The proof is immediate by induction on the complexity of A. □

Theorem 3.2. If A is derivable in $\mathbf{H_{EI}}$, then it is valid in IBNMs.

Proof. One can prove that the axioms of $\mathbf{H_{EI}}$ are valid and that the rules preserve the validity. As an example we prove the validity of axiom $\Box\top \to \neg\Diamond\bot$. To this purpose, suppose $\mathcal{M}, w \nVdash \Box\top \to \neg\Diamond\bot$. Then there is $v \geq w$ s.t. $\mathcal{M}, v \Vdash \Box\top$ and $\mathcal{M}, v \nVdash \neg\Diamond\bot$. By $\mathcal{M}, v \nVdash \neg\Diamond\bot$, there is $z \geq v$ s.t. $\mathcal{M}, z \Vdash \Diamond\bot$, *i.e.* for all $(\alpha, \beta) \in \mathcal{N}(z)$ there is $u \in \alpha$, $\mathcal{M}, u \Vdash \bot$ or there is $u \in \beta$, $\mathcal{M}, u \nVdash \bot$. But $\mathcal{M}, u \Vdash \bot$ is impossible, thus we obtain $\beta \neq \emptyset$. Now, by $\mathcal{M}, v \Vdash \Box\top$, we have in particular that there is $(\gamma, \delta) \in \mathcal{N}(z)$ s.t. for all $x \in \gamma$, $\mathcal{M}, x \Vdash \top$ and for all $x \in \delta$, $\mathcal{M}, x \nVdash \top$. Then, necessarily, $\delta = \emptyset$, which gives a contradiction. □

We strongly conjecture that the logic is also complete w.r.t. IBNM. For the time being we give an (easy) proof for its $\Box$-fragment.

A set X of formulas of $\mathcal{L}$ is called *prime* if it is consistent ($X \nvdash \bot$), closed under derivation ($X \vdash A$ implies $A \in X$) and such that if $(A \vee B) \in X$, then $A \in X$ or $B \in X$. In the following we denote with $\uparrow A$ the class of prime sets X such that $A \in X$.

Lemma 3.3. (i) Any consistent set is included in a prime set.
(ii) For any $C, D \in \mathcal{L}$, $\uparrow C = \uparrow D$ implies $\vdash C \leftrightarrow D$.

Lemma 3.4. Let the *canonical model* $\mathcal{M} = \langle W, \leq, \mathcal{N}, V \rangle$ for **EI** be defined as follows: W is the class of all prime sets of formulas of $\mathcal{L}$; for all $X, Y \in W$, $X \leq Y$ iff $X \subseteq Y$; $p \in V(X)$ iff $p \in X$; and

$$\mathcal{N}(X) = \{(\uparrow C, W \setminus \uparrow C) : \Box C \in X\}.$$

It follows immediately from the definition that $X \leq Y$ implies $V(X) \subseteq V(Y)$. By induction on the complexity of formulas one can prove that for all $X \in W$ and all $A \in \mathcal{L}$,

$$\mathcal{M}, X \Vdash A \text{ iff } A \in X.$$

Proof. We just consider the case in which $A \equiv \Box D$. Assume $\mathcal{M}, X \Vdash \Box D$, then in particular there is $(\alpha, \beta) \in \mathcal{N}(X)$ such that $\mathcal{M}, u \Vdash D$ for all $u \in \alpha$ and

$\mathcal{M}, v \not\Vdash D$ for all $v \in \beta$. By definition we have $\alpha = \uparrow C$, $\beta = W \setminus \uparrow C$ and $\Box C \in X$ for a formula C. Then by i.h. $\uparrow C \subseteq \uparrow D$ and $W \setminus \uparrow C \subseteq W \setminus \uparrow D$, *i.e.* $\uparrow C = \uparrow D$. By the properties of prime sets we then have $\vdash C \leftrightarrow D$, and by $\text{RE}_\Box$ and closure under derivation, $\Box D \in X$.

Now assume $\Box D \in X$. Then for any $Y \supseteq X$, $\Box D \in Y$. By definition, $(\uparrow D, W \setminus \uparrow D) \in \mathcal{N}(Y)$, and by i.h. we have that for all $u \in \uparrow D$, $\mathcal{M}, u \Vdash D$, and for all $v \in W \setminus \uparrow D$, $\mathcal{M}, v \not\Vdash D$. Thus $\mathcal{M}, X \Vdash \Box D$. □

As said above, we strongly conjecture that the proof can be extended to the full language including $\Diamond$.

4 A labelled calculus for IBNM semantics

We don't know yet if $\mathbf{S\text{-}Int_{EI}}$ is complete w.r.t. IBNM semantics. Lemma 3.4 implies that it is complete for the $\Box$-fragment. On the other hand, we can develop a labelled calculus matching the IBNM semantics.

The labelled calculus $\mathbf{S\text{-}Lab_{EI}}$ is based on the enriched language $\mathcal{L}_{\text{LS}}$. First we denote with $\mathcal{L}_{+\Box_l}$ the extension of language $\mathcal{L}$ with the unary operator $\Box_l$ (if A is a formula, then $\Box_l A$ is a formula). Then we consider two denumerable sets $\mathbb{WL} = \{x, y, z, ...\}$ and $\mathbb{NL} = \{a, \overline{a}, b, \overline{b}, ...\}$ of *world labels*, respectively *neighbourhood labels*. For any label $a \in \mathbb{NL}$ we also consider ist negative counterpart $\overline{a}$. The labelled formulas of $\mathcal{L}_{\text{LS}}$ are then of the following kinds, where A is any formula of $\mathcal{L}_{+\Box_l}$:

$$x \leq y,\ x : a,\ x : \overline{a},\ a : x,\ a \Vdash^{\forall} A,\ a \Vdash^{\exists} A,\ \overline{a} \Vdash^{\forall} A,\ \overline{a} \Vdash^{\exists} A,\ x : A.$$

Sequents are defined as usual as pairs $\Gamma \Rightarrow \Delta$ of multisets of formulas of $\mathcal{L}_{\text{LS}}$. In addition we stipulate that formulas of the kinds $x \leq y$, $x : a$, $x : \overline{a}$ and $a : x$ don't occur in Δ. We can now introduce the rules of $\mathbf{S\text{-}Lab_{EI}}$, which are the following:

Initial sequents:

$$(Init_x)\ \ x : p, \Gamma \Rightarrow \Delta, x : p \qquad (Init_a)\ \ x : a, x : \overline{a}, \Gamma \Rightarrow \Delta$$

$$(Init_\bot)\ \ x : \bot, \Gamma \Rightarrow \Delta \qquad (Init_\top)\ \ \Gamma \Rightarrow \Delta, x : \top$$

Prop. rules:

$$\frac{x : A, \Gamma \Rightarrow \Delta \qquad x : B, \Gamma \Rightarrow \Delta}{x : A \vee B, \Gamma \Rightarrow \Delta}\ L\vee \qquad \frac{\Gamma \Rightarrow \Delta, x : A \qquad \Gamma \Rightarrow \Delta, x : B}{\Gamma \Rightarrow \Delta, x : A \wedge B}\ R\wedge$$

$$\frac{x : A, x : B, \Gamma \Rightarrow \Delta}{x : A \wedge B, \Gamma \Rightarrow \Delta}\ L\wedge \qquad \frac{\Gamma \Rightarrow \Delta, x : A, x : B}{\Gamma \Rightarrow \Delta, x : A \vee B}\ R\vee \qquad \frac{x \leq y, y : A, \Gamma \Rightarrow \Delta, y : B}{\Gamma \Rightarrow \Delta, x : A \to B}\ R \to$$

$$\frac{x \leq y, x : A \to B, \Gamma \Rightarrow \Delta, y : A \qquad x \leq y, x : A \to B, y : B, \Gamma \Rightarrow \Delta}{x \leq y, x : A \to B, \Gamma \Rightarrow \Delta}\ L \to$$

(with y new in $R \to$)

Local forcing:

$$\frac{x \in a, a \Vdash^{\forall} A, x : A, \Gamma \Rightarrow \Delta}{x \in a, a \Vdash^{\forall} A, \Gamma \Rightarrow \Delta}\ L \Vdash^{\forall} \qquad \frac{x \in a, \Gamma \Rightarrow \Delta, x : A, a \Vdash^{\exists} A}{x \in a, \Gamma \Rightarrow \Delta, a \Vdash^{\exists} A}\ R \Vdash^{\exists}$$

$$\frac{x \in a, x : A, \Gamma \Rightarrow \Delta}{a \Vdash^{\exists} A, \Gamma \Rightarrow \Delta}\ L \Vdash^{\exists} \qquad \frac{x \in a, \Gamma \Rightarrow \Delta, x : A}{\Gamma \Rightarrow \Delta, a \Vdash^{\forall} A}\ R \Vdash^{\forall}$$

(with x new in $R \Vdash^{\forall}$ and $L \Vdash^{\exists}$)

Rules for $\leq$:

$$\frac{x \leq x, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ Ref_{\leq} \qquad \frac{x \leq y, y \leq z, x \leq z, \Gamma \Rightarrow \Delta}{x \leq y, y \leq z, \Gamma \Rightarrow \Delta}\ Tr_{\leq}$$

Rules for $\Box$ and $\Diamond$:

$$\frac{x \leq y, x : \Box A, y : \Box_l A, \Gamma \Rightarrow \Delta}{x \leq y, x : \Box A, \Gamma \Rightarrow \Delta}\ L\Box \qquad \frac{x \leq y, \Gamma \Rightarrow \Delta, y : \Box_l A}{\Gamma \Rightarrow \Delta, x : \Box A}\ R\Box$$

$$\frac{a : x, \Gamma \Rightarrow \Delta, x : \Box_l A, a \Vdash^{\forall} A \qquad a : x, \overline{a} \Vdash^{\exists} A, \Gamma \Rightarrow \Delta, x : \Box_l A}{a : x, \Gamma \Rightarrow \Delta, x : \Box_l A} R\Box_l$$

$$\frac{b : x, b \Vdash^{\forall} A, \Gamma \Rightarrow \Delta, \overline{b} \Vdash^{\exists} A}{x : \Box_l A, \Gamma \Rightarrow \Delta} L\Box_l \qquad \frac{b : x, \overline{b} \Vdash^{\forall} A, \Gamma \Rightarrow \Delta, b \Vdash^{\exists} A}{\Gamma \Rightarrow \Delta, x : \Diamond A} R\Diamond$$

$$\frac{a : x, x : \Diamond A, a \Vdash^{\exists} A, \Gamma \Rightarrow \Delta \qquad a : x, x : \Diamond A, \Gamma \Rightarrow \Delta, \overline{a} \Vdash^{\forall} A}{a : x, x : \Diamond A, \Gamma \Rightarrow \Delta} L\Diamond$$

(with b new in $L\Box_l$ and $R\Diamond$ and y new in $R\Box$)

Rules for V and $\mathcal{N}$: $\dfrac{x \leq y, x : p, y : p, \Gamma \Rightarrow \Delta}{x \leq y, x : p, \Gamma \Rightarrow \Delta} Val \qquad \dfrac{x \leq y, a : y, a : x, \Gamma \Rightarrow \Delta}{x \leq y, a : y, \Gamma \Rightarrow \Delta} Neig$

As an example we show the derivation of axiom $\Box\top \rightarrow \neg\Diamond\bot$ in $\mathbf{S\text{-}Lab_{EI}}$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{L \Vdash^{\exists}\dfrac{u : a, u : \bot, a : z, z : \Diamond\bot ... \Rightarrow ...}{a \Vdash^{\exists} \bot, a : z, z : \Diamond\bot ... \Rightarrow ...} \qquad \dfrac{\dfrac{u : \overline{a}, a : z, z : \Diamond\bot ... \Rightarrow \overline{a} \Vdash^{\exists} \top, u : \bot, u : \top}{u : \overline{a}, a : z, z : \Diamond\bot ... \Rightarrow \overline{a} \Vdash^{\exists} \top, u : \bot} R \Vdash^{\exists}}{...a : z, z : \Diamond\bot \Rightarrow \overline{a} \Vdash^{\exists} \top, \overline{a} \Vdash^{\forall} \bot} R \Vdash^{\forall}}{x \leq y, y : \Box\top, y \leq z, a : z, a \Vdash^{\forall} \top, z : \Diamond\bot, \Rightarrow \overline{a} \Vdash^{\exists} \top} L\Diamond}{x \leq y, y : \Box\top, y \leq z, z : \Box_l\top, z : \Diamond\bot, \Rightarrow} L\Box_l}{x \leq y, y : \Box\top, y \leq z, z : \Diamond\bot, \Rightarrow} L\Box}{x \leq y, y : \Box\top \Rightarrow y : \neg\Diamond\bot} R\rightarrow}{\Rightarrow x : \Box\top \rightarrow \neg\Diamond\bot} R\rightarrow$$

We can show that $\mathbf{S\text{-}Lab_{EI}}$ has the standard proof-theoretical structural properties, of which the most important is the admissibility of the Cut rule.

Definition 4.1. The weight of a labelled formula ϕ is a pair (w_f, w_l), where w_f and w_l are the weights of, respectively, the logical formula and the label occuring in ϕ, which are inductively defined as follows:

$\cdot$ $w(\bot) = 0$; $w(p) = 1$; $w(A \circ B) = w(A) + w(B) + 1$; $w(\Box_l A) = w(A) + 1$; $w(\Box A) = w(\Diamond A) = w(A) + 2$.

$\cdot$ $w(x) = 0$; $w(a) = 1$.

To formulas of the kinds $x \leq y$, $x : a$, $x : \overline{a}$ and $a : x$ we assign the weight $(0, 0)$. Weights of labelled formulas are ordered lexicographically.

Moreover we consider the following substitution function for labels:

$$x(y/z) = \begin{cases} y & \text{if } z = x \\ x & \text{if } z \neq x \end{cases} \qquad a(b/c) = \begin{cases} b & \text{if } c = a \\ a & \text{if } c \neq a \end{cases} \qquad \overline{a}(b/c) = \overline{a(b/c)}$$

Then the calculus $\mathbf{S\text{-}Lab_{EI}}$ enjoys the following properties:

Proposition 4.1. Left and right weakening, left and right contraction and labels substitution are height-preserving admissible (*hp*-admissible) and all rules are *hp*-invertible.

Theorem 4.2. The *cut* rule $\dfrac{\Gamma \Rightarrow \Delta, \phi \qquad \phi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} cut$ is admissible in $\mathbf{S\text{-}Lab_{EI}}$.

Proof. By primary induction on the weight of the cut formula and subinduction on the cut height it is possible to show how to remove any application of *cut*. We recall that the cut height is defined as the sum of the heights of the derivations of the premisses of *cut*. We just consider some significant cases, in which the cut formula is principal in the last rule applied in the derivation of both premisses of *cut*.

- The cut formula is $x : \Box A$) The case is as follows:

$$
\dfrac{R\Box\ \dfrac{\dfrac{\mathcal{D}}{x \leq y, x \leq z, \Gamma \Rightarrow \Delta, z : \Box_l A}}{x \leq y, \Gamma \Rightarrow \Delta, x : \Box A} \qquad \dfrac{x \leq y, x : \Box A, y : \Box_l A, \Gamma \Rightarrow \Delta}{x \leq y, x : \Box A, \Gamma \Rightarrow \Delta}\ L\Box}{x \leq y, \Gamma \Rightarrow \Delta}\ cut
$$

and is converted in the following derivation, where the first application of *cut* has a smaller height and the second application has a cut formula of smaller weight:

$$
\dfrac{ctr\ \dfrac{\dfrac{\mathcal{D}(y/z)}{x \leq y, x \leq y, \Gamma \Rightarrow \Delta, y : \Box_l A}}{x \leq y, \Gamma \Rightarrow \Delta, y : \Box_l A} \qquad \dfrac{wk\ \dfrac{x \leq y, \Gamma \Rightarrow \Delta, x : \Box A}{x \leq y, y : \Box_l A, \Gamma \Rightarrow \Delta, x : \Box A} \qquad x \leq y, x : \Box A, y : \Box_l A, \Gamma \Rightarrow \Delta}{x \leq y, y : \Box_l A, \Gamma \Rightarrow \Delta}\ cut}{x \leq y, \Gamma \Rightarrow \Delta}\ cut
$$

- The cut formula is $x : \Box_l A)$

$$
\dfrac{R\Box_l\ \dfrac{a : x, \Gamma \Rightarrow \Delta, x : \Box_l A, a \Vdash^{\forall} A \qquad a : x, \overline{a} \Vdash^{\exists} A, \Gamma \Rightarrow \Delta, x : \Box_l A}{a : x, \Gamma \Rightarrow \Delta, x : \Box_l A} \qquad \dfrac{a : x, b : x, b \Vdash^{\forall} A, \Gamma \Rightarrow \Delta, \overline{b} \Vdash^{\exists} A}{a : x, x : \Box_l A, \Gamma \Rightarrow \Delta}\ L\Box_l}{a : x, \Gamma \Rightarrow \Delta}\ cut
$$

and is converted in the following derivation, with two applications of *cut* with smaller height and two applications of *cut* on a formula of smaller weight:

$$
\dfrac{ctr\ \dfrac{\dfrac{\mathcal{D}(a/b)}{a : x, a : x, a \Vdash^{\forall} A, \Gamma \Rightarrow \Delta, \overline{a} \Vdash^{\exists} A}}{a : x, a \Vdash^{\forall} A, \Gamma \Rightarrow \Delta, \overline{a} \Vdash^{\exists} A} \qquad wk\ \dfrac{\dfrac{a : x, \Gamma \Rightarrow \Delta, x : \Box_l A, a \Vdash^{\forall} A \qquad \dfrac{a : x, x : \Box_l A, \Gamma \Rightarrow \Delta}{a : x, x : \Box_l A, \Gamma \Rightarrow \Delta, a \Vdash^{\forall} A}\ wk}{a : x, \Gamma \Rightarrow \Delta, a \Vdash^{\forall} A}\ cut}{a : x, \Gamma \Rightarrow \Delta, a \Vdash^{\forall} A, \overline{a} \Vdash^{\exists} A}}{(*)\ a : x, \Gamma \Rightarrow \Delta, \overline{a} \Vdash^{\exists} A}\ cut
$$

$$
\dfrac{a : x, \overline{a} \Vdash^{\exists} A, \Gamma \Rightarrow \Delta, x : \Box_l A \qquad \dfrac{a : x, x : \Box_l A, \Gamma \Rightarrow \Delta}{a : x, x : \Box_l A, \overline{a} \Vdash^{\exists} A, \Gamma \Rightarrow \Delta}\ wk}{(**)\ a : x, \overline{a} \Vdash^{\exists} A, \Gamma \Rightarrow \Delta}\ cut
$$

Then with a last application of *cut* on (*) and (**) we delete $\overline{a} \Vdash^{\exists} A$ and obtain the conclusion.

$\Box$

In the following we prove the soundness and completeness of $\mathbf{S\text{-}Lab_{EI}}$ w.r.t. IBNM semantics. In order to prove the soundness we need to give the interpretation of formulas and sequents of $\mathcal{L}_{\mathrm{LS}}$ in IBNMs. This is done as usual by means of the notion of realisation.

Definition 4.2. The relation $\Vdash$ of Def. 3.1 is extended with the following condition for $\Box_l$-formulas: $\mathcal{M}, w \Vdash \Box_l A$ iff there is $(\alpha, \beta) \in \mathcal{N}(w)$ s.t. for all $u \in \alpha$, $\mathcal{M}, u \Vdash A$ and for all $z \in \beta$, $\mathcal{M}, z \nVdash A$

Moreover, given a IBNM $\mathcal{M} = \langle W, \leq, \mathcal{N}, V \rangle$, a realisation is a pair of functions (ρ, σ), where $\rho : \mathbb{WL} \longrightarrow W$ and $\sigma : \mathbb{NL} \longrightarrow \mathcal{P}(W)$, with the condition that for any $a \in \mathbb{NL}$, $\sigma(a) \cap \sigma(\overline{a}) = \emptyset$. The relation $\mathcal{M} \models_{\rho,\sigma} \phi$, for $\phi \in \mathcal{L}_{\mathrm{LS}}$, is then defined by cases as follows:

$\mathcal{M} \models_{\rho,\sigma} x : a$ iff $\rho(x) \in \sigma(a)$;
$\mathcal{M} \models_{\rho,\sigma} x : A$ iff $\mathcal{M}, \rho(x) \Vdash A$;
$\mathcal{M} \models_{\rho,\sigma} a \Vdash^{\forall} A$ iff for all $w \in \sigma(a)$, $\mathcal{M}, w \Vdash A$;
$\mathcal{M} \models_{\rho,\sigma} a \Vdash^{\exists} A$ iff there is a $w \in \sigma(a)$ such that $\mathcal{M}, w \Vdash A$;
$\mathcal{M} \models_{\rho,\sigma} a : x$ iff $(\sigma(a), \sigma(\overline{a})) \in \mathcal{N}(\rho(x))$,

where a can be positive or negative, except for $a : x$, in which negative labels can't occur. Then given a sequent $\Gamma \Rightarrow \Delta$ we stipulate that $\mathcal{M} \models_{\rho,\sigma} \Gamma \Rightarrow \Delta$ iff whenever $\mathcal{M} \models_{\rho,\sigma} \phi$ for all formulas ϕ in Γ we also have $\mathcal{M} \models_{\rho,\sigma} \psi$ for a formula ψ in Δ. Moreover, $\Gamma \Rightarrow \Delta$ is valid in $\mathcal{M}$ iff for all realisations (ρ, σ) we have $\mathcal{M} \models_{\rho,\sigma} \Gamma \Rightarrow \Delta$, and it is valid in IBNM semantics iff it is valid in any such model.

Theorem 4.3. If a sequent $\Gamma \Rightarrow \Delta$ is derivable in $\mathbf{S\text{-}Lab_{EI}}$, then it is valid in IBNM.

Proof. It is easy to check that whenever the premiss(es) of a rule of $\mathbf{S\text{-}Lab_{EI}}$ are valid, so is the conclusion, thus the calculus is sound. □

We now show that the calculus is also complete w.r.t. IBNM semantics. As usual we prove the contrapositive, that any non-derivable sequent has a countermodel. To this purpose we need a notion of saturation of proof search branches.

Definition 4.3. Let $\mathcal{B} = \{\Gamma_i \Rightarrow \Delta_i\}$ be a (finite or infinite) branch in a proof search in $\mathbf{S\text{-}Lab_{EI}}$ for $\Gamma \Rightarrow \Delta$. We define $\Gamma^* = \bigcup \Gamma_i$ and $\Delta^* = \bigcup \Delta_i$. We say that $\mathcal{B}$ is saturated with respect to an application of rule R if condition (R) below holds, and it is saturated with respect to $\mathbf{S\text{-}Lab_{EI}}$ if it is saturated with respect to all possible applications of any rule of $\mathbf{S\text{-}Lab_{EI}}$. $(Init_x)$ For all i, there is no $x : p$ in $\Gamma_i \cap \Delta_i$. $(Init_a)$ For all i, $x : a$ and $x : \overline{a}$ are not both in Γ_i. $(Init_\bot)$ For all i, $x : \bot$ is not in Γ_i. $(Init_\top)$ For all i, $x : \top$ is not in Δ_i. (*propositional rules*) Standard. $(L \rightarrow)$ If $x \leq y$ and $x : A \rightarrow B$ are in Γ^*, then $y : A$ is in Δ^* or $y : B$ is in Γ^*. $(R \rightarrow)$ If $x : A \rightarrow B$ is in Δ^*, then for a world label y, $x \leq y$ and $y : A$ are in Γ^* and $y : B$ is in Δ^*. $(L \Vdash^\forall)$ If $a \Vdash^\forall A$ and $x : a$ are in Γ^*, then $x : A$ is in Γ^*. $(R \Vdash^\forall)$ If $a \Vdash^\forall A$ is in Δ^*, then for a label x, $x : a$ is in Γ^* and $x : A$ is in Δ^*. $(L \Vdash^\exists)$ If $a \Vdash^\exists A$ is in Γ^*, then for a world label x, $x : a$ and $x : A$ are in Γ^*. $(R \Vdash^\exists)$ If $a \Vdash^\exists A$ is in Δ^* and $x : a$ is in Γ^*, then $x : A$ is in Δ^*. $(L\Box_l)$ If $x : \Box_l A$ is in Γ^*, then for a neighbourhood label b, $b : x$ and $b \Vdash^\forall A$ are in Γ^* and $\overline{b} \Vdash^\exists A$ is in Δ^*. $(R\Box_l)$ If $x : \Box_l A$ is in Δ^* and $a : x$ is in Γ, then $a \Vdash^\forall A$ is in Δ^* or $\overline{a} \Vdash^\exists A$ is in Γ^*. $(L\Box)$ If $x \leq y$ and $x : \Box A$ are in Γ^*, then $y : \Box_l A$ is in Γ^*. $(R\Box)$ If $x : \Box A$ is in Δ^*, then for a label y, $x \leq y$ is in Γ^* and $y : \Box_l A$ is in Δ^*. $(L\Diamond)$ If $x : \Diamond A$ and $a : x$ are in Γ, then $a \Vdash^\exists A$ is in Γ^* or $\overline{a} \Vdash^\forall A$ is in Δ^*. $(R\Diamond)$ If $x : \Diamond A$ is in Γ^*, then for a neighbourhood label b, $b : x$ and $\overline{b} \Vdash^\forall A$ are in Γ^* and $b \Vdash^\exists A$ is in Δ^*. $(Ref_\leq)$ If x occurs in $\mathcal{B}$, then $x \leq x$ is in Γ^*. $(Tr_\leq)$ If $x \leq y$ and $y \leq z$ are in Γ^*, then $x \leq z$ are in Γ^*. (Val) If $x \leq y$ and $x : p$ are in Γ^*, then $y : p$ is in Γ^*. $(Neig)$ If $x \leq y$ and $a : y$ are in Γ^*, then $a : x$ is in Γ^*.

Theorem 4.4. If a sequent $\Gamma \Rightarrow \Delta$ is valid in IBNM, then is is derivable in $\mathbf{S\text{-}Lab_{EI}}$.

Proof. By contraposition we prove that if $\Gamma \Rightarrow \Delta$ is not derivable in $\mathbf{S\text{-}Lab_{EI}}$, then it is not valid. Assume that $\Gamma \Rightarrow \Delta$ is not derivable, then any proof search for $\Gamma \Rightarrow \Delta$ contains a saturated branch. Given such a saturated branch $\mathcal{B}$, we build a IBNM $\mathcal{M}$ that makes all formulas in Γ^* true and all formulas in Δ^* false.

Let the model $\mathcal{M} = \langle W, \leq, \mathcal{N}, V \rangle$ be defined as follows: $W = \{x \in \mathbb{WL} \mid x$ occurs in $\Gamma^* \cup \Delta^*\}$; for all $x, y \in W$, $x \leq y$ iff $x \leq y$ is in Γ^*; for any positive or negative label a occurring in $\mathcal{B}$, $\alpha_a = \{x \in W \mid x : a$ is in $\Gamma^*\}$; for any $x \in W$, $\mathcal{N}(x) = \{(\alpha_a, \alpha_{\overline{a}}) \mid a : x$ is in $\Gamma^*\}$; and for any $p \in \mathcal{L}$ and any $x \in W$, $p \in V(x)$ iff $x : p$ is in Γ^*. Moreover we define the realisation (ρ, σ) by choosing

$\rho(x) = x$ for any world label x, and $\sigma(a) = \alpha_a$ for any positive or negative label a occurring in $\Gamma^* \cup \Delta^*$.

First notice that $\mathcal{M}$ is well defined: by saturation of $Ref_\leq$ and $Tr_\leq$, the relation $\leq$ is a preorder on W. Moreover, by saturation of $Init_a$ and of rules Val and $Neig$ we have, respectively, that $(\alpha_a, \alpha_{\overline{a}}) \in \mathcal{N}(x)$ implies $\alpha_a \cap \alpha_{\overline{a}} = \emptyset$; that $x \leq y$ implies $V(x) \subseteq V(y)$; and that $x \leq y$ implies $\mathcal{N}(y) \subseteq \mathcal{N}(x)$.

It is easy to prove by induction on the weight of ϕ that: if ϕ is in Γ^*, then $\mathcal{M} \models_{\rho,\sigma} \phi$, and if ϕ is in Δ^*, then $\mathcal{M} \not\models_{\rho,\sigma} \phi$.

$\square$

5 Discussion and conclusion

In this paper, we have presented a preliminar proposal of a minimal intuitionistic non-normal modal logic. The logic is obtained by adding to the intuitionistic calculus the congruence rules for the modal operators, together with a very weak connection between the modalities. Then we have given a cut-free internal sequent calculus, as well as a semantic interpretation for the logic in terms of IBNMs. We have shown that the logic is sound w.r.t. the semantics, and that the $\Box$-fragment is also complete. We have also given an external calculus matching the semantics.

This is just a first step of development of intuitionistic non-normal modal logics. We aim to investigate extensions of the proposed semantics. It can be seen that stronger logics can be obtained by imposing further conditions on IBNMs, for instance assuming that in any pair $(\alpha, \beta) \in \mathcal{N}(w)$, α is upper closed w.r.t. $\leq$, the axiom $\neg\Diamond A \rightarrow \Box\neg A$ becomes valid. Of course we can also impose standard condition of classical non-normal modal logics. The study of all this, as well as the study of the precise relation between these logics and the classical ones, will be object of our future research.

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