Proof Translations in BI Logic
- extended abstract -

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Abstract. In order to study proof translations in BI logic, we consider first the bunched sequent calculus LBI and then define a new labelled sequent calculus, called GBI. We propose a procedure that translates LBI proofs into GBI proofs and prove its soundness. Moreover we discuss some steps towards the reverse translation of GBI proofs into LBI proofs and propose an algorithm that is illustrated by some examples. Further work will be devoted to its correctness and also to the design of translations of LBI proofs to TBI (labelled-tableaux) proofs.

1 Introduction

Since last years there is an increasing amount of interest for logical systems that are resource sensitive. Among the so-called resource logics, we can mention Linear Logic (LL) [4] with its resource consumption interpretation, the Bunched Implications logic (BI) [9] with its resource sharing interpretation but also the order-aware non-commutative logic (NL) [1]. As specification logics, they allow to model features like interactions, resource distribution and mobility, non-determinism, sequentiality or coordination of entities. For instance, the BI logic has been used as an assertion language for mutable data structures [6] and in this context can be seen as the logical kernel of the so-called separation logics developed in the last years [6,10].

The study of the proof theory and the related proof search methods for these resource logics must take into account the difficulty to capture some specific management of resources or sets of resources through adequate structures (like bunches in BI or NL) and also the specific interactions between connectives (additive and multiplicative connectives in BI, commutative and non-commutative connectives in NL), namely the resource semantics of these logics. Among the various calculi defined for BI we can mention first a Hilbert-style calculus and a bunched sequent calculus, called LBI, [8,9], a labelled tableaux system TBI [3] based on labels and constraints. In the context of Boolean BI, we can also mention the design of display calculi [7] and of some labelled calculi [5].

In this work we aim at focusing on label-free and labelled systems for BI logic and their relationships and then make a first step towards connecting the LBI sequent calculus to labelled proof calculi for BI. As a starting point, we define and investigate a labelled single-conclusion sequent system, called GBI, and its relationship with the LBI calculus. Intuitively, GBI can be seen as a kind of intermediate calculus between TBI and LBI. It shares with TBI the machinery of labels and constraints and with LBI some properties like being a single-conclusion sequent system. Our main result here is a procedure to systematically translate LBI proofs to GBI proofs, using a translation of bunches to sets of labels, label constraints and labelled formulas. We also start to study the other direction of translation, i.e. going from GBI proofs to LBI proofs and then we propose an algorithm, called Reconstruction Algorithm, that is a procedure that, given a GBI proof, tries to build up a corresponding LBI proof. Although the algorithm is quite simple and works well in many cases, its correctness is not obvious and remains to be shown to complete this work. As future work we want to apply the results of this paper to the study of connections between the labelled tableaux system TBI designed for BI [3] and the bunched sequent system LBI.

2 The Logic BI

In this section, we give a short introduction to BI, the logic of bunched implications. We present Kripke resource models as semantics as well as the proof system LBI. For more details see [9].

2.1 Syntax and Sequent Calculus LBI

Definition 1. *We assume a countable set of propositional letters* $\mathsf{Prop} = \{p, q, r, \dots\}$. *The formulas of BI are given by the following grammar:*

$$A ::= p \mid \mathrm{I} \mid A \star A \mid A \mathrel{-\!\!*} A \mid \top \mid \bot \mid A \wedge B \mid A \rightarrow A \mid A \vee A$$

Definition 2 (Bunches). *Bunches are labelled rooted trees that are given by the following grammar:*

$$\Gamma ::= A \mid \varnothing_a \mid \Gamma ; \Gamma \mid \varnothing_m \mid \Gamma , \Gamma$$

Equivalence, $\equiv$, is given by commutative monoid equations for ‘,’and ‘;’, with units $\varnothing_m$ and $\varnothing_a$ resp., together with the substitution congruence for subbunches.

Definition 3 (The Sequent Calculus LBI). *The sequent system LBI derives sequents of the form $\Gamma \vdash A$, where Γ is a bunch and A is a formula. It consists of the axioms and rules given in figure 1.*

Definition 4. *A formula A is a* theorem *of* LBI *iff $\varnothing_a \vdash A$ or $\varnothing_m \vdash A$ is provable in* LBI.

If we consider the valid formula $(A \star (A \mathrel{-\!\!*} B)) \mathrel{-\!\!*} B$ then a proof of this formula in LBI is the following

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A}\,\mathrm{id} \qquad \dfrac{}{B \vdash B}\,\mathrm{id}}{A, A \mathrel{-\!\!*} B \vdash B}\,{\mathrel{-\!\!*}}_R}{A \star (A \mathrel{-\!\!*} B) \vdash B}\,{\star}_L}{\varnothing_m, A \star (A \mathrel{-\!\!*} B) \vdash B}\,\equiv}{\varnothing_m \vdash (A \star (A \mathrel{-\!\!*} B)) \mathrel{-\!\!*} B}\,{\mathrel{-\!\!*}}_R$$

Where the second last inference is due to the equivalence of the bunches $\varnothing_m, A \star (A \mathrel{-\!\!*} B)$ and $A \star (A \mathrel{-\!\!*} B)$ and the corresponding rule of LBI.

2.2 Semantics of BI

Among the various semantics that BI admits, we choose to use a monoid-based Kripke semantics with two operators to better reflect the structure of bunches.

Definition 5 (Kripke Resource Monoid). *A* Kripke resource monoid *(KRM) is a structure $\mathcal{M} = (M, \otimes, 1, \oplus, 0, \infty, \sqsubseteq)$ where $(M, \otimes, 1)$ and $(M, \oplus, 0)$ are commutative monoids, $\sqsubseteq$ is a preordering relation on M with greatest element ∞ and such that*

Axioms:

$$\frac{}{A \vdash A}\,\mathrm{id} \qquad \frac{\Gamma \vdash A}{\Delta \vdash A}\,\Gamma \equiv \Delta \qquad \frac{}{\Gamma(\bot) \vdash A}\,\bot_L$$

Logical Rules:

$$\frac{\Gamma(\varnothing_m) \vdash A}{\Gamma(\mathrm{I}) \vdash A}\,\mathrm{I}_L \qquad \frac{}{\varnothing_m \vdash \mathrm{I}}\,\mathrm{I}_R \qquad \frac{\Gamma(\varnothing_a) \vdash A}{\Gamma(\top) \vdash A}\,\top_L \qquad \frac{}{\varnothing_a \vdash \top}\,\top_R$$

$$\frac{\Gamma(A) \vdash C \qquad \Gamma(B) \vdash C}{\Gamma(A \vee B) \vdash C}\,(\vee_L) \qquad \frac{\Gamma \vdash A_i \ (i = 1, 2)}{\Gamma \vdash A_1 \vee A_2}\,(\vee_R)_i$$

$$\frac{\Gamma(A; B) \vdash C}{\Gamma(A \wedge B) \vdash C}\,(\wedge_L) \qquad \frac{\Gamma \vdash A \qquad \Delta \vdash B}{\Gamma; \Delta \vdash A \wedge B}\,(\wedge_R)$$

$$\frac{\Gamma(A; B) \vdash C}{\Gamma(A \wedge B) \vdash C}\,(\wedge_L) \qquad \frac{\Gamma \vdash A \qquad \Delta \vdash B}{\Gamma; \Delta \vdash A \wedge B}\,(\wedge_R)$$

$$\frac{\Delta \vdash A \qquad \Gamma(B) \vdash C}{\Gamma(A \to B; \Delta) \vdash C}\,(\to_L) \qquad \frac{\Gamma; A \vdash B}{\Gamma \vdash A \to B}\,(\to_R)$$

$$\frac{\Gamma(A, B) \vdash C}{\Gamma(A \star B) \vdash C}\,(\star_L) \qquad \frac{\Gamma \vdash A \qquad \Delta \vdash B}{\Gamma, \Delta \vdash A \star B}\,(\star_R)$$

$$\frac{\Delta \vdash A \qquad \Gamma(B) \vdash C}{\Gamma(A \mathrel{-\!\!*} B, \Delta) \vdash C}\,(\mathrel{-\!\!*}_L) \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \mathrel{-\!\!*} B}\,(\mathrel{-\!\!*}_R)$$

Structural Rules:

$$\frac{\Gamma(\Delta) \vdash A}{\Gamma(\Delta; \Delta) \vdash A}\,\mathrm{wk} \qquad \frac{\Gamma(\Delta; \Delta) \vdash A}{\Gamma(\Delta) \vdash A}\,\mathrm{ctr} \qquad \frac{\Gamma \vdash A \qquad \Delta(A) \vdash B}{\Delta(\Gamma) \vdash B}\,\mathrm{cut}$$

Fig. 1. Rules of LBI sequent calculus.

- *for all $m \in M$, $\infty \otimes m = \infty \oplus m = \infty$,*
- *for all $m, n \in M$, $m \sqsubseteq m \oplus n$ and $m \oplus m = m$,*
- *$\otimes$ and $\oplus$ are bifunctorial (compatible) with respect to $\sqsubseteq$,* i.e.*:*

$$\textit{if } m \sqsubseteq n \textit{ and } m' \sqsubseteq n' \textit{ , then } m \otimes m' \sqsubseteq n \otimes n' \textit{ and } m \oplus m' \sqsubseteq n \oplus n'$$

Definition 6 (Kripke Resource Interpretation). *A* Kripke resource interpretation *(KRI), is a function $[\![-]\!] : \mathsf{Fm} \longrightarrow \mathcal{P}(M)$ satisfying Kripke monotonicity:*

$$\textit{for all } m, n \in M \textit{ such that } m \sqsubseteq n : \quad m \in [\![p]\!] \Rightarrow n \in [\![p]\!].$$

and

$$\infty \in [\![p]\!] \qquad \textit{for any } p \in \mathsf{Prop}.$$

Definition 7 (Kripke Resource Model). *A* Kripke resource model *(KM) is a structure $\mathcal{K} = (\mathcal{M}, \models, [\![-]\!])$ where $\mathcal{M}$ is a Kripke resource monoid, $[\![-]\!]$ is a Kripke resource interpretation and $\models$ is a forcing relation such that*

- *$m \models p$ iff $m \in [\![p]\!]$*
- *$m \models \bot$ iff $m = \infty$*
- *$m \models \top$ always*
- *$m \models \mathrm{I}$ iff $1 \sqsubseteq m$*
- *$m \models A \vee B$ iff $m \models A$ or $m \models B$*
- *$m \models A \wedge B$ iff there exist n_1, n_2 in M such that $n_1 \oplus n_2 \sqsubseteq m$, $n_1 \models A$ and $n_2 \models B$*
- *$m \models A \to B$ iff for all n in M such that $n \models A$, we have $m \oplus n \models B$*
- *$m \models A \star B$ iff there exist n_1, n_2 in M such that $n_1 \otimes n_2 \sqsubseteq m$, $n_1 \models A$ and $n_2 \models B$*
- *$m \models A \multimap B$ iff for all n in M such that $n \models A$, we have $m \otimes n \models B$*

Although the forcing clauses for $\wedge$ and $\to$ in Definition 7 are stated using the $\oplus$ operator, it is easy to prove that they are equivalent to the more usual ones:

- $m \models A \wedge B$ iff $m \models A$ and $m \models B$
- $m \models A \to B$ iff for all n in M, if $m \sqsubseteq n$ and $n \models A$, then $n \models B$

Definition 8 (Validity). *A formula A is valid in the Kripke resource semantics iff $1 \models A$ in all Kripke resource models.*

2.3 Properties of the LBI calculus

The calculus LBI itself is rather well-understood. Here we present some results from the literature, including soundness and completeness, cut-elimination and discuss semi-distributivity, a particular derivable rule which will play an important part in the Reconstruction Algorithm in the last section.

Theorem 1. *The following equivalences hold for provability in* LBI*:*

- *$\varnothing_a \vdash A$ is provable iff $\top \vdash A$ is provable.*
- *$\varnothing_m \vdash A$ is provable iff $\mathrm{I} \vdash A$ is provable.*

Corollary 1. *A is a theorem of LBI iff $\varnothing_m \vdash A$ is provable in LBI.*

It is easy to show that we can replace the rule for the monoidal eqivalence of bunches $\Gamma \equiv \Delta$ with the six rules as given in figure 2. In the proof of the Translation Theorem 4, we will prefer to use these rules, although the $\equiv$ rule could be handled directly as well using a structural induction.

$$\begin{array}{c}\Gamma(\Delta_1,\Delta_2)\\ \hline\hline \Gamma(\Delta_2,\Delta_1)\end{array}\quad\begin{array}{c}\Gamma(\Delta_1;\Delta_2)\\ \hline\hline \Gamma(\Delta_2;\Delta_1)\end{array}$$

$$\begin{array}{c}\Gamma(\varnothing_m,\Delta)\\ \hline\hline \Gamma(\Delta)\end{array}\quad\begin{array}{c}\Gamma(\varnothing_a;\Delta)\\ \hline\hline \Gamma(\Delta)\end{array}$$

$$\begin{array}{c}\Gamma(\Delta_1,(\Delta_2,\Delta_3))\\ \hline\hline \Gamma((\Delta_1,\Delta_2),\Delta_3)\end{array}\quad\begin{array}{c}\Gamma(\Delta_1;(\Delta_2;\Delta_3))\\ \hline\hline \Gamma((\Delta_1;\Delta_2);\Delta_3)\end{array}$$

Fig. 2. Alternative LBI Rules for Bunches.

Theorem 2 (Cut Elimination). *The cut rule is admissible in* LBI.

The following semi-distributivity rule will play an important role in the Reconstruction Algorithm which builds up derivations in LBI based on derivations in GBI.

Definition 9 (Semi-Distributivity). *The semi-distributivity rule is the following rule:*

$$\frac{\Gamma((A,B);(A,C))\vdash D}{\Gamma(A,(B;C))\vdash D}\,sd$$

Lemma 1. *The semi-distributivity rule is derivable in LBI.*

Proof. Let us consider the following derivation:

$$\dfrac{\dfrac{\dfrac{\Gamma((A,B);(A,C))\vdash D}{\Gamma((A,B);(A,(B;C)))\vdash D}\,\text{wk}}{\Gamma((A,(B;C));(A,(B;C)))\vdash D}\,\text{wk}}{\Gamma(A,(B;C))\vdash D}\,\text{ctr}$$

Lemma 2. *Without cut, LBI with contraction restricted on formulas is not complete.*

Proof. The following proof cannot be obtained without contracting the whole bunch on the left-hand side of the end sequent. Applying any other rule to the end sequent cannot lead to a proof.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{p\vdash p}\,\text{id}\qquad\dfrac{\dfrac{}{p\vdash p}\,\text{id}\qquad\dfrac{\dfrac{}{q\vdash q}\,\text{id}\qquad\dfrac{}{r\vdash r}\,\text{id}}{q\to r;q\vdash r}\,-\!\!*_L}{q\to r;(p-\!\!*\,q,p)\vdash r}\,-\!\!*_L}{(p-\!\!*\,(q\to r),p);(p-\!\!*\,q,p)\vdash r}\,-\!\!*_L}{(p-\!\!*\,(q\to r),p);((p-\!\!*\,(q\to r);p-\!\!*\,q),p)\vdash r}\,\text{wk}}{\dfrac{((p-\!\!*\,(q\to r);p-\!\!*\,q),p);((p-\!\!*\,(q\to r);p-\!\!*\,q),p)\vdash r}{(p-\!\!*\,(q\to r);p-\!\!*\,q),p\vdash r}\,\text{ctr}}\,\text{wk}$$

Lemma 3. *In* LBI*, we can restrict the structural rules to semi-distributivity, monoidal operations on bunches, contraction on* I *and weakening.*

Corollary 2. *Assuming a canonical form of bunches, building weakening into the rules, and considering only formulas without* I*, we can restrict the structural rules of LBI to semi-distributivity.*

Definition 10 (The system $\mathrm{LBI}_{\mathbf{sd}}$). *The system* LBI_{sd} *consists of* LBI *without weakening and contraction and the added rule of semi-distributivity.*

3 GBI - a labelled calculus for BI

In this section we define a new labelled calculus for BI and prove its soundness.

Definition 11 (Labels). *The set of* atomic labels *is* $\{\alpha, \beta, \gamma, \dots\}\{0,1\}^{*} \cup \{\epsilon\}$*. The set of* labels *is obtained by closing the set of atomic labels under* $+$ *and* $\times$*. We will write* $\ell\ell'$ *instead of* $\ell \times \ell'$*.*

Definition 12 (Constraints). *A* constraint *is a tuple* (ℓ, ℓ')*, written as* $(\ell \leq \ell')$*, where* ℓ *and* ℓ' *are labels.*

Definition 13 (Labelled Formula). *A* labelled formula *is a tuple* (A, ℓ)*, written as* $A{:}\ell$*, where* A *is a formula and* ℓ *is a label.*

The labelled sequent calculus GBI is given in Figure 3 where double lines indicate rules that work both ways (from conclusion to premisses or vice-versa). GBI deals with sequents of the form $\mathcal{G}, \Gamma \vdash C : \ell_0$ where $\mathcal{G}$ is a finite set of labels and constraints and Γ is a finite set of labelled formulas. For readability, we will often just write the constraints of $\mathcal{G}$ without listing all the occurring labels. The structural rules of GBI are depicted in Figure 4. The structural rules reflect the fact that we interpret labels and constraints up to a labelling algebra in which $\times$ and $+$ are both commutative and associative, ϵ being the unit for $\times$ and such that $(\_ \leq \_)$ is a preordering relation satisfying the following properties for all ℓ_0, ℓ_1, ℓ_2:

- if $(\ell_0 \leq \ell_1)$, then $(\ell_0 \times \ell_2 \leq \ell_1 \times \ell_2)$
- if $(\ell_0 \leq \ell_1)$, then $(\ell_0 + \ell_2 \leq \ell_1 + \ell_2)$
- $(\ell_0 \leq \ell_0 + \ell_1)$ and $(\ell_1 \leq \ell_0 + \ell_1)$
- $(\ell_0 \leq \ell_0 + \ell_0)$ and $(\ell_0 + \ell_0 \leq \ell_0)$

It is easy to see that the previous properties imply $((\ell_0 \times \ell_1) + (\ell_0 \times \ell_2) \leq \ell_0 \times (\ell_1 + \ell_2))$. All the structural rules in Figure 4 can be replaced with the following single rule:

$$\dfrac{\mathcal{G}', \mathcal{G}, \Gamma \vdash C : \ell_0}{\mathcal{G}, \Gamma \vdash C : \ell_0}\ \text{Struct}$$

provided that $\mathcal{G}'$ derives from $\mathcal{G}$ by the labelling algebra.

If we consider again the formula

$$(A \star (A \mathrel{-\!\!*} B)) \mathrel{-\!\!*} B$$

we have the following derivation in GBI

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\delta0\delta1 \leq \delta, A : \delta0 \vdash A : \delta0}\ \mathrm{id} \qquad \dfrac{\dfrac{}{B : \delta, \delta0\delta1 \leq \delta \vdash B : \delta}\ \mathrm{id}}{\delta0\delta1 \leq \delta, B : \delta0\delta1\delta \vdash B : \delta}\ K_L}{\delta0\delta1 \leq \delta, A : \delta0, A \mathrel{-\!\!*} B : \delta1 \vdash B : \delta}\ {-\!\!*}_L}{A \star (A \mathrel{-\!\!*} B) : \delta \vdash B : \delta}\ \star_L}{\vdash (A \star (A \mathrel{-\!\!*} B)) \mathrel{-\!\!*} B : \varepsilon}\ {-\!\!*}_R$$

$$\begin{array}{c} \\ \hline \mathcal{G}, \Gamma, A : \ell \vdash A : \ell \end{array}\,\mathrm{id} \qquad \begin{array}{c} \\ \hline \mathcal{G}, \Gamma, \bot : \ell \vdash C : \ell \end{array}\,\bot_{\mathrm{L}} \qquad \begin{array}{c} \\ \hline \mathcal{G}, \Gamma \vdash \top : \ell \end{array}\,\top_{\mathrm{R}} \qquad \begin{array}{c} \mathcal{G}, \Gamma, \top : \ell_1 \vdash C : \ell_0 \\ \hline\hline \mathcal{G}, \Gamma \vdash C : \ell_0 \end{array}\,\top_{\mathrm{L}}$$

$$\begin{array}{c} (\epsilon \leq \ell_1), \mathcal{G}, \Gamma \vdash C : \ell_0 \\ \hline\hline \mathcal{G}, \Gamma, \mathrm{I} : \ell_1 \vdash C : \ell_0 \end{array}\,\mathrm{I}_{\mathrm{L}} \qquad \begin{array}{c} \\ \hline (\epsilon \leq \ell), \mathcal{G}, \Gamma \vdash \mathrm{I} : \ell \end{array}\,\mathrm{I}_{\mathrm{R}} \qquad \begin{array}{c} \mathrm{I} : \varepsilon, \mathcal{G}, \Gamma \vdash \Delta \\ \hline \mathcal{G}, \Gamma \vdash \Delta \end{array}\,\mathrm{I}_{\varepsilon}$$

$$\begin{array}{c} \mathcal{G}, \Gamma \vdash A : \ell_2 \qquad \mathcal{G}, \Gamma, B : \ell_1 + \ell_2 \vdash C : \ell_0 \\ \hline \ell_1 + \ell_2, \mathcal{G}, \Gamma, A \to B : \ell_1 \vdash C : \ell_0 \end{array}\,\to_{\mathrm{L}} \qquad \begin{array}{c} \ell + a, \mathcal{G}, \Gamma, A : a \vdash B : \ell + a \\ \hline \mathcal{G}, \Gamma \vdash A \to B : \ell \end{array}\,\to_{\mathrm{R}}$$

$$\begin{array}{c} \mathcal{G}, \Gamma \vdash A : \ell_2 \qquad \mathcal{G}, \Gamma, B : \ell_1 \times \ell_2 \vdash C : \ell_0 \\ \hline \ell_1 \times \ell_2, \mathcal{G}, \Gamma, A \mathrel{-\!\!*} B : \ell_1 \vdash C : \ell_0 \end{array}\,\mathrel{-\!\!*}_{\mathrm{L}} \qquad \begin{array}{c} \ell \times a, \mathcal{G}, \Gamma, A : a \vdash B : \ell \times a \\ \hline \mathcal{G}, \Gamma \vdash A \mathrel{-\!\!*} B : \ell \end{array}\,\mathrel{-\!\!*}_{\mathrm{R}}$$

$$\begin{array}{c} (a + b \leq \ell_1), \mathcal{G}, \Gamma, A : a, B : b \vdash C : \ell_0 \\ \hline \mathcal{G}, \Gamma, A \land B : \ell_1 \vdash C : \ell_0 \end{array}\,\land_{\mathrm{L}} \qquad \begin{array}{c} \mathcal{G}, \Gamma \vdash A : \ell \qquad \mathcal{G}, \Gamma \vdash B : \ell \\ \hline \mathcal{G}, \Gamma \vdash A \land B : \ell \end{array}\,\land_{\mathrm{R}}$$

$$\begin{array}{c} (a \times b \leq \ell_1), \mathcal{G}, \Gamma, A : a, B : b \vdash C : \ell_0 \\ \hline \mathcal{G}, \Gamma, A \star B : \ell_1 \vdash C : \ell_0 \end{array}\,\star_{\mathrm{L}}$$

$$\begin{array}{c} (\ell_1 \times \ell_2 \leq \ell_0), \mathcal{G}, \Gamma \vdash A : \ell_1 \qquad (\ell_1 \times \ell_2 \leq \ell_0), \mathcal{G}, \Gamma \vdash B : \ell_2 \\ \hline (\ell_1 \times \ell_2 \leq \ell_0), \mathcal{G}, \Gamma \vdash A \star B : \ell_0 \end{array}\,\star_{\mathrm{R}}$$

$$\begin{array}{c} \mathcal{G}, \Gamma, A : \ell_1 \vdash C : \ell_0 \qquad \mathcal{G}, \Gamma, B : \ell_1 \vdash C : \ell_0 \\ \hline \mathcal{G}, \Gamma, A \lor B : \ell_1 \vdash C : \ell_0 \end{array}\,\lor_{\mathrm{L}} \qquad \begin{array}{c} \mathcal{G}, \Gamma \vdash A_i : \ell \\ \hline \mathcal{G}, \Gamma \vdash A_1 \lor A_2 : \ell \end{array}\,\lor_{\mathrm{R}_{\mathrm{i} \in \{1,2\}}}$$

$$\begin{array}{c} (\ell_1 \leq \ell_2), \mathcal{G}, \Gamma, A : \ell_2 \vdash C : \ell_0 \\ \hline (\ell_1 \leq \ell_2), \mathcal{G}, \Gamma, A : \ell_1 \vdash C : \ell_0 \end{array}\,\mathrm{K}_{\mathrm{L}} \qquad \begin{array}{c} (\ell_1 \leq \ell_0), \mathcal{G}, \Gamma \vdash C : \ell_1 \\ \hline (\ell_1 \leq \ell_0), \mathcal{G}, \Gamma \vdash C : \ell_0 \end{array}\,\mathrm{K}_{\mathrm{R}} \qquad \begin{array}{c} \mathcal{G}, \Gamma \vdash C : \ell_0 \\ \hline \mathcal{G}, \mathcal{G}_0, \Gamma, \Gamma_0 \vdash C : \ell_0 \end{array}\,\mathrm{W}_{\mathrm{L}}$$

Side conditions:
In $\star_{\mathrm{L}}$, $\land_{\mathrm{L}}$, $\mathrel{-\!\!*}_{\mathrm{R}}$, and $\to_{\mathrm{R}}$, the labels a and b are fresh atomic labels (*i.e.*, they do not occur in the conclusion).

Fig. 3. Logical rules of GBI.

$$\frac{(\ell \leq \ell), \ell, \mathcal{G}, \mathit{\Gamma} \vdash \mathit{\Delta}}{\ell, \mathcal{G}, \mathit{\Gamma} \vdash \mathit{\Delta}}\,\mathrm{R} \qquad \frac{(\ell_0 \leq \ell_2), (\ell_0 \leq \ell_1), (\ell_1 \leq \ell_2), \mathcal{G}, \mathit{\Gamma} \vdash \mathit{\Delta}}{(\ell_0 \leq \ell_1), (\ell_1 \leq \ell_2), \mathcal{G}, \mathit{\Gamma} \vdash \mathit{\Delta}}\,\mathrm{T}$$

$$\frac{\ell_0, \ell_1, (\ell_0 \leq \ell_1), \mathcal{G}, \mathit{\Gamma} \vdash \mathit{\Delta}}{(\ell_0 \leq \ell_1), \mathcal{G}, \mathit{\Gamma} \vdash \mathit{\Delta}}\,\mathrm{L} \qquad \frac{\ell_0, \ell_1, \ell_0 \times \ell_1, \mathcal{G}, \mathit{\Gamma} \vdash \mathit{\Delta}}{\ell_0 \times \ell_1, \mathcal{G}, \mathit{\Gamma} \vdash \mathit{\Delta}}\,\mathrm{SL_m} \qquad \frac{\ell_0, \ell_1, \ell_0 + \ell_1, \mathcal{G}, \mathit{\Gamma} \vdash \mathit{\Delta}}{\ell_0 + \ell_1, \mathcal{G}, \mathit{\Gamma} \vdash \mathit{\Delta}}\,\mathrm{SL_a}$$

$$\frac{(\ell_0 \leq \ell_0 + \ell_1), (\ell_1 \leq \ell_0 + \ell_1), \ell_0 + \ell_1, \mathcal{G}, \mathit{\Gamma} \vdash \mathit{\Delta}}{\ell_0 + \ell_1, \mathcal{G}, \mathit{\Gamma} \vdash \mathit{\Delta}}\,\mathrm{LUB_a} \qquad \frac{(\ell + \ell \leq \ell), \ell, \mathcal{G}, \mathit{\Gamma} \vdash \mathit{\Delta}}{\ell, \mathcal{G}, \mathit{\Gamma} \vdash \mathit{\Delta}}\,\mathrm{IP_a}$$

$$\frac{(\ell_0 \times \ell_2 \leq \ell_1 \times \ell_2), \ell_0 \times \ell_2, (\ell_0 \leq \ell_1), \ell_1 \times \ell_2, \mathcal{G}, \mathit{\Gamma} \vdash \mathit{\Delta}}{(\ell_0 \leq \ell_1), \ell_1 \times \ell_2, \mathcal{G}, \mathit{\Gamma} \vdash \mathit{\Delta}}\,\mathrm{C_m}{\downarrow}$$

$$\frac{(\ell_0 + \ell_2 \leq \ell_1 + \ell_2), \ell_0 + \ell_2, (\ell_0 \leq \ell_1), \ell_1 + \ell_2, \mathcal{G}, \mathit{\Gamma} \vdash \mathit{\Delta}}{(\ell_0 \leq \ell_1), \ell_1 + \ell_2, \mathcal{G}, \mathit{\Gamma} \vdash \mathit{\Delta}}\,\mathrm{C_a}{\downarrow}$$

$$\frac{(\ell_0 \times \ell_1 \leq \ell_2 \times \ell_3), (\ell_0 \leq \ell_2), (\ell_1 \leq \ell_3), \ell_2 \times \ell_3, \mathcal{G}, \mathit{\Gamma} \vdash \mathit{\Delta}}{(\ell_0 \leq \ell_2), (\ell_1 \leq \ell_3), \ell_2 \times \ell_3, \mathcal{G}, \mathit{\Gamma} \vdash \mathit{\Delta}}\,\mathrm{C_m}{\uparrow}$$

$$\frac{(\ell_0 + \ell_1 \leq \ell_2 + \ell_3), (\ell_0 \leq \ell_2), (\ell_1 \leq \ell_3), \ell_2 + \ell_3, \mathcal{G}, \mathit{\Gamma} \vdash \mathit{\Delta}}{(\ell_0 \leq \ell_2), (\ell_1 \leq \ell_3), \ell_2 + \ell_3, \mathcal{G}, \mathit{\Gamma} \vdash \mathit{\Delta}}\,\mathrm{C_a}{\uparrow}$$

$$\frac{(\ell_3 \times \ell_5 \leq \ell_0), (\ell_2 \times \ell_4 \leq \ell_5), (\ell_1 \times \ell_2 \leq \ell_0), (\ell_3 \times \ell_4 \leq \ell_1), \mathcal{G}, \mathit{\Gamma} \vdash \mathit{\Delta}}{(\ell_1 \times \ell_2 \leq \ell_0), (\ell_3 \times \ell_4 \leq \ell_1) \vdash \mathit{\Delta}}\,\mathrm{assoc} \times$$

$$\frac{(\ell_3 + \ell_5 \leq \ell_0), (\ell_2 + \ell_4 \leq \ell_5), (\ell_1 + \ell_2 \leq \ell_0), (\ell_3 + \ell_4 \leq \ell_1), \mathcal{G}, \mathit{\Gamma} \vdash \mathit{\Delta}}{(\ell_1 + \ell_2 \leq \ell_0), (\ell_3 + \ell_4 \leq \ell_1), \mathcal{G}, \mathit{\Gamma} \vdash \mathit{\Delta}}\,\mathrm{assoc} +$$

Fig. 4. Structural rules of GBI.

3.1 Soundness of GBI

Definition 14 (Realization). *Let $\mathcal{K} = (\mathcal{M}, \models, [\![-]\!])$ be a Kripke resource model with $\mathcal{M} = (M, \otimes, 1, \oplus, 0, \infty, \sqsubseteq)$. Let $s = \mathcal{G}, \Gamma \vdash \Delta$ be a labelled sequent. A* realization *of s* in *$\mathcal{K}$ is a total function ρ from the labels of s to the worlds in M such that:*

- *$\rho(\epsilon) = 1$, $\rho(\ell_1 \times \ell_2) = \rho(\ell_1) \otimes \rho(\ell_2)$, $\rho(\ell_1 + \ell_2) = \rho(\ell_1) \oplus \rho(\ell_2)$*
- *for all $(\ell_1 \leq \ell_2)$ in $\mathcal{G}$, $\rho(\ell_1) \sqsubseteq \rho(\ell_2)$ in $\mathcal{M}$*
- *for all $A : \ell$ in Γ, $\rho(\ell) \models A$ and for all $A : \ell$ in Δ, $\rho(\ell) \not\models A$.*

We say that s is realizable in *$\mathcal{K}$ if there exists a realization of s in $\mathcal{K}$ and that s is* realizable *if it is realizable in some Kripke resource model $\mathcal{K}$.*

Lemma 4. *If in a* GBI*-proof the sequent $s = \mathcal{G}, \Gamma \vdash \Delta$ is an initial sequent,* i.e.*, a leaf sequent that is the conclusion of a zero-premiss rule, then s is not realizable.*

Proof. Suppose that s is realizable, then we have a realization ρ of s in some Kripke resource model $\mathcal{K} = (\mathcal{M}, \models, [\![-]\!])$. We proceed by case analysis on the zero-premiss rule of which s is the conclusion and show that all cases lead to a contradiction with the definition of the Kripke resource semantics:

- Case of the rule id: then s has the form $\mathcal{G}, \Gamma, A : \ell \vdash A : \ell$, which by definition of a realization implies that we should have both $\rho(\ell) \models A$ and $\rho(\ell) \not\models A$, a contradiction.
- Case of the rule $\top_{\mathrm{R}}$: then s has the form $\mathcal{G}, \Gamma \vdash \top : \ell$ then we should have $\rho(\ell) \not\models \top$, a contradiction.
- Case of the rule $\bot_{\mathrm{L}}$: then s has the form $\mathcal{G}, \Gamma, \bot : \ell \vdash C : \ell$ and we should have both $\rho(\ell) \models \bot$ and $\rho(\ell) \not\models C$, which is a contradiction because $\rho(\ell) \models \bot$ implies $\rho(\ell) = \infty$ and since ∞ forces all formulas, we should have $\rho(\ell) \models C$.
- Case of the rule I_{L}: then s has the form $(\epsilon \leq \ell), \mathcal{G}, \Gamma \vdash \mathrm{I} : \ell$ and we should have both $\rho(\ell) \not\models \mathrm{I}$ and $1 \sqsubseteq \rho(\ell)$, a contradiction since $\rho(\ell) \not\models \mathrm{I}$ implies $1 \not\sqsubseteq \rho(\ell)$.

Lemma 5. *Every proof-rule in* GBI *preserves realizability.*

Proof. By case analysis of the proof rules in GBI.

Theorem 3 (Soundness). *If a formula A is provable in* GBI*, then it is valid in the Kripke resource semantics of* BI.

Proof. Suppose that A is provable in GBI but not valid in the Kripke resource semantics of BI. Then, the sequent $\vdash A : \epsilon$ is trivially realizable and we have a GBI-proof $\mathcal{P}$ of A. It follows from Lemma 5 that $\mathcal{P}$ contains a branch the sequents of which are all realizable. Since $\mathcal{P}$ is a proof, the branch ends with an inital (axiom) sequent and Lemma 4 implies that this initial sequent is not realizable, a contradiction. Therefore, A is valid.

4 A Translation of LBI proofs to GBI proofs

In this section, we introduce the necessary concepts for translating sequents of LBI to sequents of GBI. This is done by viewing a bunch as a tree, to label its nodes, and introducing constraints on these labels that mirror the structure of the tree.

We will also see that labels can be renamed while preserving derivations, which is a simple but important tool for combining derivations in GBI.

4.1 Main Concepts and Definitions

Definition 15 (Associated labels, constraints and labelled formulas). *We consider bunches to be rooted trees whose nodes are commas, semicolons, formulas, $\varnothing_a$ or $\varnothing_m$. Given a bunch Γ and a symbol $s \in \{\alpha, \beta, \gamma, \dots\}$ we define its associated set of labels, constraints and labelled formulas as follows:*

- *The root of the bunch is labelled with the symbol s.*
- *If a node of the bunch is labelled by ℓ, then the left child is labelled with $\ell 0$ and the right child with $\ell 1$.*
- *If a node is a leaf, labelled with ℓ and is a formula A, then its associated labelled formula is $A : \ell$.*
- *If a node is a leaf, labelled with ℓ and being $\varnothing_a$ resp. $\varnothing_m$, then its associated labelled formula is $\top : \ell$ resp. $\mathrm{I} : \ell$.*
- *If a node is a semicolon, and has label ℓ, then its associated constraints are $\ell 0 + \ell 1 \leq \ell$.*
- *If a node is a comma, and has label ℓ, then its associated constraints are $\ell 0 \times \ell 1 \leq \ell$.*
- *The set of associated labels of a bunch consists of all labels which are labeling its nodes and all labels occuring in associated constraints.*

Definition 16 (Translation from LBI to GBI). *Given a sequent $\Gamma \vdash A$ in LBI, its* translation *is defined as follows:*

$$\Theta(\Gamma \vdash A) := G^{\gamma}, \boldsymbol{\Pi} : \boldsymbol{\pi} \vdash A : \gamma$$

where G^{γ} is the set of associated labels and constraints of Γ, and $\boldsymbol{\Pi} : \boldsymbol{\pi}$ is the set of labelled formulas associated with Γ.

Remark 1. Often, we will use a slight abuse of notation and write $\boldsymbol{\Gamma} : \boldsymbol{\gamma}$ for the associated set of labelled formulas of Γ.

Consider the sequent

$$(\varnothing_m, p); q \vdash r$$

Its translation (omitting listing the labels) is

$$\mathrm{I} : \delta 00, r : \delta 01, q : \delta 1, \delta 0 + \delta 1 \leq \delta, \delta 00 \delta 01 \leq \delta 0 \vdash r : \delta$$

Definition 17 (Renaming). *A* renaming *is a mapping from initial segments of labels (different from ε) to arbitrary labels. We write renamings in the form*

$$\ell \mapsto \ell'$$

Lemma 6 (Renaming Lemma). *Derivations in* GBI *are preserved under renaming, i.e. If $\mathcal{D}$ is a derivation in* GBI*, then replacing arbitrary labels with fresh labels results in a derivation in* GBI.

4.2 The Translation Theorem

Theorem 4 (Translation from LBI to GBI). *For each LBI proof, there is a translation which is admissible in the GBI calculus.*

Proof. We proceed by induction on the length of the derivation, using a case distinction on the rules applied in LBI. We show that if we have translations of the premisses of a rule, then there is a translation such that the translated conclusion is derivable in GBI from the translated premisses.
We only consider here some cases, the other being similar.

- If the rule (id) is applied, then we have:

  $$\dfrac{}{A \vdash A}\ \text{id}$$

  Then we have

  $$\dfrac{}{A : \delta \vdash A : \delta}\ \text{id}$$

  in GBI.
- If a rule for associativity is applied, then we have

  $$\dfrac{\varDelta_1, (\varDelta_2, \varDelta_3) \vdash A}{(\varDelta_1, \varDelta_2), \varDelta_3 \vdash A}$$

  The translation of the premis is

  $$\boldsymbol{\varDelta_1} : \boldsymbol{\delta 0}, \boldsymbol{\varDelta_2} : \boldsymbol{\delta 10}, \boldsymbol{\varDelta_3} : \boldsymbol{\delta 11}, \delta 10\delta 11 \leq \delta 1, \delta 0\delta 1 \leq \delta \vdash A : \delta$$

  We choose $\gamma \neq \delta$, use a renaming

  $$\delta 1 \mapsto \gamma, \quad \delta 11x \mapsto \delta 1x, \quad \delta 10x \mapsto \delta 01x, \quad \delta 0x \mapsto \delta 00x \quad \text{ for all } x \in \{0,1\}^*$$

  so we get a derivation of

  $$\boldsymbol{\varDelta_1} : \boldsymbol{\delta 00}, \boldsymbol{\varDelta_2} : \boldsymbol{\delta 01}, \boldsymbol{\varDelta_3} : \boldsymbol{\delta 1}, \delta 01\delta 1 \leq \gamma, \delta 00\gamma \leq \delta \vdash A : \delta$$

  and we continue with

  $$\dfrac{\dfrac{\boldsymbol{\varDelta_1} : \boldsymbol{\delta 00}, \boldsymbol{\varDelta_2} : \boldsymbol{\delta 01}, \boldsymbol{\varDelta_3} : \boldsymbol{\delta 1}, \delta 01\delta 1 \leq \gamma, \delta 00\gamma \leq \delta \vdash A : \delta}{\boldsymbol{\varDelta_1} : \boldsymbol{\delta 00}, \boldsymbol{\varDelta_2} : \boldsymbol{\delta 01}, \boldsymbol{\varDelta_3} : \boldsymbol{\delta 1}, \delta 00\delta 01 \leq \delta 0, \delta 0\delta 1 \leq \delta, \delta 01\delta 1 \leq \gamma, \delta 00\gamma \leq \delta \vdash A : \delta}\ \text{wk}}{\boldsymbol{\varDelta_1} : \boldsymbol{\delta 00}, \boldsymbol{\varDelta_2} : \boldsymbol{\delta 01}, \boldsymbol{\varDelta_3} : \boldsymbol{\delta 1}, \delta 00\delta 01 \leq \delta 0, \delta 0\delta 1 \leq \delta \vdash A : \delta}\ \text{assoc} \times$$

  where the last line is the translation of the conclusion.
- If the rule $\star_L$ is applied, then we have

  $$\dfrac{\dfrac{\vdots}{\varGamma(A, B) \vdash C}}{\varGamma(A \star B) \vdash C}\ \star_L$$

  We proceed similarly to the last case. By the I.H., we have a derivation of the following sequent in GBI:

  $$G^{\delta}, A : \delta x0, B : \delta x1, \delta x0\delta x1 \leq \delta x, \boldsymbol{\varGamma} : \boldsymbol{\gamma} \vdash C : \delta$$

  where $x \in \{0,1\}^+$ indicates the position of the occurrence of , in the corresponding tree.
  We continue with

$$\cfrac{\cfrac{\cfrac{\vdots}{G^{\delta}, A : \delta x0, B : \delta x1, \delta x0\delta x1 \leq \delta x, \boldsymbol{\Gamma} : \boldsymbol{\gamma} \vdash C : \delta}}{G^{\delta}, A \star B : \delta x, \delta x0\delta x1 \leq \delta x, \boldsymbol{\Gamma} : \boldsymbol{\gamma} \vdash C : \delta}\,\star_L}{G^{\delta}, A \star B : \delta x, \boldsymbol{\Gamma} : \boldsymbol{\gamma} \vdash C : \delta}\,\text{structural rule}$$

where the last sequent is the translation of $\Gamma(A \star B) \vdash C$.

- If the rule $\star_R$ is applied, then we have

$$\cfrac{\cfrac{\vdots}{\Gamma \vdash A} \qquad \cfrac{\vdots}{\Delta \vdash B}}{\Gamma, \Delta \vdash A \star B}\,\star_R$$

By the I.H. we have derivations of the following in GBI:

$$G_1^{\delta}, \boldsymbol{\Pi}_1 : \boldsymbol{\pi}_1 \vdash A : \delta$$

and

$$G_2^{\sigma}, \boldsymbol{\Pi}_2 : \boldsymbol{\pi}_2 \vdash B : \sigma$$

We perform a renaming ($\delta \mapsto \delta 0$ and $\sigma \mapsto \delta 1$), so by Lemma 6 and have

$$G_1^{\delta_0}, \boldsymbol{\Pi}_1 : \boldsymbol{\pi}'_1 \vdash A : \delta_0$$

and

$$G_2^{\delta 1}, \boldsymbol{\Pi}_2 : \boldsymbol{\pi}'_2 \vdash B : \delta 1.$$

Consider the derivations $\mathcal{D}_1$

$$\cfrac{\cfrac{\vdots}{G_1^{\delta 0}, \boldsymbol{\Pi}_1 : \boldsymbol{\pi}'_1 \vdash A : \delta_0}}{G_1^{\delta 0}, G_2^{\delta 1}, \delta 0\delta 1 \leq \delta, \boldsymbol{\Pi}_1 : \boldsymbol{\pi}_1, \boldsymbol{\Pi}_2 : \boldsymbol{\pi}'_2 \vdash A : \delta_0}\,\text{wk}$$

and $\mathcal{D}_2$

$$\cfrac{\cfrac{\vdots}{G_2^{\delta 1}, \boldsymbol{\Pi}_2 : \boldsymbol{\pi}'_2 \vdash B : \delta 1}}{G_1^{\delta 0}, G_2^{\delta 1}, \delta 0\delta 1 \leq \delta, \boldsymbol{\Pi}_1 : \boldsymbol{\pi}'_1, \boldsymbol{\Pi}_2 : \boldsymbol{\pi}'_2 \vdash B : \delta 1}\,\text{wk}$$

so we can continue with

$$\cfrac{\mathcal{D}_1 \qquad \mathcal{D}_2}{G_1^{\delta 0}, G_2^{\delta 1}, \delta 0\delta 1 \leq \delta, \boldsymbol{\Pi}_1 : \boldsymbol{\pi}'_1, \boldsymbol{\Pi}_2 : \boldsymbol{\pi}'_2 \vdash A \star B : \delta}\,\star_R$$

Observe here that

$$G_1^{\delta 0}, G_2^{\delta 1}, \delta 0\delta 1 \leq \delta, \boldsymbol{\Pi}_1 : \boldsymbol{\pi}'_1, \boldsymbol{\Pi}_2 : \boldsymbol{\pi}'_2 \vdash A \star B : \delta \quad = \quad \Theta(\Gamma, \Delta \vdash A \star B).$$

4.3 An Example of the translation

We consider again a formula of the form $(A \star (A \mathrel{-\!\!*} B)) \mathrel{-\!\!*} B$ and the following derivation

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A}\,\mathrm{id} \qquad \dfrac{}{A \vdash A}\,\mathrm{id}}{A, A \mathrel{-\!\!*} B \vdash B}\,{\mathrel{-\!\!*}}_L}{A \star (A \mathrel{-\!\!*} B) \vdash B}\,\star_L}{\varnothing_m, A \star (A \mathrel{-\!\!*} B) \vdash B}\,\equiv}{\varnothing_m \vdash (A \star (A \mathrel{-\!\!*} B)) \mathrel{-\!\!*} B}\,{\mathrel{-\!\!*}}_R$$

Translating the axioms $A \vdash A$ and $B \vdash B$ we get $A : \gamma \vdash A : \gamma$ and $B : \delta \vdash B : \delta$. Using renamings $(\gamma \mapsto \delta 0)$, we have

$$A : \delta 0 \vdash A : \delta 0$$

using weakening, we have

$$A : \delta 0, \delta 0 \delta 1 \leq \delta, \vdash A : \delta 0$$

and

$$B : \delta, \delta 0 \delta 1 \leq \delta \vdash B : \delta$$

using K_L, we have

$$B : \delta 0 \delta 1, \delta 0 \delta 1 \leq \delta, \vdash B : \delta$$

A translation of

$$A, A \mathrel{-\!\!*} B \vdash B$$

is

$$A : \delta 0, A \mathrel{-\!\!*} B : \delta 1, \delta 0 \delta 1 \leq \delta \vdash B : \delta$$

which is what we get by using ${\mathrel{-\!\!*}}_L$ on two of the previous sequents:

$$\dfrac{\dfrac{\vdots}{A : \delta 0, \delta 0 \delta 1 \leq \delta \vdash A : \delta 0} \qquad \dfrac{\vdots}{B : \delta 0 \delta 1, \delta 0 \delta 1 \leq \delta \vdash B : \delta}}{A : \delta 0, A \mathrel{-\!\!*} B : \delta 1, \delta 0 \delta 1 \leq \delta \vdash B : \delta}\,{\mathrel{-\!\!*}}_L$$

Now, a translation of

$$A \star (A \mathrel{-\!\!*} B) \vdash B$$

is

$$A \star (A \mathrel{-\!\!*} B) : \delta \vdash B : \delta$$

which is exactly what we get by continuing the derivation above with the rule $\star_L$.

Using a renaming

$$\delta \mapsto \delta 1$$

we get

$$A \star (A \mathrel{-\!\!*} B) : \delta 1 \vdash B : \delta 1$$

Next, the translation of

$$\varnothing_m, A \star (A \mathrel{-\!\!*} B) \vdash B$$

is

$$\mathrm{I} : \delta 0, A \star (A \mathrel{-\!\!*} B) : \delta 1, \delta 0 \delta 1 \leq \delta \vdash B : \delta.$$

so we continue with

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \star (A \mathrel{-\!\!*} B) : \delta 1 \vdash B : \delta 1}{\delta 1 \leq \delta, \delta 1 \leq \delta 0 \delta 1, \varepsilon \leq \delta 0, \delta 0 \delta 1 \leq \delta, \mathrm{I} : \delta 0, A \star (A \mathrel{-\!\!*} B) : \delta 1 \vdash B : \delta 1}\,\text{wk}}{\delta 1 \leq \delta, \delta 1 \leq \delta 0 \delta 1, \varepsilon \leq \delta 0, \delta 0 \delta 1 \leq \delta, \mathrm{I} : \delta 0, A \star (A \mathrel{-\!\!*} B) : \delta 1 \vdash B : \delta}\,K_R}{\delta 1 \leq \delta 0 \delta 1, \varepsilon \leq \delta 0, \delta 0 \delta 1 \leq \delta, \mathrm{I} : \delta 0, A \star (A \mathrel{-\!\!*} B) : \delta 1 \vdash B : \delta}\,\text{str. rules}}{\varepsilon \leq \delta 0, \delta 0 \delta 1 \leq \delta, \mathrm{I} : \delta 0, A \star (A \mathrel{-\!\!*} B) : \delta 1 \vdash B : \delta}\,\text{str. rules}}{\delta 0 \delta 1 \leq \delta, \mathrm{I} : \delta 0, A \star (A \mathrel{-\!\!*} B) : \delta 1 \vdash B : \delta}\,\mathrm{I}
$$

next, we use another renaming:

$$\delta 1 \mapsto a, \quad \delta \mapsto \delta a, \quad \delta 0 \mapsto \delta$$

then it follows that there is a derivation of

$$\delta a \leq \delta a, \mathrm{I} : \delta, A \star (A \mathrel{-\!\!*} B) : a \vdash B : \delta a$$

we continue with

$$
\dfrac{\dfrac{\delta a \leq \delta a, \mathrm{I} : \delta, A \star (A \mathrel{-\!\!*} B) : a \vdash B : \delta a}{\mathrm{I} : \delta, A \star (A \mathrel{-\!\!*} B) : a \vdash B : \delta a}\,\text{str. rules}}{I : \delta \vdash (A \star (A \mathrel{-\!\!*} B)) \mathrel{-\!\!*} B : \delta}\,{\mathrel{-\!\!*}}_R
$$

where the last sequent is the translation of

$$\varnothing_m \vdash (A \star (A \mathrel{-\!\!*} B)) \mathrel{-\!\!*} B.$$

5 From GBI proofs to LBI proofs

In this section, we will discuss some first steps towards a translation in the other direction, i.e. translating derivations from GBI to LBI.

5.1 The Reconstruction Algorithm

The Reconstruction Algorithm is a procedure that, given a derivation of a formula in GBI, builds up a derivation of this formula in LBI. We call this a reconstruction since the algorithm does not directly translate the syntactic elements of the derivation in GBI, but tries to build up a new derivation in LBI using some information extracted from the given derivation in GBI. Conceptually, the procedure is simple: It just tries to apply the logical rules of LBI corresponding to the rules of GBI in the same order, starting from similar axioms.

Definition 18 (Reconstruction Algorithm). *This Algorithm builds up a derivation in LBI, using a derivation in GBI.*
Input: A derivation in GBI.

Output: A derivation in LBI_{sd}

- *Bottom-up mark the active formulas and their predecessor(s),Annotate with the applied rule, and add a link between them.*
- *Also add links between the occurrences of marked formulas and in a line and the same formula in the previous line.*
- *Try to apply the corresponding rules to the marked formulas in LBI.*
- *If the rule is* $\to_R$ *and it is not applicable, then add* $\varnothing_a$*; in the appropriate place in the bunch.*
- *If the rule is* ${\mathrel{-\!\!*}}_R$ *and it is not applicable, then add* $\varnothing_m$*, in the appropriate place in the bunch.*
- *Otherwise, apply semi-distributivity.*

5.2 Some examples

Now we proceed to look at some examples of derivations in GBI and build up a derivation in LBI, following the Reconstruction Algorithm.

As a first example we consider the semi-distributivity formula

$$A \star (B \wedge C) \to (A \star B) \wedge (A \star C)$$

It has the following derivation in GBI:

We consider the derivations $\mathcal{D}_1$

$$\dfrac{\dfrac{\dfrac{}{bc \leq a, a \leq \varepsilon + a, bc \leq \varepsilon + a, \boxed{A} : b, \boxed{B} : c, C : c \vdash \boxed{A} : b}\,\text{id} \qquad \dfrac{}{bc \leq a, a \leq \varepsilon + a, bc \leq \varepsilon + a, \boxed{A} : b, \boxed{B} : c, C : c \vdash \boxed{B} : c}\,\text{id}}{bc \leq a, a \leq \varepsilon + a, bc \leq \varepsilon + a, \boxed{A} : b, \boxed{B} : c, C : c \vdash \boxed{A \star B} : \varepsilon + a}\,\star_R}{bc \leq a, \boxed{A} : b, \boxed{B} : c, C : c \vdash \boxed{A \star B} : \varepsilon + a}\,\text{str. rules}$$

as well as $\mathcal{D}_2$

$$\dfrac{\dfrac{\dfrac{}{bc \leq a, a \leq \varepsilon + a, bc \leq \varepsilon + a, \boxed{A} : b \vdash \boxed{A} : b}\,\text{id} \qquad \dfrac{}{bc \leq a, a \leq \varepsilon + a, bc \leq \varepsilon + a, \boxed{C} : c \vdash \boxed{C} : c}\,\text{id}}{bc \leq a, a \leq \varepsilon + a, bc \leq \varepsilon + a, \boxed{A} : b, B : c, \boxed{C} : c \vdash \boxed{A \star C} : \varepsilon + a}\,\star_R}{bc \leq a, \boxed{A} : b, B : c, \boxed{C} : c \vdash \boxed{A \star C} : \varepsilon + a}\,\text{str. rules}$$

and finally

$$\dfrac{\dfrac{\dfrac{\dfrac{\mathcal{D}_1 \qquad \mathcal{D}_2}{bc \leq a, \boxed{A} : b, \boxed{B} : c, \boxed{C} : c \vdash \boxed{(A \star B) \wedge (A \star C)} : \varepsilon + a}\,\wedge_R}{bc \leq a, \boxed{A} : b, \boxed{B \wedge C} : c \vdash \boxed{(A \star B) \wedge (A \star C)} : \varepsilon + a}\,\wedge_L}{\boxed{A \star (B \wedge C)} : a \vdash \boxed{(A \star B) \wedge (A \star C)} : \varepsilon + a}\,\star_L}{\vdash \boxed{A \star (B \wedge C) \to (A \star B) \wedge (A \star C)} : \varepsilon}\,\to_R$$

We observe that we use the structural rules to reason about labels, which allows us to apply the rule $(\star_R)$. Informally, the reasoning is:

$$a \leq \varepsilon + a \leq b \times c \Rightarrow a \leq b \times c$$

and the corresponding derivation in LBI is

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{id}
\UnaryInfC{$A \vdash A$}
\AxiomC{}
\RightLabel{id}
\UnaryInfC{$B \vdash B$}
\RightLabel{$\star_R$}
\BinaryInfC{$A, B \vdash A \star B$}
\AxiomC{}
\RightLabel{id}
\UnaryInfC{$A \vdash A$}
\AxiomC{}
\RightLabel{id}
\UnaryInfC{$C \vdash C$}
\RightLabel{$\star_R$}
\BinaryInfC{$A, C \vdash A \star C$}
\RightLabel{$\wedge_R$}
\BinaryInfC{$(A, B); (A, C) \vdash (A \star B) \wedge (A \star C)$}
\RightLabel{\textbf{s.d.}}
\UnaryInfC{$A, (B; C) \vdash (A \star B) \wedge (A \star C)$}
\RightLabel{$\wedge_L$}
\UnaryInfC{$A, (B \wedge C) \vdash (A \star B) \wedge (A \star C)$}
\RightLabel{$\star_L$}
\UnaryInfC{$A \star (B \wedge C) \vdash (A \star B) \wedge (A \star C)$}
\RightLabel{$\equiv$}
\UnaryInfC{$\varnothing_a; A \star (B \wedge C) \vdash (A \star B) \wedge (A \star C)$}
\RightLabel{$\to_R$}
\UnaryInfC{$\varnothing_a \vdash A \star (B \wedge C) \to (A \star B) \wedge (A \star C)$}
\end{prooftree}
$$

where in the second last step, we use the equivalence of bunches

$$\varnothing_a; A \star (B \wedge C) \quad \text{and} \quad A \star (B \wedge C)$$

and the corresponding rule of LBI.

As another example we consider a formula of the form

$$A \mathrel{-\!\!*} (B \to C) \to ((A \mathrel{-\!\!*} B) \to (A \mathrel{-\!\!*} C)).$$

We consider the derivation $\mathcal{D}$

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{id}
\UnaryInfC{$\boxed{B \to C} : ac, \boxed{A} : c \vdash \boxed{A} : c$}
\AxiomC{}
\RightLabel{id}
\UnaryInfC{$\boxed{B} : (\varepsilon + a + b)c \vdash \boxed{B} : (\varepsilon + a + b)c$}
\AxiomC{}
\RightLabel{id}
\UnaryInfC{$\boxed{C} : (\varepsilon + a + b)c \vdash \boxed{C} : (\varepsilon + a + b)c$}
\RightLabel{$\to_L$}
\BinaryInfC{$\boxed{B \to C} : (\varepsilon + a + b)c, \boxed{B} : (\varepsilon + a + b)c \vdash \boxed{C} : (\varepsilon + a + b)c$}
\RightLabel{str. rules}
\UnaryInfC{$\boxed{B \to C} : ac, \boxed{B} : bc \vdash \boxed{C} : (\varepsilon + a + b)c$}
\RightLabel{$\mathrel{-\!\!*}_L$}
\BinaryInfC{$A : c, \boxed{A \mathrel{-\!\!*} B} : b, B \to C : ac \vdash \boxed{C} : (\varepsilon + a + b)c$}
\RightLabel{str. rules}
\UnaryInfC{$A : c, \boxed{A \mathrel{-\!\!*} B} : b, \boxed{B \to C} : ac \vdash \boxed{C} : (\varepsilon + a + b)c$}
\end{prooftree}
$$

and we continue with

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{id}
\UnaryInfC{$\boxed{A \mathrel{-\!\!*} B} : b, \boxed{A} : c \vdash \boxed{A} : c$}
\AxiomC{$\mathcal{D}$}
\RightLabel{$\mathrel{-\!\!*}_L$}
\BinaryInfC{$\boxed{A \mathrel{-\!\!*} (B \to C)} : a, \boxed{A \mathrel{-\!\!*} B} : b, \boxed{A} : c \vdash \boxed{C} : (\varepsilon + a + b)c$}
\RightLabel{$\mathrel{-\!\!*}_R$}
\UnaryInfC{$\boxed{A \mathrel{-\!\!*} (B \to C)} : a, \boxed{A \mathrel{-\!\!*} B} : b \vdash \boxed{A \mathrel{-\!\!*} C} : \varepsilon + a + b$}
\RightLabel{$\to_R$}
\UnaryInfC{$\boxed{A \mathrel{-\!\!*} (B \to C)} : a \vdash \boxed{(A \mathrel{-\!\!*} B) \to (A \mathrel{-\!\!*} C)} : \varepsilon + a$}
\RightLabel{$\to_R$}
\UnaryInfC{$\vdash \boxed{(A \mathrel{-\!\!*} (B \to C)) \to ((A \mathrel{-\!\!*} B) \to (A \mathrel{-\!\!*} C))} : \varepsilon$}
\end{prooftree}
$$

and the corresponding derivation in LBI is

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A}\,\text{id} \qquad \dfrac{\dfrac{}{A \vdash A}\,\text{id} \qquad \dfrac{\dfrac{}{B \vdash B}\,\text{id} \qquad \dfrac{}{C \vdash C}\,\text{id}}{B \to C; B \vdash C}\,{\to_L}}{(A \mathrel{-\!\!*} B, A); B \to C \vdash C}\,{\mathrel{-\!\!*}_L}}{(A \mathrel{-\!\!*} (B \to C), A); (A \mathrel{-\!\!*} B, A) \vdash C}\,{\mathrel{-\!\!*}_L}}{A, (A \mathrel{-\!\!*} (B \to C); (A \mathrel{-\!\!*} B) \vdash C}\,\textbf{s.d.}}{(A \mathrel{-\!\!*} (B \to C); (A \mathrel{-\!\!*} B) \vdash A \mathrel{-\!\!*} C}\,{\mathrel{-\!\!*}_R}}{A \mathrel{-\!\!*} (B \to C) \vdash (A \mathrel{-\!\!*} B) \to (A \mathrel{-\!\!*} C)}\,{\to_R}}{\varnothing_a; A \mathrel{-\!\!*} (B \to C) \vdash (A \mathrel{-\!\!*} B) \to (A \mathrel{-\!\!*} C)}\,\equiv}{\varnothing_a \vdash (A \mathrel{-\!\!*} (B \to C)) \to ((A \mathrel{-\!\!*} B) \to (A \mathrel{-\!\!*} C))}\,{\to_R}$$

Again, observe that in the second last step, we use the equivalence of the bunches

$$\varnothing_a; A \mathrel{-\!\!*} (B \to C) \quad \text{and} \quad A \mathrel{-\!\!*} (B \to C)$$

and the corresponding rule in LBI.

6 Conclusion and perspectives

From the definition of a new labelled sequent calculus, called GBI we have proposed a procedure that translates LBI proofs into GBI proofs and proved its soundness. We have also discussed the steps towards the reverse translation of GBI proofs into LBI proofs and proposed an algorithm to define it. In the continuation of these works and results we have identified several directions, like the correctness proof of the Reconstruction Algorithm and also the connections between the labelled tableaux system TBI designed for BI [3] and with the non-labelled bunched sequent system LBI [9]. Let us give more details about these perspectives.

As simple as the Reconstruction Algorithm is in its formulation, its correctness is far from obvious. By construction, whenever the Reconstruction Algorithm manages to apply all corresponding rules in LBI, the result is a correct derivation of the required formula in LBI. The challenge however lies in the task of establishing that there exists no formula and derivation of it in GBI such that the Reconstruction Algorithm gets stuck. We intend to show its general correctness using a more complicated procedure which we call the Translation Algorithm, where we recursively translate a derivation in GBI to a derivation in LBI, using the quasi-semantic structure given by the labels and their relations to build up corresponding bunches. Then we expect to prove the following result: *Let A be a formula and $\mathcal{D}$ a derivation of A in* GBI*. If there exists a derivation of A in* LBI*, then the Reconstruction Algorithm applied to $\mathcal{D}$ does not get stuck and produces a derivation of A in* LBI$_{sd}$*.*

We aim at finding transparent and efficient procedures for translating TBI proofs to LBI proofs as well as from LBI proofs to TBI proofs, establishing the equivalence of these systems.
TBI itself is fairly well studied and understood [2,3] and for instance TBI has been proved sound and complete with respect to Kripke resource models.
We can first observe that GBI and TBI are similar systems, both using the machinery of labels and constraints. Actually, a multi-conclusion version of GBI is nothing but a reformulation of TBI in the form of a sequent calculus. Therefore, it will be easy to show that there is a procedure

to translate derivations from GBI to TBI. Using this result, we can chain the translations from LBI proofs to GBI proofs and then from GBI proofs to TBI proofs to get the following result : *There is a procedure to translate LBI proofs to TBI proofs.*
We can also turn towards the translation in the other direction. As it is straightforward to translate derivations from TBI into a multi-conclusion version of GBI, the problem lies in showing that a single conclusion is enough. For that we could start to show that there is a procedure to translate TBI proofs to proofs in a multi-conclusion version of GBI. From this result we will prove the following result: *Proofs in the multi-conclusion version of GBI can be restricted to GBI proofs.* Thus from both results we expect to define a procedure that translates TBI proofs to GBI proofs. As a consequence, since TBI was shown to be complete with respect to Kripke resource models, this would also imply that GBI is complete with respect to Kripke resource models as well. Moreover having verified the correctness of the Reconstruction Algorithm we will be able to deduce that LBI_{sd} is complete for Kripke resource models.

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