Hypersequent calculus for the logic of conditional belief: preliminary results

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Abstract

The logic of Conditional Beliefs (CDL) has been introduced by Board, Baltag and Smets to reason about knowledge and revisable beliefs in a multi-agent setting. Our aim is to develop standard internal calculi for this logic. As a preliminary result we propose an internal hypersequent calculus for the logic in the single agent case.

1 Introduction

Knowledge and belief are the most important propositional attitudes to reason about epistemic interaction among agents. Conditional Doxastic Logic (CDL) has been proposed by Board [4] and Baltag and Smets [2, 1, 3] for modelling both belief and knowledge in a multi-agent setting (refer to [9] for a survey). Differently from knowledge, the essential feature of beliefs is that they are *revisable* whenever the agent learns new information. To capture the revisable nature of beliefs, CDL contains the conditional belief operator $Bel(C|B)$, the meaning of which is that agent i would believe C in case she learnt B. Both unconditional beliefs and knowledge can be defined in CDL : $BelB$ (agent i believes B) as $Bel(B|\top)$, K_iB (agent i knows B) as $Bel(\bot|\neg B)$, the latter meaning that i considers impossible (inconsistent) to learn $\neg B$.

The logic of conditional belief has been significantly employed in game theory [10], and it has been used as the basic formalism to study further dynamic extensions of epistemic logics, determined by several kinds of epistemic/doxastic actions. Not surprisingly, the axiomatization of the operator Bel in CDL internalises the well-known AGM postulates of belief revision.

The original semantics of this logic is defined in terms of the so-called Plausibility Models: these are standard epistemic models, where each agent is further equipped by a "comparative plausibility" relation between worlds needed to evaluate her own (conditional) beliefs. However, as shown in [6], it is possible to give an alternative semantics of this logic in terms of multi-agent neighbourhood models, which are essentially a multi-agent version of Lewis' sphere models for counterfactual logics. In particular, it turns out that the semantics of CDL coincides with a multi-agent version of Lewis' logic $\mathbb{VTA}$.

From a proof-theoretical point of view the logic CDL has not been studied very much, the only existing calculus for it being the one given in [6]: a labelled sequent calculus based on the neighbourhood semantics mentioned above.

Our aim here is to design an internal calculus for this logic. By an internal calculus we mean a calculus where the syntactic structures (sequents) can be directly interpreted as formulas of CDL. An internal calculus for CDL would be particularly significant,

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as the logic admits two rather different semantics, and the internal calculus would be independent from the choice of the semantics.

As a preliminary result, we treat here the single-agent case: we develop an internal hypersequent calculus for single agent CDL. The calculus we propose is similar to the hypersequent calculi given given in [5] for Lewis' counterfactual logics with both Uniformity and Reflexivity, more precisely to the calculus for the logic $\mathbb{VTA}$. In both cases, the idea is to take as primitive the operator of comparative plausibility $A \preccurlyeq B$ (meaning: "A is at least as plausible as B"), in terms of which all the others operators can been defined.

The calculus presented here is different from the calculus in [5], due to the fact that the Absoluteness condition replaces Uniformity. Moreover, the calculus contains rules for all epistemic operators. We shall give a semantical proof of completeness for the calculus, a result that is missing in [5] for the corresponding logic $\mathbb{VTA}$.

We briefly discuss how the calculus could be possibly extended to the multi-agent setting, a non-trivial task that will be the object of our future research.

2 Logic CDL

The language of (mono-agent) CDL extends ordinary propositional logic with operators for (conditional) belief, knowledge, and comparative plausibility.

Definition 2.1 (Language)**.** Formulas of CDL are generated as follows, for P propositional formula:

$$A ::= P \mid \neg A \mid A \to B \mid A \preccurlyeq B \mid Bel(A|A)$$

A conditional belief $Bel(C|B)$ is read "the agent believes C, given B". The meaning of a formula $A \preccurlyeq B$ is that the agent considers A at least as plausible as B. The operators of Bel and $\preccurlyeq$ are interdefinable:

$$Bel(B|A) \;\equiv\; (\bot \preccurlyeq A) \vee \neg((A \wedge \neg B) \preccurlyeq (A \wedge B))$$

$$A \preccurlyeq B \;\equiv\; (Bel(\bot|A \vee B) \vee \neg Bel(\neg A|A \vee B)$$

Intuitively, by the first equivalence, an agent conditionally believes B given A whenever she considers A impossible or she considers that $A \wedge \neg B$ is less plausible than $A \wedge B$. As mentioned in the introduction, unconditional belief and knowledge may be defined in terms of conditional belief:

$BelA =_{def} Bel(A|\top)$ (belief)
$KA =_{def} Bel(\bot|\neg A)$ (knowledge)

If we take as primitive the operator of conditional belief, we can give an axiomatisation of CDL as follows [4, 3]:

(1) If $\vdash B$, then $\vdash Bel(B|A)$
(2) If $\vdash A \supset\subset B$, then $\vdash Bel(C|A) \supset\subset Bel(C|B)$
(3) $(Bel(B|A) \wedge Bel(B \supset C|A)) \supset Bel(C|A)$
(4) $Bel(A|A)$
(5) $Bel(B|A) \supset (Bel(C|A \wedge B) \supset\subset Bel(C|A))$
(6) $\neg Bel(\neg B|A) \supset (Bel(C|A \wedge B) \supset\subset Bel(B \supset C|A))$
(7) $Bel(B|A) \supset Bel(Bel(B|A)|C)$
(8) $\neg Bel(B|A) \supset Bel(\neg Bel(B|A)|C)$
(9)$A \supset \neg Bel(\bot|A)$

On the other hand, we can give an alternative of CDL is an equivalent axiomatization taking $\preccurlyeq$ as primitive [5], the axiomatization coincides with the one for Lewis' system $\mathbb{VTA}$:

CPR $\dfrac{\vdash B \rightarrow A}{\vdash A \preccurlyeq B}$

CPA $(A \preccurlyeq A \vee B) \vee (B \preccurlyeq A \vee B)$

TR $(A \preccurlyeq B) \wedge (B \preccurlyeq C) \rightarrow (A \preccurlyeq C)$

CO $(A \preccurlyeq B) \vee (B \preccurlyeq A)$

N $\neg(\bot \preccurlyeq \top)$

T $(\bot \preccurlyeq \neg A) \rightarrow A$

A1 $(A \preccurlyeq B) \rightarrow \big(\bot \preccurlyeq \neg(A \preccurlyeq B)\big)$

A2 $\neg(A \preccurlyeq B) \rightarrow \big(\bot \preccurlyeq (A \preccurlyeq B)\big)$

The semantics of CDL is defined in terms of plausibility models, as recalled in the introduction; however, an alternative semantics can be given in terms of neighbourhood models as defined in [6]. We give here the definition for the single-agent case.

Definition 2.2. A *neighbourhood model* has the form $\mathcal{M} = \langle W, S, [\![\ ]\!] \rangle$ where W is a non empty set of elements; S is a function $S_i : W \rightarrow \mathcal{P}(\mathcal{P}(W))$, and $[\![\ ]\!] : Atm \rightarrow \mathcal{P}(W)$ is the propositional evaluation. Moreover, S satisfies the following properties:

- *Non-emptiness*: $\forall \alpha \in S(x)$ it holds that $\alpha \neq \emptyset$;
- *Nesting*: $\forall \alpha, \beta \in S(x)$ it holds that $\alpha \subseteq \beta$ or $\beta \subseteq \alpha$;
- *Total reflexivity*:[1] $\exists \alpha \in S(x)$ such that $x \in \alpha$;
- *Absoluteness*: $\forall x, y \in W$ it holds that $S(x) = S(y)$.

The truth conditions for Boolean combinations of formulas are the standard ones; to state truth conditions for conditional belief, unconditional (simple) belief and knowledge we use the local forcing notation introduced in [8]: $\alpha \Vdash^{\forall} A$ iff $\forall y \in \alpha.\ y \Vdash A$ and $\alpha \Vdash^{\exists} A$ iff $\exists y \in \alpha.\ y \Vdash A$:

$x \Vdash Bel(B|A)$ iff $\forall \alpha \in S(x).\quad \alpha \Vdash^{\forall} \neg A$ or $\exists \beta \in S(x).\quad \beta \Vdash^{\exists} A$ and $\beta \Vdash^{\forall} A \supset B$
$x \Vdash BelB$ iff $\exists \beta \in S(x).\beta \Vdash^{\forall} B$
$x \Vdash KB$ iff $\forall \beta \in S(x).\beta \Vdash^{\forall} B$

A formula A is *valid* in $\mathcal{M}$ if for all $w \in W$, $w \Vdash A$. We say that A *is valid in the class of neighbourhood models* if A is valid in every multi-agent neighbourhood model. The operators of unconditional belief and knowledge correspond to standard modalities in neighbourhood models, in particular knowledge modality satisfies standard S5 axioms and unconditional beliefs standard K45D axioms.

Observe that by condition Absoluteness, the definition can be simplified: S is a constant function, that is just a subset of $\mathcal{P}(W)$ since it does not depend on worlds. But we have kept the definition close to the multi-agent case, where to each agent i (within a set of agents $\mathcal{A}$) is associated a neighbourhood function $S_i : W \rightarrow \mathcal{P}(\mathcal{P}(W))$, which only satisfies the following weaker condition, called Local Absoluteness:

if $\alpha \in S(x)$ and $y \in \alpha$ then $S(x) = S(y)$

In the following we develop a calculus for the single-agent case, but the above generalisation must be kept in mind if we want to handle the multi-agent case.

3 Hypersequent calculus $\mathcal{H}_{\mathbb{VTA}}$

In this section we provide an hypersequent calculus for single-agent CDL. Hypersequents are disjunction of ordinary sequents. In this case the structure of hypersequents is

[1] Total reflexivity entails $\forall x \in W, S(x) \neq \emptyset$.

enriched with an additional part, called the *head*, similarly as in [7]. This component is used to store comparative plausibility formulas; in the single-agent case, these formulas are globally valid in a model. Moreover, plausibility formulas may interact with each other: for this reason they are structured in *blocks*, each block representing a disjunction of them.

Definition 3.1. Hypersequents have the following structure:

$$\Theta \mid\mid \Gamma_1 \Rightarrow \Delta_1 \mid \ \ldots \mid \Gamma_n \Rightarrow \Delta_n$$

where $\Gamma_1 \ldots \Gamma$ and $\Delta_1 \ldots \Delta_n$ are multisets of formulas, and Θ is a multiset of blocks. A block is a syntactic structure representing a disjunction of comparative plausibility formulas: $[A_1 \ldots A_n \lhd B]$ is interpreted as: $A_1 \preccurlyeq B \vee \cdots \vee A_n \preccurlyeq B$. Θ is called the *head* of the hypersequent; each $\Sigma_i \Rightarrow \Pi_i$ is called a *component* of the hypersequent.

Since we are in the single-agent case, the operator K plays the role of the universal modality, and the formula interpretation of hypersequents is given :

$$([\Sigma \lhd C_1], \ldots, [\Sigma \lhd C_k] \mid\mid \Gamma_1 \Rightarrow \Delta_1 \mid \ \ldots \mid \Gamma_n \Rightarrow \Delta_n)^i =$$

$$= \bigvee_{1 \leqslant i \leqslant k} \bigvee_{A \in \Sigma_i} (A \preccurlyeq C_i) \vee K(\bigwedge \Gamma_1 \Rightarrow \bigvee \Delta_1) \vee \cdots \vee K(\bigwedge \Gamma_n \Rightarrow \bigvee \Delta_n)$$

Basically, we store in the head of the hypersequents *only* comparative plausibility formulas which hold at all the world of the model. The components of the hypersequent store formulas which are true at some world "seen" from the actual world where, for x actual world, a world y is seen from x if there exists a neighbourhood $\alpha \in S(x)$ and $y \in \alpha$. Thus, for an hypersequent $\Theta \mid\mid \Gamma_1 \Rightarrow \Delta_1 \mid \ \ldots \mid \Gamma_n \Rightarrow \Delta_n$ to be valid in a model $\mathcal{M}$ at a world x, it must hold that either $x \Vdash A \preccurlyeq C$ for $A \in \Sigma$ and some block $[\Sigma \lhd C]$ occurring in Θ, *or* $x \nVdash F$ for some F occurring in Γ_i, *or* $x \Vdash G$ for some G occurring in Δ_i, for $1 \leqslant i \leqslant n$.
The rules of the calculus are given in figure 1. The calculus $\mathcal{H}_{\mathbb{VTA}}$ can be extended with the rules for knowledge and simple belief:

$$\frac{\Theta \mid\mid \neg A \preccurlyeq \top, \Sigma \Rightarrow \Pi \mid \mathcal{G}}{\Theta \mid\mid \Sigma \Rightarrow \Pi, Bel(A) \mid \mathcal{G}}\ \mathsf{RS} \qquad \frac{\Theta, [\neg A \lhd \top] \mid\mid Bel(A), \Sigma \Rightarrow \Pi \mid \mathcal{G}}{\Theta \mid\mid Bel(A), \Sigma \Rightarrow \Pi \mid \mathcal{G}}\ \mathsf{LS}$$

$$\frac{\Theta, [\bot \lhd \neg A] \mid\mid \Sigma \Rightarrow \Pi \mid \mathcal{G}}{\Theta \mid\mid \Sigma \Rightarrow \Pi, K(A) \mid \mathcal{G}}\ \mathsf{RK} \qquad \frac{\Theta \mid\mid \bot \preccurlyeq \neg A, \Sigma \Rightarrow \Pi \mid \mathcal{G}}{\Theta \mid\mid K(A), \Sigma \Rightarrow \Pi \mid \mathcal{G}}\ \mathsf{LK}$$

It is easy to check the above rules are admissible in $\mathcal{H}_{\mathbb{VTA}}$; for this reason, they will not be further considered.

Example 3.1. Derivation of Axiom (T): $(\bot \preccurlyeq \neg A) \rightarrow A$

$$\dfrac{\dfrac{\dfrac{[\bot \lhd \bot] \mid\mid \ \Rightarrow A \mid \bot \Rightarrow \bot}{[\bot \lhd \bot] \mid\mid \ \Rightarrow A}\ \mathsf{jump} \qquad \dfrac{\mid\mid \bot \preccurlyeq \neg A, A \Rightarrow A}{\mid\mid \bot \preccurlyeq \neg A \Rightarrow A, \neg A}\ R\neg}{\mid\mid \bot \preccurlyeq \neg A \Rightarrow A}\ \mathsf{T}}{\mid\mid \ \Rightarrow \bot \preccurlyeq \neg A) \rightarrow A}\ R\rightarrow$$

Theorem 3.1 (Soundness)**.** If an hypersequent $\mathcal{G}$ is derivable in $\mathcal{H}_{\mathbb{VTA}}$ then $(\mathcal{G})^i$ is valid.

Proof. By induction on the height of the derivation of the premisses, assuming that the premisses are valid in all $\mathbb{VTA}$ sphere models, while the conclusion is not. We show only the case of rule T.
Rule T: the conclusion is false at model $\mathcal{M}$ and world x. Thus, we have:

Initial sequents

$$\Theta \,||\, A, \Sigma \Rightarrow \Pi, A \,|\, \mathcal{G} \qquad\qquad \Theta \,||\, \bot \Sigma \Rightarrow \Pi \,|\, \mathcal{G}$$

Rules for the head

$$\frac{\Theta, [A \lhd B] \,||\, \Sigma \Rightarrow \Pi \,|\, \mathcal{G}}{\Theta \,||\, \Sigma \Rightarrow \Pi, A \preccurlyeq B \,|\, \mathcal{G}} \; \mathsf{H}_R \qquad\qquad \frac{\Theta, [\Sigma \lhd C] \,||\, \mathcal{G} \,|\, C \Rightarrow \Sigma}{\Theta, [\Sigma \lhd C] \,||\, \mathcal{G}} \; \mathsf{jump}$$

$$\frac{\Theta, [B, \Sigma \lhd C] \,||\, A \preccurlyeq B, \Sigma \Rightarrow \Pi \,|\, \mathcal{G} \quad \Theta, [\Sigma \lhd C], [\Sigma \lhd A] \,||\, A \preccurlyeq B, \Sigma \Rightarrow \Pi \,|\, \mathcal{G}}{\Theta, [\Sigma \lhd C] \,||\, A \preccurlyeq B, \Sigma \Rightarrow \Pi \,|\, \mathcal{G}} \; \mathsf{H}_L$$

$$\frac{\Theta, [\Sigma, \Pi \lhd A], [\Pi \lhd B] \,||\, \mathcal{G} \quad \Theta, [\Sigma \lhd A], [\Sigma, \Pi \lhd B] \,||\, \mathcal{G}}{\Theta, [\Sigma \lhd A], [\Pi \lhd B] \,||\, \mathcal{G}} \; \mathsf{com}$$

Rule for reflexivity

$$\frac{\Theta, [\bot \lhd A] \,||\, A \preccurlyeq B, \Gamma_1 \Rightarrow \Delta_1 \,|\, \ldots \,|\, \Gamma_n \Rightarrow \Delta_n \quad \Theta \,||\, A \preccurlyeq B, \Gamma_1 \Rightarrow \Delta_1, B \,|\, \ldots \,|\, \Gamma_n \Rightarrow \Delta_n, B}{\Theta \,||\, A \preccurlyeq B, \Gamma_1 \Rightarrow \Delta_1 \,|\, \Gamma_2 \Rightarrow \Delta_2 \,|\, \ldots \,|\, \Gamma_n \Rightarrow \Delta_n} \; \mathsf{T}$$

Propositional rules

$$\frac{\Theta \,||\, B, \Gamma \Rightarrow \Delta \,|\, \mathcal{G} \quad \Theta \,||\, \Gamma \Rightarrow \Delta, A \,|\, \mathcal{G}}{\Theta \,||\, A \rightarrow B, \Gamma \Rightarrow \Delta \,|\, \mathcal{G}} \; \mathsf{L}\rightarrow \qquad\qquad \frac{\Theta \,||\, A, \Gamma \Rightarrow \Delta, B \,|\, \mathcal{G}}{\Theta \,||\, \Gamma \Rightarrow \Delta, A \rightarrow B \,|\, \mathcal{G}} \; \mathsf{R}\rightarrow$$

Rules for conditional belief

$$\frac{\Theta \,||\, \bot \preccurlyeq A, Bel(B|A), \Sigma \Rightarrow \Pi \mid \mathcal{G} \quad \Theta, [A \wedge \neg B \lhd A] \,||\, Bel(B|A), \Sigma \Rightarrow \Pi \mid \mathcal{G}}{\Theta \,||\, Bel(B|A), \Sigma \Rightarrow \Pi \mid \mathcal{G}} \; \mathsf{LB}$$

$$\frac{\Theta, [\bot \lhd A] \,||\, A \wedge \neg B \preccurlyeq A, \Sigma \Rightarrow \Pi \mid \mathcal{G}}{\Theta \,||\, \Sigma \Rightarrow \Pi, Bel(B|A) \mid \mathcal{G}} \; \mathsf{RB}$$

Figure 1: Hypersequent calculus $\mathcal{H}_{\mathbb{VTA}}$

1. for all $y \in W$, for all $T \in \Theta$, $y \not\Vdash T$;
2. $x \Vdash K(A \preccurlyeq B)$, that is $\forall \beta \in S(x)(\beta \Vdash^{\forall} A \preccurlyeq B)$; thus, for all $y \in \beta$ it holds that $\forall \alpha \in S(y)(\alpha \Vdash^{\exists} B \rightarrow \alpha \Vdash^{\exists} A)$;
3. for all $G \in \{\Gamma_1, \ldots, \Gamma_n\}$, $x \Vdash K(G)$, that is $\forall \beta \in S(x)(\beta \Vdash^{\forall} G)$;
4. for all $D \in \{\Delta_1, \ldots, \Delta_n\}$, $x \Vdash K(D)$, that is $\forall \beta \in S(x)(\beta \Vdash^{\forall} D)$.

Since the premisses are valid, it holds that:

5. for all $y \in W$, $y \Vdash \bot \preccurlyeq A$ (from validity of the left premiss, 1, 2, 3 and 4); thus, $\forall \beta \in S(y)(\beta \Vdash^{\forall} \bot \preccurlyeq A)$, i.e. for all $y \in \beta$ it holds that $\forall \alpha \in S(y)(\alpha \Vdash^{\exists} A \rightarrow \alpha \Vdash^{\exists} \bot)$;
6. $x \Vdash K(B)$, that is $\forall \beta \in S(x)(\beta \Vdash^{\forall} B)$ (from validity of the second premiss, 1, 2, 3 and 4).

By total reflexivity we have that $\exists \alpha \in S(x)$ such that $x \in \alpha$. From 6 we have $\alpha \Vdash^{\forall} B$, which implies $\alpha \Vdash^{\exists} B$, since α is not empty. By 2 and absoluteness, we have $\alpha \Vdash^{\exists} A$. By 5 and absoluteness, we have $\alpha \Vdash^{\exists} \bot$, and we reached a contradiction. □

4 Completeness

In this section, we prove the semantic completeness of $\mathcal{H}_{\mathbb{VTA}}$. To this purpose we introduce a cumulative version of the calculus, which keeps all formulas and blocks in the premisses. We call this cumulative calculus $\mathcal{H}^{i}_{\mathbb{VTA}}$, its rules are given in Figure 2.

Lemma 4.1. The structural rules of weakening on formulas are admissible in $\mathcal{H}_{\mathbb{VTA}}$. Also the following rules of weakening on blocks are admissible in $\mathcal{H}_{\mathbb{VTA}}$.

$$\frac{\Theta \,||\, \Gamma \Rightarrow \Delta \,|\, \mathcal{G}}{\Theta, [\Sigma \lhd C] \,||\, \Gamma \Rightarrow \Delta \,|\, \mathcal{G}} \; \mathsf{WkB} \qquad \frac{\Theta, [\Sigma \lhd C] \,||\, \Gamma \Rightarrow \Delta \,|\, \mathcal{G}}{\Theta, [A, \Sigma \lhd C] \,||\, \Gamma \Rightarrow \Delta \,|\, \mathcal{G}} \; \mathsf{WkI}$$

Lemma 4.2. The structural rules of contraction on formulas are admissible in $\mathcal{H}^i_{\mathbb{VTA}}$. The rules of contraction on blocks are admissible in $\mathcal{H}^i_{\mathbb{VTA}}$:

$$\frac{\Theta, [\Sigma \lhd C], [\Sigma \lhd C] \,||\, \Gamma \Rightarrow \Delta \,|\, \mathcal{G}}{\Theta, [\Sigma \lhd C] \,||\, \Gamma \Rightarrow \Delta \,|\, \mathcal{G}} \; \mathsf{CtrB} \qquad \frac{\Theta, [A, A, \Sigma \lhd C] \,||\, \Gamma \Rightarrow \Delta \,|\, \mathcal{G}}{\Theta, [A, \Sigma \lhd C] \,||\, \Gamma \Rightarrow \Delta \,|\, \mathcal{G}} \; \mathsf{CtrI}$$

Proofs of both lemmas are by standard induction on the height of the derivation. From the two lemmas immediately follows that:

Proposition 4.3. The calculi $\mathcal{H}_{\mathbb{VTA}}$ and $\mathcal{H}^i_{\mathbb{VTA}}$ are equivalent.

Initial sequents

$$\Theta \,||\, A, \Gamma \Rightarrow \Delta, A \,|\, \mathcal{G} \qquad\qquad \Theta \,||\, \bot, \Gamma \Rightarrow \Delta \,|\, \mathcal{G}$$

Rules for the head

$$\frac{\Theta, [A \lhd B] \,||\, \Gamma \Rightarrow \Delta, A \preccurlyeq B \,|\, \mathcal{G}}{\Theta \,||\, \Gamma \Rightarrow \Delta, A \preccurlyeq B \,|\, \mathcal{G}} \; \mathsf{H}_R \qquad \frac{\Theta, [\Sigma \lhd C] \,||\, \mathcal{G} \,|\, C \Rightarrow \Sigma}{\Theta, [\Sigma \lhd C] \,||\, \mathcal{G}} \; \mathsf{jump}$$

$$\frac{\Theta, [B, \Sigma \lhd C], [\Sigma \lhd C] \,||\, A \preccurlyeq B, \Gamma \Rightarrow \Delta \,|\, \mathcal{G} \quad \Theta, [\Sigma \lhd A], [\Sigma \lhd C] \,||\, A \preccurlyeq B, \Gamma \Rightarrow \Delta \,|\, \mathcal{G}}{\Theta, [\Sigma \lhd C] \,||\, A \preccurlyeq B, \Gamma \Rightarrow \Delta \,|\, \mathcal{G}} \; \mathsf{H}_L$$

$$\frac{\Theta, [\Sigma, \Pi \lhd A], [\Sigma \lhd A], [\Pi \lhd B] \,||\, \mathcal{G} \quad \Theta, [\Sigma \lhd A], [\Sigma, \Pi \lhd B], [\Pi \lhd B] \,||\, \mathcal{G}}{\Theta, [\Sigma \lhd A], [\Pi \lhd B] \,||\, \mathcal{G}} \; \mathsf{com}$$

Rule for reflexivity

$$\frac{\Theta, [\bot \lhd A] \,||\, A \preccurlyeq B, \Gamma_1 \Rightarrow \Delta_1 \,|\, \dots \,|\, \Gamma_n \Rightarrow \Delta_n \quad \Theta \,||\, A \preccurlyeq B, \Sigma_1 \Rightarrow \Pi_1, B \,|\, \dots \,|\, \Sigma_n \Rightarrow \Pi_n, B}{\Theta \,||\, A \preccurlyeq B, \Gamma_1 \Rightarrow \Delta_1 \,|\, \Gamma_2 \Rightarrow \Delta_2 \,|\, \dots \,|\, \Gamma_n \Rightarrow \Delta_n} \; \mathsf{T}$$

Propositional rules

$$\frac{\Theta \,||\, A \to B, B, \Gamma \Rightarrow \Delta \,|\, \mathcal{G} \quad \Theta \,||\, A \to B, \Gamma \Rightarrow \Delta, A \,|\, \mathcal{G}}{\Theta \,||\, A \to B, \Gamma \Rightarrow \Delta \,|\, \mathcal{G}} \; \mathsf{L}\to \qquad \frac{\Theta \,||\, A, \Gamma \Rightarrow \Delta, A \to B, B \,|\, \mathcal{G}}{\Theta \,||\, \Gamma \Rightarrow \Delta, A \to B \,|\, \mathcal{G}} \; \mathsf{R}\to$$

Rules for conditional belief

$$\frac{\Theta \,||\, \bot \preccurlyeq A, Bel(B|A), \Gamma \Rightarrow \Delta \,|\, \mathcal{G} \quad \Theta, [A \wedge \neg B \lhd A] \,||\, Bel(B|A), \Gamma \Rightarrow \Delta \,|\, \mathcal{G}}{\Theta \,||\, Bel(B|A), \Gamma \Rightarrow \Delta \,|\, \mathcal{G}} \; \mathsf{LB}$$

$$\frac{\Theta, [\bot \lhd A] \,||\, A \wedge \neg B \preccurlyeq A, \Gamma \Rightarrow \Delta, Bel(B|A) \,|\, \mathcal{G}}{\Theta \,||\, \Gamma \Rightarrow \Delta, Bel(B|A) \,|\, \mathcal{G}} \; \mathsf{RB}$$

Figure 2: Hypersequent calculus $\mathcal{H}^i_{\mathbb{VTA}}$

Definition 4.1 (Saturated hypersequent)**.** Let $\mathcal{G}$ be a $\mathcal{H}^i_{\mathbb{VTA}}$ hypersequent of the form $\Theta \,||\, \Sigma_1 \Rightarrow \Pi_1 \,|\, \dots \,|\, \Sigma_n \Rightarrow \Pi_n$. $\mathcal{G}$ is *saturated* if it satisfies the following conditions:

1. (H_R) if $\Gamma \Rightarrow A \preccurlyeq B \in \mathcal{G}$, then $[\Sigma, A \lhd B] \in \Theta$, for some Σ;
2. (HL) if $[\Sigma \lhd C] \in \Theta$ and $A \preccurlyeq B, \Gamma \Rightarrow \Delta \in \mathcal{G}$, then either $B \in \Sigma$ or $[\Sigma \lhd A] \in \Theta$;
3. (com) if $[\Sigma \lhd A], [\Sigma \lhd B] \in \mathcal{G}$, then $\Sigma \subseteq \Pi$ or $\Pi \subseteq \Sigma$;
4. (jump) if $[\Sigma \lhd C] \in \Theta$, then $C, \Lambda \Rightarrow \Omega, \Sigma \in \mathcal{G}$;
5. (T) if $A \preccurlyeq B, \Gamma \Rightarrow \Delta \in \mathcal{G}$, then either $[\bot \lhd A] \in \Theta$ or $\Lambda \Rightarrow \Omega, B$ for all components of $\mathcal{G}$;
6. ($\mathsf{L}\to$) if $A \to B, \Gamma \Rightarrow \Delta \in \mathcal{G}$, then $B \in \Gamma$ or $A \in \Delta$;
7. ($\mathsf{R}\to$) if $\Gamma \Rightarrow \Delta, A \to B \in \mathcal{G}$, then $A \in \Gamma$ and $B \in \Delta$;
8. (LB) if $Bel(B|A), \Gamma \Rightarrow \Delta \in \mathcal{G}$, then $\bot \preccurlyeq A \in \Gamma$ or $[A \wedge \neg B \lhd A] \in \Theta$;

9. (RB) if $\Gamma \Rightarrow \Delta, Bel(B|A) \in \mathcal{G}$, then $[\bot \lhd A] \in \Theta$ and $A \wedge \neg B \preccurlyeq A \in \Gamma$.

We say a $\mathcal{H}^i_{\mathbb{VTA}}$ hypersequent to be *unprovable* if the hypersequent is saturated and it is not an initial sequent. Let $\mathcal{G}$ be such an unprovable hypersequent, of the form $\Theta \,||\, \Sigma_1 \Rightarrow \Pi_1 \,|\, \dots \,|\, \Sigma_n \Rightarrow \Pi_n$. We construct a countermodel for it as follows:

- $W := \{1, \dots, n\}$
- $[\![p]\!] := \{i \leqslant n : p \in \Gamma_i\}$

So, we introduce a world for each component, satisfying atomic formulas occurring in the antecedent of the component. For the condition of absoluteness, the system of neighbourhood is the same for all the worlds. Thus, for each $i \leqslant n$, we define the same system of neighbourhood $S(i)$, looking at the blocks occurring in the head of the hypersequent:

$$\Theta = [\Sigma_1 \lhd C_1], \dots, [\Sigma_n \lhd C_n]$$

By the saturation condition associated to com, $\Sigma_1 \subseteq \Sigma_2 \subseteq \dots \subseteq \Sigma_n$. By the saturation condition associated to jump, for every $j \leqslant k$ there is a component m_j such that $\Lambda_{m_j} \Rightarrow \Omega_{m_j} \in \mathcal{G}$, and $A_j \in \Lambda_{m_j}$ and $\Sigma_j \subseteq \Omega_{m_j}$. Thus, for any $i \leqslant k$, set:

$$S(i) := \{\{m_k\}, \{m_k, m_{k-1}\}, \dots, \{m_k, \dots, m_1\}, W\}$$

Call the resulting structure $\mathscr{M}_{\mathcal{G}}$.

Lemma 4.4. For a saturated hypersequent $\mathcal{G}$ the structure $\mathscr{M}_{\mathcal{G}}$ is a neighbourhood model.

Proof. Nesting of neighbourhoods holds in $\mathscr{M}_{\mathcal{G}}$, since $\{m_k\} \subseteq \{m_k, m_{k-1}\} \subseteq \dots \subseteq \{m_k, \dots, m_1\} \subseteq W$. Reflexivity holds, since $W \in S(i)$, for all $i \in W$. Absoluteness holds, since for every $v, w \in W$ it holds that $S(v) = S(w)$. □

Definition 4.2. The complexity of a formula is defined as follows:

- $w(P) = w(\bot) = 1$;
- $w(A \circ B) = w(A) + w(B) + 1$ for $\circ$ implication and comparative plausibility;
- $w(Bel(B|A)) = w(A) + w(B) + 2$.

Lemma 4.5. Let $\mathcal{G}$ be an unprovable hypersequent, as defined in 4.1, and let $\mathscr{M}_{\mathcal{G}}$ be a $\mathbb{VTA}$ neighbourhood model as defined above. For all $i \leqslant n$ world of the countermodel, corresponding to a component $\Gamma_i \Rightarrow \Delta_i \in \mathcal{G}$, we show that:

1. for all $A \in \Gamma_i$, $\mathscr{M}_{\mathcal{G}}, i \Vdash A$;
2. for all $A \in \Delta_i$, $\mathscr{M}_{\mathcal{G}}, i \nVdash A$;
3. for all $[\Sigma \lhd C] \in \Theta$, $\mathscr{M}_{\mathcal{G}}, i \nVdash \bigvee_{A \in \Sigma}(A \preccurlyeq C)$.

Proof. Cases 1 and 2 are proved by mutual induction on the complexity of the formula A. Propositional cases are immediate; we consider only cases $A \equiv E \preccurlyeq F$ and $A \equiv Bel(F|E)$.

- Suppose $E \preccurlyeq F \in \Gamma_i$. For all $\alpha \in S(i)$, we have to show that $\alpha \nVdash^{\forall} \neg F$ or $\alpha \Vdash^{\exists} E$. In case $\alpha \neq W$ we have $\alpha = \{m_k, \dots, m_j\}$, for some $j \leqslant k$. Each $m_l \in \alpha$ comes from a component $C_l, \Lambda_l \Rightarrow \Omega_l, \Sigma_l$ generated from a block $[\Sigma_l \lhd C_l] \in \Theta$. Let us consider m_j, and its associated component $C_j, \Lambda_j \Rightarrow \Omega_j, \Sigma_j$. For the saturation condition associated to H_L, either $F \in \Gamma_j$ or $E = C_j$. In the former case we have that $\Sigma_j \subseteq \dots \subseteq \Sigma_k$; by inductive hypothesis $m_i \nVdash F$, for $i = \{j, \dots, k\}$. Thus, $\alpha \Vdash^{\forall} \neg F$. Otherwise, if $E = C_j$, let us consider the component $C_j, \Lambda_j \Rightarrow \Omega_j, \Sigma_j$. By inductive hypothesis, $\mathscr{M}_{\mathcal{G}}, m_j \Vdash^{\exists} E$; thus, $\alpha \Vdash^{\exists} E$.

In case $\alpha = W$, by the saturation condition associated to T we have that either $[\bot \lhd E] \in \Theta$ or $F \in P_i$ for all components $\Sigma_i \Rightarrow \Pi_i \in \mathcal{G}$. In the former case, by the saturation condition associated to jump there will be a component $E, \Lambda_j \Rightarrow \Omega_j, \bot$; thus, by inductive hypothesis $m_j \Vdash^{\exists} E$, and $W \Vdash^{\exists} E$. In the latter case, we get by inductive hypothesis that for all $m_i \in W$, $m_j \nVdash^{\exists} F$; thus,$W \Vdash^{\forall} \neg F$.

- Suppose $E \preccurlyeq F \in \Delta_i$. For some $\alpha \in S(i)$, we have to show that $\alpha \nVdash^{\forall} \neg F$ and $\alpha \nVdash^{\exists} E$. Recall that $S(i) := \{\{m_k\}, \{m_k, m_{k-1}\}, \ldots, \{m_k, \ldots, m_1\}, W\}$, and that to each m_l is associated a component $C_l, \Lambda_l \Rightarrow \Omega_l, \Sigma_l \in \mathcal{G}$ generated from a block $[\Sigma_l \lhd C_l] \in \Theta$. By saturation condition associated to $\mathsf{R}\preccurlyeq$, there is a $j \leqslant k$ such that $C_j = F$ and $E \in \Gamma_j$. Consider the world m_j associated to the component $C_j, \Lambda_j \Rightarrow \Omega_j, \Sigma_j$. By inductive hypothesis it holds that $m_j \Vdash^{\exists} F$; moreover, since $\Sigma_j \subseteq \Sigma_j + 1 \subseteq \cdots \subseteq \Sigma_k$, it holds by inductive hypothesis that $m_i \nVdash^{\exists} E$ for $i = \{j, \ldots, k\}$. Thus for $\alpha = \{m_j, \ldots, m_k\}$ we get that $\alpha \nVdash^{\forall} \neg F$ and $\alpha \nVdash^{\exists} E$.
- Suppose $Bel(F|E) \in \Gamma_i$. For all $\alpha \in S(i)$, we have to show that either $\alpha \Vdash^{\forall} \neg E$ (first disjunct) or that there exists $\beta \in S(i)$ such that $\beta \Vdash^{\exists} A$ and $\beta \Vdash^{\forall} A \rightarrow B$ (second disjunct).
  By the saturation condition associated to LB, we have that either $\bot \preccurlyeq E \in \Gamma_i$ or $[E \wedge \neg F \lhd E] \in \Theta$. In the former case, by case a) and inductive hypothesis we have that $\mathscr{M}_{\mathcal{G}}, i \Vdash \bot \preccurlyeq E$, and thus for all $\alpha \in S(i)(\alpha \Vdash^{\exists} \neg E)$, and the first disjunct hold. In the latter case, by saturation there is a component $E, \Lambda_j \Rightarrow \Omega_j, E \wedge \neg F \in \mathcal{G}$. Let us consider $\alpha = \{m_k, \ldots, m_j\}$. By inductive hypothesis, $\alpha \Vdash^{\forall} E$. Moreover, by inductive hypothesis and since $\Sigma_j = \Omega_j, E \wedge \neg F$ is included in Σ_k, we have that $\alpha \nVdash^{\forall} \neg(E \wedge \neg F)$, thus $\alpha \Vdash^{\forall} E \rightarrow F$, and the second disjunct holds.
- Suppose $Bel(F|E) \in \Delta_i$. We have to show that there is a $\alpha \in S(i)$ such that $\alpha \Vdash^{\exists} E$ and for all $\beta \in S(i)$, $\beta \Vdash^{\exists} E$ and $\beta \Vdash^{\exists} E \vee \beta \nVdash^{\forall} (A \rightarrow B)$.
  By the saturation condition associated to RB, we have that $[\bot \lhd E] \in \Theta$ and $E \wedge \neg B \preccurlyeq E \in \Gamma_i$. From the former condition we have by saturation that there is a component $E, \Lambda_j \Rightarrow \Omega, \bot \in \mathcal{G}$; by inductive hypothesis, $m_j \Vdash E$, and for $\alpha = \{m_k, \ldots, m_j\}$, $\alpha \Vdash^{\exists} E$. From the latter condition we get, by inductive hypothesis and case a), that $\forall \alpha \in S(i)$ it holds that either $\alpha \Vdash^{\exists} E$ or $\alpha \Vdash^{\exists} E \wedge \neg F$, and we are done.

The proof of 3 uses 2, recalling the fact that a block is a disjunction of comparative plausibility formulas.

□

5 Towards a calculus for multi-agent CDL

Our aim is to extend the calculus to the multi-agent case. In the multi-agent case the semantics is different, since for each agent i the neighbourhood function S_i splits worlds into several equivalence classes. As a consequence plausibility formulas, whence all modal formulas, are no longer global. The situation is similar to the one of multi-agent S5 modal logic, which can be embedded into multi-agent CDL. In this case, it is easily seen that the structure of hypersequents is no longer sufficient, since (a) only plausibility formulas of the the same agent can interact, and (b) they may interact only if they are in the same equivalence class for that agent. Thus, we likely need a nested structure to capture the multi-agent version of the logic. On the other hand a labelled calculus for CDL has been given in [6]. We can adapt it to the language based on comparative plausibility (we only write the most significant rules), as shown in Figure 3.

In future work we aim at developing an internal calculus in the form of nested sequents and its relation with the labelled calculus given here. In particular we would like to devise a translation from the internal calculus into the labelled one.

Rules for local forcing

$$\frac{x \in a, \Gamma \Rightarrow \Delta, x : A}{\Gamma \Rightarrow \Delta, a \Vdash^{\forall} A} \; R\Vdash^{\forall} \; (x\,fresh) \qquad \frac{x : A, x \in a, a \Vdash^{\forall} A, \Gamma \Rightarrow \Delta}{x \in a, a \Vdash^{\forall} A, \Gamma \Rightarrow \Delta} \; L\Vdash^{\forall}$$

$$\frac{x \in a, \Gamma \Rightarrow \Delta, x : A, a \Vdash^{\exists} A}{x \in a, \Gamma \Rightarrow \Delta, a \Vdash^{\exists} A} \; R\Vdash^{\exists} \qquad \frac{x \in a, x : A, \Gamma \Rightarrow \Delta}{a \Vdash^{\exists} A, \Gamma \Rightarrow \Delta} \; L\Vdash^{\exists} \; (x\,fresh)$$

Rules for comparative plausibility

$$\frac{a \in S_i(x), a \Vdash^{\exists} B, \Gamma \Rightarrow \Delta, a \Vdash^{\exists} A}{\Gamma \Rightarrow \Delta, x : A \preccurlyeq_i B} \; RB \; (a\,fresh)$$

$$\frac{a \in S_i(x), x : A \preccurlyeq_i B, \Gamma \Rightarrow \Delta, a \Vdash^{\exists} B \quad a \in S_i(x), x : A \preccurlyeq_i B, a \Vdash^{\exists} A, \Gamma \Rightarrow \Delta}{a \in S_i(x), x : A \preccurlyeq_i B, \Gamma \Rightarrow \Delta} \; LB$$

Rules for inclusion

$$\frac{a \subseteq a, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} \; Ref \qquad \frac{c \subseteq a, c \subseteq b, b \subseteq a, \Gamma \Rightarrow \Delta}{c \subseteq b, b \subseteq a, \Gamma \Rightarrow \Delta} \; Tr$$

$$\frac{x \in a, a \subseteq b, x \in b, \Gamma \Rightarrow \Delta}{x \in a, a \subseteq b, \Gamma \Rightarrow \Delta} \; L\subseteq$$

Rules for semantic conditions

$$\frac{a \subseteq b, a \in S_i(x), b \in S_i(x), \Gamma \Rightarrow \Delta \quad b \subseteq a, a \in S_i(x), b \in S_i(x), \Gamma \Rightarrow \Delta}{a \in S_i(x), b \in S_i(x), \Gamma \Rightarrow \Delta} \; S$$

$$\frac{x \in a, a \in S_i(x), \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} \; T \; (a\,fresh)$$

$$\frac{a \in S_i(x), y \in a, b \in S_i(x), b \in S_i(y), \Gamma \Rightarrow \Delta}{a \in S_i(x), y \in a, b \in S_i(x), \Gamma \Rightarrow \Delta} \; A_1 \qquad \frac{a \in S_i(x), y \in a, a \in S_i(y), \Gamma \Rightarrow \Delta}{a \in S_i(x), y \in a, \Gamma \Rightarrow \Delta} \; A_2$$

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6 Conclusion

We have presented here an internal calculus for single-agent CDL in the form of an hypersequent calculus. We have shown that the calculus is semantically complete. This is a first step to the definition of an internal calculus for the multi-agent case, a step which will require an extension of the proof-theoretical framework and will be the object of our future research.

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