Sequent Calculi for Logic that Includes an Infinite Number of Modalities

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Abstract. Multimodal logic has been studied for various purposes. Although some studies have considered an infinite number of modalities, propositions for the quantification of modalities, such as "For all modalities [i] in an infinite set of modalities S, [i]p is true" have not been discussed in general. In this paper, a simple method for expressing these propositions is discussed and deduction systems for new logic are established. The conditions on Kripke frames that include these notions are also discussed.

1 Introduction

Basic modal logic has only one primitive modal operator. A *multimodal logic* has more than one modal operator, and it has been researched in various ways. For example, in *epistemic logic* [4] [5] [6], we use the same number of modalities as the number of agents, and we use a *Kripke model* that has the same number of relations as the number of agents. $[i]p$ is translated as "Agent i knows that p is true."

Some logic includes a proposition, such as "For all $i \in I$, $[i]p$ is true" or "For all $i \in I'$ ($I' \subseteq I$), $[i]p$ is true." If the numbers of modalities are finite, then we can represent such a proposition using a formula composed of traditional modal logic. For example, in epistemic logic, if the number of agents is five {1,2,3,4,5}, then we can represent "All three agents in {1,2,4} know p" as $[1]p \wedge [2]p \wedge [4]p$. However, if the number of modalities is infinite, then we cannot represent such a proposition because we cannot use a formula that includes an infinite number of symbols. These types of propositions are important in some areas of logics. For example, *dynamic quantum logic* [1] [2] [3] has propositions based on closed subspaces of Hilbert space H, and $\Box A$ represents "For all relations $P?$ that represent a projection to a closed subspace of H, $[P?]A$ is true." The number of relations $P?$ may be infinite because a Hilbert space may have an infinite number of closed subspaces. Another example is *dynamic logic with program quantifiers* [9]. Dynamic logic considers dynamic propositions of computer programs. In [9], the proposition "After executing any possible program a, p is true" is used. As a fundamental study of these types of propositions, in this paper, we discuss a simple expansion of multimodal logic.

Another important aspect of multiple relational models is the condition on relations. In modal logic, basic conditions such as *reflexive*, *transitive*, and *sym-*

metric are studied. In multiple relational models, we can introduce more complicated and developed conditions. In a model of dynamic quantum logic, infinite numbers of types of relations are included and the following type of condition is used:

For all elements x, y and z in a model, there exist relations R and R' such that xRy, $yR'z$.

In this paper we use a specific symbol for each subset of $\mathbb{N}$ in the same manner as basic multimodal logic. In this method, if S represents the set of all odd numbers $\{1, 3, 5, 7, ...\}$, then $[S]p$ means "For all $i \in \{1, 3, 5, 7, ...\}$, $[i]p$ is true." We can consider formulas for conditions using these symbols in the same manner as original modal logic.

In section 2, we discuss the basics of multimodal logic and models that have an infinite number of modalities. The complete axiomatization of the logic is also discussed. We call this infinite modal logic (**IML**).

In section 3, we discuss *labeled sequent calculi* for **IML**, which is useful for managing the conditions of frames. A labeled sequent includes not only formulas but also relations based on a Kripke frame. This type of sequent calculus has been used to prove the cut-elimination theorem and other useful theorems [8] [12]. The development of labeled sequent calculi is summarized in [11].

2 IML

We use a language that has a denumerable infinite set of propositional variables. The propositional variables are denoted by $p, q, r, ...$ and composite formulas are denoted by $A, B, C, ...$ We express each element of the power set $\mathfrak{P}(\mathbb{N})$ using symbols, and use $S, T, ...$ to denote these symbols. Formulas are defined as follows:

$$A ::= p \mid \bot \mid \neg A \mid A \wedge A \mid [S]A.$$

We use $A \vee B$ as an abbreviation of $\neg(\neg A \wedge \neg B)$, and $A \rightarrow B$ as an abbreviation of $\neg A \vee B$.

An ∞-*relational frame* is a pair (X, R), where X is a non-empty set and R is a set $\{R_n | n \in \mathbb{N}\}$. Each R_n is a binary relation on X. We write $x(n)y$ if $(x, y) \in R_n$. We write $x(S)y$ if there exists $n \in S$ such that $x(n)y$, or if $S = \emptyset$ and for all $n \in \mathbb{N}$, $x(n)y$.

An ∞-*relational model* is a triple (X, R, V), where (X, R) is an ∞-relational frame and V is a function that assigns each propositional variable p to a subset of X. We simply use frame and model to refer to an ∞-relational frame and ∞-relational model.

Set $\|A\|$ for (X, R, V) is defined by induction on the composition of A as follows.

$$\|p\| = V(p)$$

$\|A \wedge B\| = \|A\| \cap \|B\|$
$\|\neg A\| = \|A\|^c$
$\|[S]A\| = \{x \in X |$for every $y \in X$ and $n \in \mathbb{N}$, if $x(n)y$ and $n \in S$, then $y \in \|A\|\} (S \neq \emptyset)$
$\|[\emptyset]A\| = \{x \in X |$for every $y \in X$, if $x(n)y$ for all $n \in \mathbb{N}$, then $y \in \|A\|\}$

$\|[\emptyset]A\|$ is a upper bound of $\|[S]A\|$. We adopt the above definition of $\|[\emptyset]A\|$ to satisfy the completeness theorem of a system in later sections. It is difficult to create natural rules for a modal symbol if we adopt another upper bound such as $\|[\emptyset]A\| = X$. If we use $\|[\emptyset]A\| = X$, then we have to add another axiom to the calculus because $[\emptyset]A$ is always true.

We say formula A is *true* at $x \in X$ if $x \in \|A\|$ and write $x \models A$. We say formula A is *false* at $x \in X$ if A is not true at x. We say formula A is *valid* in a model (X, R, V) if A is true at each $x \in X$. Logic **IML** is defined as the set of all formulas that is valid in all models.

A *sequent* is defined as $\Gamma \Rightarrow \Delta$, where Γ and Δ are finite sets of formulas. We say a sequent $\Gamma \Rightarrow \Delta$ is *true* at $x \in X$ if $x \in \| \wedge \Gamma \to \vee \Delta\|$, where $\wedge \Gamma$ is a conjunction of all $A \in \Gamma$ and $\vee \Delta$ is a disjunction of all $A \in \Delta$. We say a sequent $\Gamma \Rightarrow \Delta$ is *false* at $x \in X$ if $\Gamma \Rightarrow \Delta$ is not true at x. We say a sequent $\Gamma \Rightarrow \Delta$ is *valid* in model (X, R, V) if $\Gamma \Rightarrow \Delta$ is true at all $x \in X$.

Because Γ and Δ are finite sets, the number of symbols for sets $\{S, T, ...\}$ that appear in $\Gamma \Rightarrow \Delta$ is also finite. We denote by $\{\beta_1, \beta_2, ..., \beta_l\}$ the set of such symbols. We define *base set* $\mathcal{B}$ of $\Gamma \Rightarrow \Delta$ and $\mathcal{B}_{\beta k}$ of $\Gamma \Rightarrow \Delta$ as follows:

$\mathcal{B}(\Gamma \Rightarrow \Delta) = \{\beta_1^* \cap \beta_2^* \cap ... \cap \beta_l^* |$ each β_i^* is β_i or $\beta_i^c\}$
$\mathcal{B}_{\beta k}(\Gamma \Rightarrow \Delta) = \{\beta_1^* \cap \beta_2^* \cap ... \cap \beta_k \cap ... \cap \beta_l^* |$ each β_i^* is β_i or $\beta_i^c\}$

The base set provides minimum units of the set of all modalities in $\Gamma \Rightarrow \Delta$.

Sequent calculus for **IML** (**SIML**) is given as follows:

Axiom:

$$A \Rightarrow A$$

Rules:

$$\frac{\Gamma \Rightarrow \Delta, A \quad A, \Pi \Rightarrow \Sigma}{\Gamma, \Pi \Rightarrow \Delta, \Sigma} \text{ (cut)}$$

$$\frac{\Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta} \text{ (wL)} \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, A} \text{ (wR)}$$

$$\frac{A, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta} \text{ }(\wedge\text{L}) \qquad \frac{B, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta} \text{ }(\wedge\text{L}) \qquad \frac{\Gamma \Rightarrow \Delta, A \quad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B} \text{ }(\wedge\text{R})$$

$$\frac{\Gamma \Rightarrow \Delta, A}{\neg A, \Gamma \Rightarrow \Delta} \text{ }(\neg\text{L}) \qquad \frac{A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg A} \text{ }(\neg\text{R})$$

$$\frac{\Gamma \Rightarrow A}{[S]\Gamma \Rightarrow [S]A} \text{ }([S]) \qquad \frac{[S]A, \Gamma \Rightarrow \Delta}{[T]A, \Gamma \Rightarrow \Delta} \text{ }(\cup\text{L})^{(1)}$$

$$\frac{(\forall m \in I) \ \Gamma \Rightarrow \Delta, [S_m]A}{\Gamma \Rightarrow \Delta, [T]A} \text{ }(\cup\text{R})^{(2)}$$

(1) $S \subseteq T$.
(2) I is a finite set. $T = \bigcup_{m \in I} S_m$.

SIML consists of sequent calculus for classical logic and the rules for $[S]A$.

Theorem 1 (Soundness theorem for SIML). *If $\Gamma \Rightarrow \Delta$ is provable in* **SIML***, then $\Gamma \Rightarrow \Delta$ is valid for all models.*

For the completeness theorem, we prove a contrapositive; that is, we prove that if sequent $\Gamma \Rightarrow \Delta$ is not provable, then there exists model (X, R, V) such that $\Gamma \Rightarrow \Delta$ is false at $x \in X$.

Suppose $\Gamma \Rightarrow \Delta$ is not provable. A canonical model (X_c, R_c, V_c) can be constructed by executing following procedure. We start from $X = \{\Gamma \Rightarrow \Delta\}$ and $R = \{\emptyset\}$. The procedure adds formulas to sequents in a model, and creates new elements and relations to a model. X_c and R_c are defined by X and R at the end of this procedure.

Step 1:
For all $\Gamma' \Rightarrow \Delta' \in X$:

- Provided the resulting sequent is not identical to the original sequent: if $A \wedge B \in \Gamma'$, then we add A and B to Γ'. This new sequent is not provable because of rule ($\wedge$L).
- Provided the resulting sequent is not identical to the original sequent: if $A \wedge B \in \Delta'$, then at least one of $\Gamma' \Rightarrow \Delta', A$ and $\Gamma' \Rightarrow \Delta', B$ is not provable because of rule ($\wedge$R). We add A or B to Δ', which preserves unprovability.
- Provided the resulting sequent is not identical to the original sequent: if $\neg A \in \Gamma'$, then $\Gamma' \Rightarrow \Delta', A$ is not provable because of rule ($\neg$L). We add A to Δ'.
- Provided the resulting sequent is not identical to the original sequent: if $\neg A \in \Delta'$, then $A, \Gamma' \Rightarrow \Delta'$ is not provable because of rule ($\neg$R). We add A to Γ'.
- If $[S]A \in \Delta'$ and if we did not execute this procedure on this formula and if $S \notin \mathcal{B}(\Gamma \Rightarrow \Delta)$, then we add $[T]A$ to Δ' such that $\Gamma' \Rightarrow \Delta', [T]A$ is not provable and $T \in \mathcal{B}_S(\Gamma \Rightarrow \Delta)$. If every $\Gamma' \Rightarrow \Delta', [T]A$ such that $T \in \mathcal{B}_S(\Gamma \Rightarrow \Delta)$ are provable, then by ($\cup$R), $\Gamma' \Rightarrow \Delta', [S]A$ would be provable since $\bigcup \mathcal{B}_S(\Gamma \Rightarrow \Delta) = S$. Therefore, there exists $T \in \mathcal{B}_S(\Gamma \Rightarrow \Delta)$ such that $\Gamma' \Rightarrow \Delta', [T]A$ is not provable.

Step 1 ends at some point because Γ' and Δ' are finite sets, sub(A) is also finite, and $\mathcal{B}_S(\Gamma \Rightarrow \Delta) \subseteq \mathcal{B}(\Gamma \Rightarrow \Delta)$. Then, we move to the next step.

Step 2:
For all $\Gamma' \Rightarrow \Delta' \in X$ that already existed before this step:

- If $[S]A \in \Delta'$ and $S \in \mathcal{B}(\Gamma \Rightarrow \Delta) \cup \{\emptyset\}$, and if we did not execute this procedure on this formula, then we add a new sequent $\Rightarrow A$ to X. We say that this new sequent is created by $[S]A$. For all $[T]B \in \Gamma'$, if $S \subseteq T$, then we add B to the premise of this new sequent. This new sequent is not provable because of rules ($[S]$) and ($\cup$L). We denote by $\Gamma'' \Rightarrow \Delta''$ this new sequent. If $S \neq \emptyset$, then we choose element n from S randomly and add $(\Gamma' \Rightarrow \Delta', \Gamma'' \Rightarrow \Delta'')$ to R_n. If $S = \emptyset$, for all $n \in \mathbb{N}$, then we add $(\Gamma' \Rightarrow \Delta', \Gamma'' \Rightarrow \Delta'')$ to R_n.

Step 2 ends at some point for the same reason provided above. Because new sequents are created by Step 2, we return to Step 1. This repeated procedure ends at some point for same reason provided above.

We define V_c as follows:

$$V_c(p) = \{\Gamma' \Rightarrow \Delta' \mid p \in \Gamma'\}$$

Lemma 1. *For all $(\Gamma' \Rightarrow \Delta') \in X_c$, if $A \in \Gamma'$, then A is true at $(\Gamma' \Rightarrow \Delta')$, and if $A \in \Delta'$, then A is false at $(\Gamma' \Rightarrow \Delta')$. Therefore, $\Gamma \Rightarrow \Delta$ is false at some $(\Gamma' \Rightarrow \Delta') \in X_c$.*

Lemma 1 provides the following theorem.

Theorem 2 (Cut-free completeness theorem for SIML). *If $\Gamma \Rightarrow \Delta$ is valid for all models, then $\Gamma \Rightarrow \Delta$ is provable in* **SIML** *without using the rule (cut).*

A Hilbert-style axiomatization for **IML** (**HIML**) is defined as follows:

All axiom and rules of traditional modal logic for all $[S]$ and
$[S_1]A \wedge [S_2]A \wedge ... \wedge [S_n]A \to [T]A$ for all S_i and T such that $T \subseteq \bigcup S_i$.

Theorem 3 (Soundness and completeness theorem for HIML). *A is valid for all models iff A is provable in* **HIML**.

3 Labeled sequent calculus LSIML

In this section, we discuss labeled sequent calculus for **IML** to manage conditions for frames. Conditions for frames can be defined as natural expansions of conditions for basic modal logic.

We say frame (X, R) satisfies the $\{S\}$-*reflexive* condition if (X, R) satisfies the following condition:

For all $x \in X$, $x(S)x$.

We say frame (X, R) satisfies the $\{S, T, U\}$-*transitive* condition if (X, R) satisfies the following condition:

For all $x, y, z \in X$, if $x(S)y$ and $y(T)z$, then $x(U)z$.

We say frame (X, R) satisfies the $\{S, T\}$-*symmetric* condition if (X, R) satisfies the following condition:

For all $x, y \in X$, if $x(S)y$, then $y(T)x$.

The above conditions can be summarized as parts of the following *Geach formula*-type conditions. For Geach formula, see [7] [10]. We say frame (X, R) satisfies the $\{S_1, ..., S_e, T_1, ..., T_f, U_1, .., U_g, W_1, ..., W_h\}$- condition if (X, R) satisfies the following condition: e, f, g, h are zero or natural numbers.

For all $\{x_0, ..., x_e, y_1, ..., y_f\} \subseteq X$, if $x_0(S_1)x_1, x_1(S_2)x_2, ..., x_{e-1}(S_e)x_e$, $x_0(T_1)y_1, y_1(T_2)y_2, ..., y_{f-1}(T_f)y_f$, then there exists $\{z_1, .., z_g$, $w_1, ..., w_{h-1}\} \subseteq X$ such that $x_e(U_1)z_1, z_1(U_2)z_2, ..., z_{g-1}(U_G)z_g$, $y_f(W_1)w_1, w_1(W_2)w_2, ..., w_{h-1}(W_h)z_g$.

Although we proved the completeness theorem for **SIML**, this type of sequent is not suitable for **IML** with conditions. For example, we can introduce the axiom for $\{S, T, U\}$-transitive as follows:

$[U]A \Rightarrow [S][T]A$.

This axiom is a generalization of the basic transitive axiom $\Box A \Rightarrow \Box\Box A$. However, we cannot prove the cut-elimination theorem if we add this axiom to the calculus. We have to introduce a rule for $\{S, T, U\}$-transitive not an axiom. The rule for a basic transitive condition is as follows:

$$\frac{\Box\Gamma \Rightarrow A.}{\Box\Gamma \Rightarrow \Box A}$$

It is difficult to introduce a rule for $\{S, T, U\}$-transitive by modifying this rule because each square has a different role. In this section, we use labeled sequent calculus to overcome this problem.

We define the infinite set of *labels*, which is denoted by $\{a, b, c, ...\}$. A *labeled formula* is denoted by $a : A$, where a is a label and A is a formula. Sets of labeled formulas are denoted by $\Gamma, \Delta, \Sigma, ...$ We say label a appears in Γ if $a : A$ is included in Γ for some A. We define a *set relation of* S on labels a and b by $a(S)b$.

A *labeled sequent* is defined as $\{R\}\Gamma \Rightarrow \Delta$, where $\{R\}$ is a set of set relations on labels, and all $\{R\}$, Γ, and Δ are finite sets. $\{R\} \cup \{a(S)b\}$ is denoted by $\{R, a(S)b\}$. We say that letter a appears in set $\{R\}$ if $a(S)b$ or $b(S)a$ are included in $\{R\}$ for some b and S. We define set $\{R\}_L$, which is a set of letters that appears in $\{R\}$. We say set S appears in $\{R\}\Gamma \Rightarrow \Delta$ if $a(S)b \in \{R\}$ for some a and b, or $[S]A$ is a sub-formula of B such that $a : B \in \Gamma \cup \Delta$ for some a. We say a labeled sequent $\{R'\}\Gamma' \Rightarrow \Delta'$ is a *subsequent* of $\{R\}\Gamma \Rightarrow \Delta$ if $\{R\}' \subseteq \{R\}, \Gamma' \subseteq \Gamma$, and $\Delta' \subseteq \Delta$.

An *embedding* of labeled sequent $\{R\}\Gamma \Rightarrow \Delta$ to model (X, R, V) is function E from $\{R\}_L$ to X that satisfies the following conditions:

If $a(S)b \in \{R\}$, $S \neq \emptyset$, $E(a) = x$, and $E(b) = y$, then there exists $n \in S$ such that $x(n)y$.

If $a(\emptyset)b \in \{R\}$, $E(a) = x$, and $E(b) = y$, then for all $n \in \mathbb{N}$, $x(n)y$.

We say labeled sequent $\{R\}\Gamma \Rightarrow \Delta$ is *true* in $(X, \perp, V)$ under E if for some $a : A \in \Gamma$, A is false at $E(a)$, or for some $b : B \in \Delta$, B is true at $E(b)$. We say labeled sequent $\{R\}\Gamma \Rightarrow \Delta$ is *false* in $(X, \perp, V)$ under E if $\{R\}\Gamma \Rightarrow \Delta$ is not true in $(X, \perp, V)$ by E. We say labeled sequent $\{R\}\Gamma \Rightarrow \Delta$ is *valid* in $(X, \perp, V)$ if for all E, $\{R\}\Gamma \Rightarrow \Delta$ is true in $(X, \perp, V)$ under E. An *infinite labeled sequent* is defined as $\{R\}\Gamma \Rightarrow \Delta$, where $\{R\}$, Γ, or Δ is an infinite set, and the other conditions are the same as those for labeled sequents. We say that labeled sequent $\{R'\}\Gamma' \Rightarrow \Delta'$ is a *subsequent* of $\{R\}\Gamma \Rightarrow \Delta$ if $\{R\}' \subseteq \{R\}, \Gamma' \subseteq \Gamma$, and $\Delta' \subseteq \Delta$.

The labeled sequent calculus **LSIML** is defined as follows. $\{R\}$, Γ, and Δ are finite sets.

Axiom:

$$\{R\}a : A \Rightarrow a : A$$

Rules:

$$\frac{\{R\}\Gamma \Rightarrow \Delta, a : A \quad \{R\}\ a : A, \Gamma \Rightarrow \Delta}{\{R\}\Gamma \Rightarrow \Delta}\ (\text{cut})$$

$$\frac{\{R\}\Gamma \Rightarrow \Delta}{\{R\}a : A, \Gamma \Rightarrow \Delta}\ (\text{wL}) \qquad \frac{\{R\}\Gamma \Rightarrow \Delta}{\{R\}\Gamma \Rightarrow \Delta, a : A}\ (\text{wR})$$

$$\frac{\{R\}a : A, a : B, \Gamma \Rightarrow \Delta}{\{R\}a : A \wedge B, \Gamma \Rightarrow \Delta}\ (\wedge\text{L}) \qquad \frac{\{R\}\Gamma \Rightarrow \Delta, a : A \quad \{R\}\Gamma \Rightarrow \Delta, a : B}{\{R\}\Gamma \Rightarrow \Delta, a : A \wedge B}\ (\wedge\text{R})$$

$$\frac{\{R\}\Gamma \Rightarrow \Delta, a : A}{\{R\}a : \neg A, \Gamma \Rightarrow \Delta}\ (\neg\text{L}) \qquad \frac{\{R\}a : A, \Gamma \Rightarrow \Delta}{\{R\}\Gamma \Rightarrow \Delta, a : \neg A}\ (\neg\text{R})$$

$$\frac{\{R, a(S)b\}b : A, \Gamma \Rightarrow \Delta}{\{R, a(S)b\}a : [T]A, \Gamma \Rightarrow \Delta}\ ([]\ \text{L})^{(1)} \qquad \frac{\{R, a(S)b\}\Gamma \Rightarrow \Delta, b : A}{\{R\}\Gamma \Rightarrow \Delta, a : [S]A}\ ([]\ \text{R})^{(2)}$$

$$\frac{(\forall m \in I) \quad \{R, a(S_m)b\}\Gamma \Rightarrow \Delta}{\{R, a(T)b\}\Gamma \Rightarrow \Delta}\ (\cup)^{(3)}$$

(1) $S \subseteq T$.
(2) b does not appear in $\{R\}$, Γ, and Δ.
(3) I is a finite set. $T = \bigcup S_m$.

Theorem 4 (Soundness theorem for LSIML). *If $\{R\}\Gamma \Rightarrow \Delta$ is provable in* **LSIML**, *then $\{R\}\Gamma \Rightarrow \Delta$ is valid in all models.*

Because $\{R\}$, Γ, and Δ are finite sets, the number of symbols for sets $\{S, T, ...\}$ that appear in $\{R\}\Gamma \Rightarrow \Delta$ is also finite. We denote by $\{\beta_1, \beta_2, ..., \beta_l\}$ the set of such symbols. The definitions of set $\mathcal{B}$ of $\{R\}\Gamma \Rightarrow \Delta$ and set $\mathcal{B}_{\beta k}$ of $\{R\}\Gamma \Rightarrow \Delta$ are the same as those in the previous section.

For the completeness theorem, suppose $\{R\}\Gamma \Rightarrow \Delta$ is not provable. Canonical model (X_c, R_c, V_c) can be created from $\{R\}\Gamma \Rightarrow \Delta$ by adding new labels and relations in this sequent while preserving unprovability. These changes of the sequent are denoted by $\{R\}_0\Gamma_0 \Rightarrow \Delta_0 (= \{R\}\Gamma \Rightarrow \Delta), \{R\}_1\Gamma_1 \Rightarrow \Delta_1, ..., \{R\}_i\Gamma_i \Rightarrow \Delta_i, \{R\}_{i+1}\Gamma_{i+1} \Rightarrow \Delta_{i+1}, ...$

We execute following procedure until the sequent does not change.

- If $a : (A \wedge B) \in \Gamma_i$, then we define $\Gamma_{i+1} = \Gamma_i \cup \{a : A, a : B\}$, $\Delta_{i+1} = \Delta_i$ and $\{R\}_{i+1} = \{R\}_i$. $\{R\}_{i+1}\Gamma_{i+1} \Rightarrow \Delta_{i+1}$ is not provable because of rule ($\wedge$L).
- If $a : (A \wedge B) \in \Delta_i$, then at least one of $\{R\}_i\Gamma_i \Rightarrow \Delta_i, a : A$ and $\{R\}_i\Gamma_i \Rightarrow \Delta_i, a : B$ is not provable because of rule ($\wedge$R). Therefore, we define $\Delta_{i+1} = \Delta_i \cup \{a : A\}$ or $\Delta_i \cup \{a : B\}$, which preserve unprovability. In either case, we define $\Gamma_{i+1} = \Gamma_i$ and $\{R\}_{i+1} = \{R\}_i$.
- If $a : \neg A \in \Gamma_i$, $\{R\}_i\Gamma_i \Rightarrow \Delta_i, thena : A$ is not provable because of rule ($\neg$L). Therefore, we define $\Gamma_{i+1} = \Gamma_i$, $\Delta_{i+1} = \Delta_i \cup \{a : A\}$ and $\{R\}_{i+1} = \{R\}_i$.
- If $a : \neg A \in \Delta_i$, then $\{R\}_i\Gamma_i, a : A \Rightarrow \Delta_i$ is not provable because of rule ($\neg$R). Therefore, we define $\Gamma_{i+1} = \Gamma_i \cup \{a : A\}$, $\Delta_{i+1} = \Delta_i$ and $\{R\}_{i+1} = \{R\}_i$.
- If $a : [T]A \in \Gamma_i$, $a(S)b \in \{R\}_i$, and $S \subseteq T$, then $\{R\}_i b : A, \Gamma_i \Rightarrow \Delta_i$ is not provable because of rule ([]L). Therefore, we define $\Gamma_{i+1} = \Gamma_i \cup \{b : A\}$, $\Delta_{i+1} = \Delta_i$ and $\{R\}_{i+1} = \{R\}_i$.
- If $a : [S]A \in \Delta_i$ and if we did not execute this procedure on this formula, then we consider label b that does not appear in $\{R\}_i\Gamma_i \Rightarrow \Delta_i$ and we define $\Gamma_{i+1} = \Gamma_i$, $\Delta_{i+1} = \Delta_i \cup \{b : A\}$ and $\{R\}_{i+1} = \{R, a(S)b\}_i$. $\{R\}_{i+1}\Gamma_{i+1} \Rightarrow \Delta_{i+1}$ is not provable because of rule ([]R).
- If $a(S)b \in \{R\}_i$ and $S \notin \mathcal{B}(\{R\}\Gamma \Rightarrow \Delta)$, from $S \in \{\beta_1, \beta_2, ..., \beta_l\}$ and $\bigcup \mathcal{B}_S(\{R\}\Gamma \Rightarrow \Delta) = S$, at least one of $\{R - (a(S)b), a(\{U\})b\}_i\Gamma_i \Rightarrow \Delta_i$ $(U \in \mathcal{B}_S(\{R\}\Gamma \Rightarrow \Delta))$ is not provable because of rule ($\cup$). We define $\Gamma_{i+1} = \Gamma_i$, $\Delta_{i+1} = \Delta_i$ and $\{R\}_{i+1} = \{R - (a(S)b), a(\{U\})b\}_i$ such that $\{R\}_{i+1}\Gamma_{i+1} \Rightarrow \Delta_{i+1}$ is not provable and $U \in \mathcal{B}_S(\{R\}\Gamma \Rightarrow \Delta)$.

This procedure ends at some point because of the following reasons: The numbers of labels, relations, and propositional variables that appear in $\{R\}_i\Gamma_i \Rightarrow \Delta_i$ are finite. All these procedures decrease the complexity of the formulas. We can add only one label b for each $a : [S]A \in \Delta_i$ because of the condition $\mathcal{B}_S(\{R\}\Gamma \Rightarrow \Delta) \subseteq \mathcal{B}(\{R\}\Gamma \Rightarrow \Delta)$.

We write the final sequent as $\{R\}_c\Gamma_c \Rightarrow \Delta_c$. Note that $\{R\}_c$ only includes the set relations of $S \in \mathcal{B}(\{R\}\Gamma \Rightarrow \Delta))$ because of the final procedure. The canonical model (X_c, R_c, V_c) $(R_c = \{R_{1c}, R_{2c}, ...\})$ of $\{R\}\Gamma \Rightarrow \Delta$ and the embedding E_c are defined as follows:

$X_c = \{a \mid a \text{ appears in } \{R\}_c\}$.
Starting from $R_{nc} = \emptyset$ for all n, if $a(S)b \in \{R\}_c$ and $S \neq \emptyset$, then we choose element n from S randomly and add (a, b) to R_{nc}. If $a(\emptyset)b \in \{R\}_c$, then we add (a, b) to all R_{nc}.
$V_c(p) = \{a \mid a : p \in \Gamma_c\}$.
$E_c(a) = a$.

Lemma 2. *If $a : A \in \Gamma_c$, then A is true at $a \in X_c$. If $a : A \in \Delta_c$, then A is false at $a \in X_c$. Therefore, $\{R\}\Gamma \Rightarrow \Delta$ is false in $(X_c, \perp_c, V_c)$ under E_c.*

Theorem 5 (Cut-free completeness theorem for LSIML). *If $\{R\}\Gamma \Rightarrow \Delta$ is valid for all models, then $\{R\}\Gamma \Rightarrow \Delta$ is provable in* **LSIML** *without using the rule (cut).*

For a frame with conditions, we add the following rules to **LSIML**.

$\{S\}$-reflexive

$$\frac{\{R, a(S)a\}\Gamma \Rightarrow \Delta}{\{R\}\Gamma \Rightarrow \Delta}\ (\{S\}\text{-ref})$$

$\{S, T, U\}$-transitive

$$\frac{\{R, a(S')b, b(T')c, a(U)c\}\Gamma \Rightarrow \Delta}{\{R, a(S')b, b(T')c\}\Gamma \Rightarrow \Delta}\ (\{S, T, U\}\text{-tra})^{(1)}$$

$\{S, T\}$-symmetric

$$\frac{\{R, a(S')b, b(T)a\}\Gamma \Rightarrow \Delta}{\{R, a(S')b\}\Gamma \Rightarrow \Delta}\ (\{S, T\}\text{-sym})^{(2)}$$

$\{S_1, ..., S_e, T_1, ..., T_f, U_1, .., U_g, W_1, ..., W_h\}$-condition

$$\frac{\{R, G_1, G_2\}\Gamma \Rightarrow \Delta}{\{R, G_1\}\Gamma \Rightarrow \Delta}\ (\{G\}\text{-con})^{(3)}$$

$G = \{S_1, ..., S_e, T_1, ..., T_f, U_1, .., U_g, W_1, ..., W_h\}$
$G_1 = \{a_0(S_1')a_1, a_1(S_2')a_2, ..., a_{e-1}(S_e')a_e, a_0(T_1')b_1, b_1(T_2')b_2, ..., b_{f-1}(T_f')b_f\}$
$G_2 = \{a_e(U_1')c_1, c_1(U_2')c_2, ..., c_{g-1}(U_g')c_g, b_f(W_1')d_1, d_1(W_2')d_2, ..., d_{h-1}(W_h')c_g\}$

(1) $S' \subseteq S$. (2) $S' \subseteq S$ and $T' \subseteq T$.
(3) All labels in G_2 except a_e and b_f do not appear in the under sequent. $S_j' \subseteq S_j$, $T_j' \subseteq T_j$, $U_j' \subseteq U_j$, and $W_j' \subseteq W_j$ for all j.

We simply use G-condition to refer to $\{S_1, ..., S_e, T_1, ..., T_f, U_1, .., U_g, W_1, ..., W_h\}$-condition. We say $\{R\}$ satisfies the G-condition if for all a_j, b_j, S_j', T_j', if $G_1 \subseteq \{R\}$, then there exists $G_2 \in \{R\}$.

Theorem 6. *If $\{R\}\Gamma \Rightarrow \Delta$ is provable in* **LSIML**+*($\{G\}$-con), then $\{R\}\Gamma \Rightarrow \Delta$ is valid for all models that satisfy the G-condition.*

For the completeness theorem for **LSIML**+($\{G\}$-con), we change the definition of $\mathcal{B}$ and $\mathcal{B}_{\beta k}$ to manage new relations created by ($\{G\}$-con).

$\mathcal{B}(\{R\}\Gamma \Rightarrow \Delta) = \{\beta_1^* \cap ... \cap \beta_l^* \cap S_1^* \cap ... \cap S_e^* \cap T_1^* \cap ... T_f^* \cap U_1^* \cap ... \cap U_g^* \cap W_1^* \cap ... \cap W_h^* |$ each $*$ is empty or $c\}$
$\mathcal{B}_{\beta k}(\{R\}\Gamma \Rightarrow \Delta) = \{\beta_1^* \cap ... \cap \beta_k \cap ... \cap \beta_l^* \cap S_1^* \cap ... \cap S_e^* \cap T_1^* \cap ... T_f^* \cap U_1^* \cap ... \cap U_g^* \cap W_1^* \cap ... \cap W_h^* |$ each $*$ is empty or $c\}$

We add the following step to procedure to create a canonical model.

- If $\{R\}$ does not satisfy the G-condition and $G_1 \subseteq \{R\}$ is a cause, then we consider labels $c_1, ...c_g, d_1, ..., d_{h-1}$ that do not appear in $\{R\}_i\Gamma_i \Rightarrow \Delta_i$ and define $\Gamma_{i+1} = \Gamma_i$, $\Delta_{i+1} = \Delta_i$ and $\{R\}_{i+1} = \{R, G_2\}_i$. $\{R\}_{i+1}\Gamma_{i+1} \Rightarrow \Delta_{i+1}$ is not provable because of rule ($\{G\}$-con).

Different from the case of **LSIML**, there is a possibility that the procedure does not end at some point. Therefore, the final sequent $\{R\}_c\Gamma_c \Rightarrow \Delta_c$ is defined as $\{R\}_c = \bigcup_{i=0}^{\infty}\{R\}_i$, $\Gamma_c = \bigcup_{i=0}^{\infty}\Gamma_i$ and $\Delta_c = \bigcup_{i=0}^{\infty}\Delta_i$.

Theorem 7 (Cut-free completeness theorem for LSIML+ ($\{G\}$-con)). *If $\{R\}\Gamma \Rightarrow \Delta$ is valid in all G-conditional models, then $\{R\}\Gamma \Rightarrow \Delta$ is provable in* **LSIML** *+ ($\{G\}$-con) without using the rule (cut).*

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