Effective Translations between Display and Labelled Proofs for Tense Logics

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Abstract

The expansive use of logic in various domains, from areas of computer science and artificial intelligence, to philosophy and epistemology, has resulted in the development of many new logics. Along with the introduction of each new logic, it has proven crucial to design an analytic calculus for the logic. A calculus is said to be analytic when every formula occurring in a proof of the calculus is a subformula of the formula to be proved. Due to this property, such a calculus allows for the stepwise decomposition of formulae, which has proven useful in decidability and automated deduction procedures. The sequent calculus, introduced by Gerhard Gentzen in the 1930's, has since been the preferred formalism for constructing analytic calculi due to its simplicity and ease of use.

Nevertheless, for many modal and related logics the sequent calculus formalism proves itself *too simple*, and so, many new extensions of the sequent calculus have been proposed over the last 30 years. Such extensions are achieved by augmenting Gentzen's sequent calculus with additional structure, thus increasing the complexity of the formalism but allowing for the construction of analytic calculi for a broader range of logics.

The multitude of new calculi were found to largely fall into two different camps: (i) internal calculi, where each object can be read as a formula of the logic and (ii) external calculi, where each object is a formula in a more expressive language partially encoding the semantics of the logic. Internal calculi have proven themselves useful in proving certain properties of logics such as interpolation and optimal complexity, whereas external calculi have proven useful in establishing results such as completeness and cut-admissibility.

The seeming bifurcation between the types of results obtained from internal and external calculi gives rise to questions concerning the core differences between the two formalisms. By studying the interrelationships between the two categories of calculi via translations, we can obtain results regarding the relative expressivity of the proof-theoretic languages, the relative complexity of proofs in each calculus, and transfer of results between the different calculi (See [2, 4, 7]). Moreover, defining translations between calculi opens up the possibility of switching to a formalism better suited for the task at hand.

My topic falls within the scope of this project, aiming to focus on two different calculi for the minimal tense logic Kt extended with any finite set of tense axioms of the form $\Pi p \rightarrow \Sigma p$ (with $\Pi, \Sigma \in \{\Diamond, \blacklozenge\}^*$). These logics have practical value, having been applied in verification and model checking. Moreover,

these axioms cover a wide range of interesting logics, such as logics for reasoning about relational properties such as transitivity, density, symmetry, and reflexivity. My work focuses on constructing effective translations for two proof calculi that concern these logics: a display calculus and a labelled calculus.

The display calculus for the logics of interest is an internal calculus (See [5, 9, 3]), whereas the labelled calculus is an external calculus (See [8, 6]). The former is constructed with the syntax of the logics in mind, while the latter is motivated and influenced by the Kripke semantics of the logics. Due to the different approaches to constructing the two calculi, and the different nature of each, designing a bi-directional translation from one to the other provides a fruitful case study of bridging the gap between internal and external calculi and between the multitude of different formalisms more generally.

The work of Goré *et al.* [3] offers a simplified display calculus for the tense logic Kt by making use of two-typed nested sequents. I show that each of these sequents can be viewed as a directed tree with two types of edges, and refer to such a structure as a *labelled UT* (where *UT* means *underlying tree graph*). Using this representation of display sequents, I then provide a bi-directional embedding between display calculus proofs and proofs in Negri's [6] labelled sequent calculus for Kt. Translating display proofs for Kt extended with tense axioms into labelled proofs for such logics is also shown and is based on published work (See [1]). Providing a reverse translation–translating labelled proofs into display proofs for Kt extended with tense axioms–has proven to be much more difficult and is a topic of current research. In the talk I will discuss the above translations as well as for which tense logics it is known that labelled proofs can be effectively translated into display proofs. The latter topic is of interest to those studying differences between internal and external calculi because the first step of the translation offers an efficient method of "internalizing" Negri's labelled calculi for such logics.

References

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