Relating Labelled and Label-Free Bunched Calculi in BI Logic*

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Abstract. In this paper we study proof translations between labelled and label-free calculi for the logic of Bunched Implications (BI). We first consider the bunched sequent calculus LBI and define a labelled sequent calculus, called GBI, in which labels and constraints reflect the properties of a specifically tailored Kripke resource semantics of BI with two total resource composition operators and explicit internalization of inconsistency. After showing the soundness of GBI w.r.t. our specific Kripke frames, we show how to translate any LBI-proof into a GBI-proof. Building on the properties of that translation we devise a tree property that every LBI-translated GBI-proof enjoys. We finally show that any GBI-proof enjoying this tree property (and not only LBI-translated ones) can systematically be translated to an LBI-proof.

1 Introduction

The ubiquitous notion of resource is a basic one in many fields but has become more and more central in the design and validation of modern computer systems over the past twenty years. Resource management encompasses various kinds of behaviours and interactions including consumption and production, sharing and separation, spatial distribution and mobility, temporal evolution, sequentiality or non-determinism, ownership and access control, etc.

Dealing with the various aspects of resource management is mostly in the territory of substructural logics, and more precisely, resource-aware logics such as Linear Logic (LL) [10] with its resource consumption interpretation, the logic of Bunched Implications (BI) [17,18] with its resource sharing interpretation, or order-aware non-commutative logic (NL) [1]. As specification logics, they model features like resource distribution and mobility, non-determinism, sequentiality or coordination of entities [4]. Separation Logic, of which BI is the logical kernel, has proved itself very successful as an assertion language for verifying programs that handle mutable data structures via pointers [12,19].

From a semantic point of view, resource interactions such as production and consumption, or separation and sharing are handled in resource models at the level of resource composition. For example, various semantics have been proposed to capture the resource sharing interpretation of BI including categorical, topological, relational and monoidal models [9]. From a proof-theoretic and

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purely syntactic point of view, the subtleties of a particular resource composition usually lead to the definition of distinct sets of connectives (*e.g.*, additive vs multiplicative, commutative vs non-commutative).

Capturing the interaction between various kinds of connectives often results in label-free calculi that deal with structures more elaborated than sets or multisets of formulas. For example, the standard label-free sequent calculus for BI, which is called LBI, admits sequents the left-hand part of which are structured as binary trees called bunches [15,18]. Resource interaction is usually much simpler to handle in labelled calculi since labels and label constraints are allowed to reflect and mimic, inside the calculus, the fundamental properties of the resource models they are drawn from. Several labelled tableaux or sequent-style systems have been proposed for BI and its variants [9,11,14].

Categorical, relational, topological and monoidal resource models with a Beth interpretation of the additive disjunction have all been proven sound and complete w.r.t. both LBI and TBI in [9,17,18]. Unfortunately, although by far the most widely used models of BI in the literature, monoidal resource models with a more usual Kripke interpretation of the additive disjunction have only been proven complete w.r.t. TBI. Their status w.r.t. LBI is not known and still a difficult open problem as many attempts at solving it from a semantic point of view have failed over the past fifteen years. Therefore, a better understanding of how LBI relates to labelled calculi could be very helpful as a first step towards solving the problem from the more syntactic standpoint of proof translations.

Our work takes place in the general context of studying the relationships between labelled and label-free calculi. In this paper we more particularly focus on the relationships between GBI, a sequent-style reworking of the labelled tableaux calculus TBI [8], and the label-free bunched sequent calculus LBI [18].

In Section 2 we recall the basic notions about BI and its label-free bunch sequent calculus LBI. We also introduce a non-standard resource semantics for BI based on two total monoidal operators with an explicit treatment of inconsistency from which we derive a new sequent-style labelled calculus called GBI in Section 3. GBI can be seen as an intermediate calculus between TBI and LBI as both calculi share the idea of sets of labels and constraints arranged as a resource graph, but the resource graph in GBI is partially constructed on the fly using explicit structural rules rather than being obtained as the result of a closure operator [8].

Section 4 is devoted to our first contribution which is a translation of any LBI-proof into a GBI-proof. This translation is not a one-to-one correspondence sending each LBI-rule occurring in the original proof to its corresponding GBI counterpart in the translated proof. Indeed, most of the translations patterns require several additional structural steps to obtain an actual GBI-proof. However, these patterns are such that the rule-application strategy of the original proof will be contained in the translated proof, making our translation structure preserving in that particular sense.

Section 5 investigates how GBI-proofs relate to LBI-proofs. We first restrict GBI to have a single formula on the right-hand side. This is justified by the fact that, contrary to related works on translating labelled or prefixed calculi to label-free sequent calculi, mainly in modal and (bi-)intuitionistic logics [16,20], we cannot rely on the existence of a multi-conclusioned variant of LBI. Such a variant would require bunches on the right-hand side of sequents, and thus the definition of an intuitionistic dual to multiplicative conjunction, which seems problematic, although there exists a multi-conclusioned display calculus for Boolean BI [3].

We define a tree property for single-conclusioned GBI labelled sequents which allows us to translate the left-hand side of a labelled sequent to a bunch according to the label of the formula on its right-hand side. Refining our analysis of the LBI-translation, we show that every sequent in a GBI-proof obtained by translation of an LBI-proof satisfies our tree property.

The second and main contribution finally follows the definition of a restricted variant of GBI the proofs of which always satisfy the tree property and can moreover systematically be translated into LBI-proofs. Let us remark that this result does not depend on a GBI-proof being some translated image of an LBI-proof. We thus observe that our tree property can serve as a criterion for defining a notion of normal GBI-proofs for which normality also means LBI-translatability.

2 The Logic BI

In this section, we give a short introduction to BI (see [18] for more details). We recall the bunched sequent calculus LBI and introduce a variant of the usual Kripke resource semantics.

2.1 Syntax and Sequent Calculus LBI

Let $\mathsf{Prop} = \{\, \mathrm{p}, \mathrm{q}, \dots \}$ be a countable set of propositional letters. The formulas of BI, the set of which is denoted Fm, are given by the grammar:

$$\mathrm{A} ::= \mathrm{p} \mid \top_{\mathrm{m}} \mid \mathrm{A} * \mathrm{A} \mid \mathrm{A} \mathrel{-\!\!*} \mathrm{A} \mid \top_{\mathrm{a}} \mid \bot \mid \mathrm{A} \wedge \mathrm{A} \mid \mathrm{A} \vee \mathrm{A} \mid \mathrm{A} \to \mathrm{A}$$

Bunches are rooted trees given by the following grammar:

$$\Gamma ::= \mathrm{A} \mid \varnothing_{\mathrm{a}} \mid \Gamma \,;\Gamma \mid \varnothing_{\mathrm{m}} \mid \Gamma \,, \Gamma$$

Equivalence of bunches $\equiv$ is given by commutative monoid equations for ";" and "," with units $\varnothing_{\mathrm{a}}$ and $\varnothing_{\mathrm{m}}$ respectively, together with the substitution congruence for subbunches.

The LBI sequent calculus is depicted in Fig. 1. LBI derives sequents of the form $\Gamma \vdash \mathrm{C}$, where Γ is a bunch and C is a formula. The notation $\Gamma(\Delta)$ denotes a bunch Γ that contains the bunch Δ as a subtree.

A formula C is a *theorem* of LBI iff $\varnothing_{\mathrm{m}} \vdash \mathrm{C}$ is provable in LBI. Let us remark that the cut rule is admissible in LBI [18]. In order to make LBI-proofs shorter, we often skip explicit uses of the exchange rules. We thus consider bunches up to commutativity of "," and ";". However, we do not consider associativity of bunches as implicit (*i.e.*, we do not consider "," and ";" as n-ary functors) since

$$\frac{}{A \vdash A}\,\mathrm{id} \qquad \frac{}{\varnothing_m \vdash \top_m}\,\top_m\mathrm{R} \qquad \frac{}{\varnothing_a \vdash \top_a}\,\top_a\mathrm{R} \qquad \frac{}{\Gamma(\bot) \vdash A}\,\bot\mathrm{L}$$

$$\frac{\Gamma(\varnothing_m) \vdash A}{\Gamma(\top_m) \vdash A}\,\top_m\mathrm{L} \qquad \frac{\Gamma(\varnothing_a) \vdash A}{\Gamma(\top_a) \vdash A}\,\top_a\mathrm{L} \qquad \frac{\Gamma(B) \vdash A \quad \Gamma(C) \vdash A}{\Gamma(B \vee C) \vdash A}\,\vee\mathrm{L} \qquad \frac{\Gamma \vdash A_{i \in \{1,2\}}}{\Gamma \vdash A_1 \vee A_2}\,\vee^i_\mathrm{R}$$

$$\frac{\Delta \vdash B \quad \Gamma(C) \vdash A}{\Gamma(B \mathrel{-\!\!*} C \,,\, \Delta) \vdash A}\,{-\!\!*}\mathrm{L} \qquad \frac{\Gamma \,,\, A \vdash B}{\Gamma \vdash A \mathrel{-\!\!*} B}\,{-\!\!*}\mathrm{R} \qquad \frac{\Gamma(B \,,\, C) \vdash A}{\Gamma(B * C) \vdash A}\,{*}\mathrm{L} \qquad \frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma \,,\, \Delta \vdash A * B}\,{*}\mathrm{R}$$

$$\frac{\Delta \vdash B \quad \Gamma(C) \vdash A}{\Gamma(B \to C \,;\, \Delta) \vdash A}\,{\to}\mathrm{L} \qquad \frac{\Gamma \,;\, A \vdash B}{\Gamma \vdash A \to B}\,{\to}\mathrm{R} \qquad \frac{\Gamma(B \,;\, C) \vdash A}{\Gamma(B \wedge C) \vdash A}\,\wedge\mathrm{L} \qquad \frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma \,;\, \Delta \vdash A \wedge B}\,\wedge\mathrm{R}$$

$$\frac{\Gamma(\Delta_1) \vdash A}{\Gamma(\Delta_1 \,;\, \Delta_2) \vdash A}\,\mathrm{W} \qquad \frac{\Gamma(\Delta \,;\, \Delta) \vdash A}{\Gamma(\Delta) \vdash A}\,\mathrm{C} \qquad \frac{\Gamma \vdash A}{\Delta \vdash A}\,\Gamma \equiv \Delta \qquad \frac{\Delta \vdash B \quad \Gamma(B) \vdash A}{\Gamma(\Delta) \vdash A}\,\mathrm{Cut}$$

Rules replacing $\equiv$:

$$\frac{\Gamma(\Delta_1 \,,\, \Delta_2) \vdash A}{\Gamma(\Delta_2 \,,\, \Delta_1) \vdash A}\,\mathrm{E}_m \qquad \frac{\Gamma((\Delta_1 \,,\, \Delta_2) \,,\, \Delta_3) \vdash A}{\Gamma(\Delta_1 \,,(\Delta_2 \,,\, \Delta_3)) \vdash A}\,\mathrm{A}_m \qquad \frac{\Gamma(\Delta) \vdash A}{\Gamma(\varnothing_m \,,\, \Delta) \vdash A}\,\mathrm{U}_m$$

$$\frac{\Gamma(\Delta_1 \,;\, \Delta_2) \vdash A}{\Gamma(\Delta_2 \,;\, \Delta_1) \vdash A}\,\mathrm{E}_a \qquad \frac{\Gamma((\Delta_1 \,;\, \Delta_2) \,;\, \Delta_3) \vdash A}{\Gamma(\Delta_1 \,;(\Delta_2 \,;\, \Delta_3)) \vdash A}\,\mathrm{A}_a \qquad \frac{\Gamma(\Delta) \vdash A}{\Gamma(\varnothing_a \,;\, \Delta) \vdash A}\,\mathrm{U}_a$$

Fig. 1. The Sequent Calculus LBI.

it easily leads to unexpected difficulties when adapting results from unassociative systems, *e.g.* in [13] where the decidability of BI is erroneously concluded from the decidability of the Lambek calculus using length and depth arguments on the representation of bunches from [7] that actually fail in the presence of associativity (and contraction).

The rule for equivalence of bunches can easily be replaced with the last six rules given in Fig. 1, where double lines indicate rules that work both ways (*i.e.*, rules for which the premiss and the conclusion can be swapped). We distinguish bottom-up and top-down uses of such rules in LBI-proofs with up and down arrows respectively. For technical reasons, the rules replacing $\equiv$ will be prefered in the proofs of the forthcoming translation theorems.

Fig. 2 gives an example of a proof in LBI, which also shows that the set of derivable sequents in cut-free LBI gets strictly smaller if contraction is removed or restricted to a single formula.

Lemma 1. *The following semi-distributivity rule is derivable in* LBI*:*

$$\frac{\Gamma((\Delta_1 \,,\, \Delta_2) \,;(\Delta_1 \,,\, \Delta_3)) \vdash A}{\Gamma(\Delta_1 \,,(\Delta_2 \,;\, \Delta_3)) \vdash A}\,\mathrm{Sd}$$

Proof. Use contraction on $(\Delta_1 \,,(\Delta_2 \,;\, \Delta_3))$ followed by two weakenings.

$$
\dfrac{
  \dfrac{}{\mathrm{p} \vdash \mathrm{p}}\,\mathrm{id}
  \qquad
  \dfrac{
    \dfrac{}{\mathrm{p} \vdash \mathrm{p}}\,\mathrm{id}
    \qquad
    \dfrac{
      \dfrac{}{\mathrm{q} \vdash \mathrm{q}}\,\mathrm{id}
      \qquad
      \dfrac{}{\mathrm{r}\,;\mathrm{q} \vdash \mathrm{r}}\,\mathrm{id}
    }{\mathrm{q} \to \mathrm{r}\,;\mathrm{q} \vdash \mathrm{r}}\,\to_{\mathrm{L}}
  }{\mathrm{q} \to \mathrm{r}\,;(\mathrm{p} \mathrel{-\!\!*} \mathrm{q}\,,\mathrm{p}) \vdash \mathrm{r}}\,\mathrel{-\!\!*}_{\mathrm{L}}
}{
  \dfrac{
    \dfrac{(\mathrm{p} \mathrel{-\!\!*} (\mathrm{q} \to \mathrm{r})\,,\mathrm{p})\,;(\mathrm{p} \mathrel{-\!\!*} \mathrm{q}\,,\mathrm{p}) \vdash \mathrm{r}}
    {(\mathrm{p} \mathrel{-\!\!*} (\mathrm{q} \to \mathrm{r})\,,\mathrm{p})\,;((\mathrm{p} \mathrel{-\!\!*} (\mathrm{q} \to \mathrm{r})\,;\mathrm{p} \mathrel{-\!\!*} \mathrm{q})\,,\mathrm{p}) \vdash \mathrm{r}}\,\mathrm{W}
  }{
    \dfrac{((\mathrm{p} \mathrel{-\!\!*} (\mathrm{q} \to \mathrm{r})\,;\mathrm{p} \mathrel{-\!\!*} \mathrm{q})\,,\mathrm{p})\,;((\mathrm{p} \mathrel{-\!\!*} (\mathrm{q} \to \mathrm{r})\,;\mathrm{p} \mathrel{-\!\!*} \mathrm{q})\,,\mathrm{p}) \vdash \mathrm{r}}
    {(\mathrm{p} \mathrel{-\!\!*} (\mathrm{q} \to \mathrm{r})\,;\mathrm{p} \mathrel{-\!\!*} \mathrm{q})\,,\mathrm{p} \vdash \mathrm{r}}\,\mathrm{C}
  }\,\mathrm{W}
}\,\mathrel{-\!\!*}_{\mathrm{L}}
$$

Fig. 2. A Proof in LBI.

Lemma 2. *Adding semi-distributivity to* LBI *while restricting contraction to* $\varnothing_{\mathrm{m}}$ *(or* $\top_{\mathrm{m}}$*) leads to the same set of derivable sequents.*

Proof. Contraction is derivable from contraction on $\varnothing_{\mathrm{m}}$ and semi-distributivity:

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{\Gamma(\Delta\,;\Delta) \vdash \mathrm{A}}
      {\Gamma((\Delta\,,\varnothing_{\mathrm{m}})\,;(\Delta\,,\varnothing_{\mathrm{m}})) \vdash \mathrm{A}}\,\mathrm{U_m}\!\uparrow
    }{\Gamma((\varnothing_{\mathrm{m}}\,;\varnothing_{\mathrm{m}})\,,\Delta) \vdash \mathrm{A}}\,\mathrm{E_m} + \mathrm{Sd}
  }{\Gamma(\varnothing_{\mathrm{m}}\,,\Delta) \vdash \mathrm{A}}\,\mathrm{C}
}{\Gamma(\Delta) \vdash \mathrm{A}}\,\mathrm{U_m}\!\downarrow
$$

2.2 Semantics of BI

BI admits various semantics: monoidal, relational, topological, categorical, with or without explicit inconsistency [18]. We introduce a variant of the total (*i.e.*, with an explicit treatment of inconsistency) monoidal semantics [9] that makes use of two monoidal functors to better reflect the syntactic structure of bunches. Although the labelled tableau calculus TBI is known to be complete w.r.t. this semantics [9], whether it is also the case for LBI is still an open problem.

Definition 1 (Resource Monoid). *A* resource monoid (RM) *is a structure* $\mathcal{M} = (M, \otimes, 1, \oplus, 0, \infty, \sqsubseteq)$ *where* $(M, \otimes, 1)$, $(M, \oplus, 0)$ *are commutative monoids and* $\sqsubseteq$ *is a preordering relation on* M *such that:*

- *for all* $m \in M$, $m \sqsubseteq \infty$ *and* $\infty \sqsubseteq \infty \otimes m$,
- *for all* $m, n \in M$, $m \sqsubseteq m \oplus n$ *and* $m \oplus m \sqsubseteq m$,
- *if* $m \sqsubseteq n$ *and* $m' \sqsubseteq n'$, *then* $m \otimes m' \sqsubseteq n \otimes n'$ *and* $m \oplus m' \sqsubseteq n \oplus n'$.

Let us remark that the conditions of Definition 1 imply that ∞ and 0 respectively are greatest and least elements and that $\oplus$ is idempotent.

Definition 2 (Resource Interpretation). *Given a resource monoid* $\mathcal{M}$, *a* resource interpretation (RI) for $\mathcal{M}$, *is a function* $[\,-\,] : \mathsf{Fm} \longrightarrow \mathcal{P}(M)$ *satisfying* $\forall\, \mathrm{p} \in \mathsf{Prop}$, $\infty \in [\,\mathrm{p}\,]$ *and* $\forall\, m, n \in M$ *such that* $m \sqsubseteq n$, $m \in [\,\mathrm{p}\,] \Rightarrow n \in [\,\mathrm{p}\,]$.

Definition 3 (Kripke Resource Model). *A* Kripke resource model (KRM) *is a structure* $\mathcal{K} = (\mathcal{M}, \models, [\,-\,])$ *where* $\mathcal{M}$ *is a resource monoid,* $[\,-\,]$ *is a resource interpretation and* $\models$ *is a forcing relation such that:*

- $m \models \mathrm{p}$ *iff* $m \in [\,\mathrm{p}\,]$*,*
- $m \models \bot$ *iff* $\infty \sqsubseteq m$*,* $m \models \top_{\mathfrak{a}}$ *iff* $0 \sqsubseteq m$*,* $m \models \top_{\mathfrak{m}}$ *iff* $1 \sqsubseteq m$*,*
- $m \models \mathrm{A} * \mathrm{B}$ *iff for some* n, n' *in* M *such that* $n \otimes n' \sqsubseteq m$*,* $n \models \mathrm{A}$ *and* $n' \models \mathrm{B}$*,*
- $m \models \mathrm{A} \wedge \mathrm{B}$ *iff for some* n, n' *in* M *such that* $n \oplus n' \sqsubseteq m$*,* $n \models \mathrm{A}$ *and* $n' \models \mathrm{B}$*,*
- $m \models \mathrm{A} \mathrel{-\!\!*} \mathrm{B}$ *iff for all* n, n' *in* M *such that* $n \models \mathrm{A}$ *and* $m \otimes n \sqsubseteq n'$*,* $n' \models \mathrm{B}$*,*
- $m \models \mathrm{A} \rightarrow \mathrm{B}$ *iff for all* n, n' *in* M *such that* $n \models \mathrm{A}$ *and* $m \oplus n \sqsubseteq n'$*,* $n' \models \mathrm{B}$*,*
- $m \models \mathrm{A} \vee \mathrm{B}$ *iff* $m \models \mathrm{A}$ *or* $m \models \mathrm{B}$.

The semantic clauses for the additive connectives are stated so as to be perfectly symmetric with their multiplicative counterparts (as is the case of their corresponding syntactic rules in LBI). Although such clauses might seem strange at first sight, they are easily proven equivalent to their more usual definitions.

A formula A is valid in the Kripke resource semantics iff $1 \models \mathrm{A}$ in all Kripke resource models.

3 The Labelled Calculus GBI

In this section we define a new labelled calculus for BI in the spirit of [2,5,6] and prove its soundness w.r.t. the resource semantics given in Section 2.

A countable set L of symbols is a set of *label letters* if it is disjoint from the set $U = \{\, \mathrm{m}, \mathrm{a}, \varpi \,\}$ of *label units.* $\mathcal{L}^0_L = L \cup U$ is the set of *atomic labels over L*. The set $\mathcal{L}_L$ of *labels over L* is defined as $\bigcup_{n \in \mathbb{N}} \mathcal{L}^n_L$ where

$$\mathcal{L}^{n+1}_L = \mathcal{L}^n_L \cup \{\, \mathfrak{r}(\ell, \ell') \mid \ell, \ell' \in \mathcal{L}^n_L \text{ and } \mathfrak{r} \in \{\, \mathfrak{m}, \mathfrak{a} \,\}\,\}.$$

For readability, we often drop the subscript L when L is clear from the context. A *label constraint* is an expression $\ell \leqslant \ell'$, where ℓ and ℓ' are labels. A *labelled formula* is an expression $\mathrm{A} : \ell$, where A is a formula and ℓ is a label.

In full generality, the labelled sequent calculus GBI deals with sequents of the form $\Gamma \vdash \Delta$, where Γ is a multiset mixing both labelled formulas and label constraints and Δ is a multiset of labelled formulas. From now on, we only deal with the single-conclusioned variant of GBI where Δ is restricted to exactly one labelled formula. This restriction is justified by the fact that this paper is a first step at understanding how purely syntactic LBI-proofs relate to GBI-proofs and LBI is a single-conclusioned calculus. Similarly to bunches, we use the notation $\Gamma(\Delta)$ for a multiset Γ which contains Δ as a sub-multiset.

The structural rules of GBI are given in Fig. 3. They syntactically reflect the semantic properties of the binary operators $\otimes$, $\oplus$ and the binary relation $\sqsubseteq$ into the binary functors $\mathfrak{m}$, $\mathfrak{a}$ and the binary relation $\leqslant$. The units 1, 0 and ∞ are reflected into the labels units m, a and ϖ. We generically write $\mathfrak{r}$ (resp. r) to denote either $\mathfrak{m}$ or $\mathfrak{a}$ (resp. m and a) in contexts where the multiplicative or additive nature of the functor (resp. unit) is not important (*e.g.*, for properties that hold in both cases).

$$\frac{\ell \leqslant \ell, \Gamma \vdash \Delta}{\Gamma \vdash \Delta}\,\mathrm{R} \qquad \frac{\ell_0 \leqslant \ell, \ell_0 \leqslant \ell_1, \ell_1 \leqslant \ell, \Gamma \vdash \Delta}{\ell_0 \leqslant \ell_1, \ell_1 \leqslant \ell, \Gamma \vdash \Delta}\,\mathrm{T}$$

$$\frac{\mathfrak{r}(\ell, \mathrm{r}) \leqslant \ell, \Gamma \vdash \Delta}{\Gamma \vdash \Delta}\,\mathrm{U}^1_{\mathfrak{r}} \qquad \frac{\mathfrak{r}(\mathrm{r}, \ell) \leqslant \ell, \Gamma \vdash \Delta}{\Gamma \vdash \Delta}\,\mathrm{U}^2_{\mathfrak{r}} \qquad \frac{\mathfrak{r}(\ell_2, \ell_1) \leqslant \ell, \Gamma \vdash \Delta}{\mathfrak{r}(\ell_1, \ell_2) \leqslant \ell, \Gamma \vdash \Delta}\,\mathrm{E}_{\mathfrak{r}}$$

$$\frac{\mathfrak{r}(\ell_3, \ell_2) \leqslant \ell_0, \mathfrak{r}(\ell_4, \ell_0) \leqslant \ell, \Gamma \vdash \Delta}{\mathfrak{r}(\ell_4, \ell_3) \leqslant \ell_1, \mathfrak{r}(\ell_1, \ell_2) \leqslant \ell, \Gamma \vdash \Delta}\,\mathrm{A}^1_{\mathfrak{r}} \qquad \frac{\mathfrak{r}(\ell_1, \ell_4) \leqslant \ell_0, \mathfrak{r}(\ell_0, \ell_3) \leqslant \ell, \Gamma \vdash \Delta}{\mathfrak{r}(\ell_4, \ell_3) \leqslant \ell_2, \mathfrak{r}(\ell_1, \ell_2) \leqslant \ell, \Gamma \vdash \Delta}\,\mathrm{A}^2_{\mathfrak{r}}$$

$$\frac{\mathfrak{a}(\ell, \ell) \leqslant \ell, \Gamma \vdash \Delta}{\Gamma \vdash \Delta}\,\mathrm{I}_{\mathfrak{a}} \qquad \frac{\ell_i \leqslant \ell, \mathfrak{a}(\ell_1, \ell_2) \leqslant \ell, \Gamma \vdash \Delta}{\mathfrak{a}(\ell_1, \ell_2) \leqslant \ell, \Gamma \vdash \Delta}\,\mathrm{P}^{\mathrm{i}}_{\mathfrak{a}} \qquad \frac{\ell_i \leqslant \ell, \mathfrak{m}(\ell_1, \ell_2) \leqslant \ell, \Gamma \vdash \Delta}{\mathfrak{m}(\ell_1, \ell_2) \leqslant \ell, \Gamma \vdash \Delta}\,\mathrm{P}^{\mathrm{i}}_{\mathfrak{m}}$$

$$\frac{\mathfrak{r}(\ell_0, \ell_2) \leqslant \ell, \ell_0 \leqslant \ell_1, \mathfrak{r}(\ell_1, \ell_2) \leqslant \ell, \Gamma \vdash \Delta}{\ell_0 \leqslant \ell_1, \mathfrak{r}(\ell_1, \ell_2) \leqslant \ell, \Gamma \vdash \Delta}\,\mathrm{C}^1_{\mathfrak{r}} \qquad \frac{\ell \leqslant \ell_1, \Gamma, \mathrm{A} : \ell_1 \vdash \Delta}{\ell \leqslant \ell_1, \Gamma, \mathrm{A} : \ell \vdash \Delta}\,\mathrm{K_L}$$

$$\frac{\mathfrak{r}(\ell_1, \ell_0) \leqslant \ell, \ell_0 \leqslant \ell_2, \mathfrak{r}(\ell_1, \ell_2) \leqslant \ell, \Gamma \vdash \Delta}{\ell_0 \leqslant \ell_2, \mathfrak{r}(\ell_1, \ell_2) \leqslant \ell, \Gamma \vdash \Delta}\,\mathrm{C}^2_{\mathfrak{r}} \qquad \frac{\ell_1 \leqslant \ell, \Gamma \vdash \mathrm{A} : \ell_1, \Delta}{\ell_1 \leqslant \ell, \Gamma \vdash \mathrm{A} : \ell, \Delta}\,\mathrm{K_R}$$

$$\frac{\Gamma_0 \vdash \Delta}{\Gamma_0, \Gamma_1 \vdash \Delta}\,\mathrm{W_L} \qquad \frac{\Gamma \vdash \Delta_0}{\Gamma \vdash \Delta_0, \Delta_1}\,\mathrm{W_R} \qquad \frac{\Gamma_0, \Gamma_1, \Gamma_1 \vdash \Delta}{\Gamma_0, \Gamma_1 \vdash \Delta}\,\mathrm{C_L} \qquad \frac{\Gamma \vdash \Delta_0, \Delta_1, \Delta_1}{\Gamma \vdash \Delta_0, \Delta_1}\,\mathrm{C_R}$$

Side conditions:
$i \in \{1, 2\}$ and $\mathfrak{r} \in \{\mathfrak{m}, \mathfrak{a}\}$.
ℓ_0 is a fresh label letter in $\mathrm{A}^{\mathrm{i}}_{\mathfrak{r}}$. ℓ_i in $\mathrm{P}^{\mathrm{i}}_{\mathfrak{m}}$ must be in $\{\mathrm{m}, \varpi\}$.
ℓ in R and $\mathrm{I}_{\mathfrak{a}}$, ℓ_1, ℓ_2 in $\mathrm{P}^{\mathrm{i}}_{\mathfrak{a}}$ and ℓ_{3-i} in $\mathrm{P}^{\mathrm{i}}_{\mathfrak{m}}$ must occur in Γ, Δ or $\{\mathrm{m}, \mathrm{a}, \varpi\}$.

Fig. 3. Structural Rules of GBI.

We begin with rules R and T to capture the reflexivity and transitivity of the accessibility relation. Then we continue with rules $\mathrm{U}^{\mathrm{i}}_{\mathfrak{r}}$ that capture the identity of the functors $\mathfrak{m}$ and $\mathfrak{a}$ w.r.t. m and a. The superscript $i \in \{1, 2\}$ in GBI-rule names denotes which argument of an underlying $\mathfrak{r}$-functor is treated by the rule. We then proceed with rules $\mathrm{A}^{\mathrm{i}}_{\mathfrak{r}}$ and $\mathrm{E}_{\mathfrak{r}}$ for associativity and commutativity of the $\mathfrak{r}$-functors. In the presence of explicit exchange rules $\mathrm{E}_{\mathfrak{r}}$, or if we implicitly consider the $\mathfrak{r}$-functors as commutative (which we do not), the superscript variants of the rules are not needed. We nevertheless keep them as they help drastically reduce explicit uses of $\mathrm{E}_{\mathfrak{r}}$. The rule $\mathrm{I}_{\mathfrak{a}}$ reflects the idempotency of $\oplus$ into the $\mathfrak{a}$-functor. The projection rules $\mathrm{P}^{\mathrm{i}}_{\mathfrak{a}}$ reflect into the $\mathfrak{a}$-functor the fact that $\oplus$ is increasing, *i.e.*, $m \sqsubseteq m \oplus n$. The projection rules $\mathrm{P}^{\mathrm{i}}_{\mathfrak{m}}$ capture the fact that $m \sqsubseteq m \otimes n$ generally only holds if n is ∞ or 1. The compatibility rules $\mathrm{C}^{\mathrm{i}}_{\mathfrak{r}}$ reflect that $\oplus$ and $\otimes$ are both order preserving. Finally the last six rules simply express Kripke monotonicity, weakening and contraction.

The logical rules of GBI are given in Fig. 4. and are direct translations of their semantic clauses. Fig. 7 gives an example of a proof in GBI, where the notation "$-$" subsumes all the elements we omit to keep proofs more compact.

Definition 4. *A formula* A *is a theorem of* GBI *if the sequent* $\mathrm{m} \leqslant \ell \vdash \mathrm{A} : \ell$ *is provable in* GBI *for some label letter* ℓ*.*

$$\frac{}{\Gamma, \varpi \leqslant \ell \vdash \mathrm{A} : \ell, \Delta}\ \bot_{\mathrm{R}} \qquad \frac{}{\Gamma, \mathrm{A} : \ell \vdash \mathrm{A} : \ell, \Delta}\ \mathrm{id} \qquad \frac{}{\Gamma, \mathrm{m} \leqslant \ell \vdash \top_{\mathfrak{m}} : \ell, \Delta}\ \top_{\mathfrak{m}\mathrm{R}}$$

$$\frac{\Gamma, \varpi \leqslant \ell \vdash \Delta}{\Gamma, \bot : \ell \vdash \Delta}\ \bot_{\mathrm{L}} \quad \frac{\Gamma, \mathrm{m} \leqslant \ell \vdash \Delta}{\Gamma, \top_{\mathfrak{m}} : \ell \vdash \Delta}\ \top_{\mathfrak{m}\mathrm{L}} \quad \frac{\Gamma, \mathrm{a} \leqslant \ell \vdash \Delta}{\Gamma, \top_{\mathfrak{a}} : \ell \vdash \Delta}\ \top_{\mathfrak{a}\mathrm{L}} \quad \frac{}{\Gamma, \mathrm{a} \leqslant \ell \vdash \top_{\mathfrak{a}} : \ell, \Delta}\ \top_{\mathfrak{a}\mathrm{R}}$$

$$\frac{\mathfrak{a}(\ell, \ell_1) \leqslant \ell_2, \Gamma \vdash \mathrm{A} : \ell_1, \Delta \qquad \mathfrak{a}(\ell, \ell_1) \leqslant \ell_2, \Gamma, \mathrm{B} : \ell_2 \vdash \Delta}{\mathfrak{a}(\ell, \ell_1) \leqslant \ell_2, \Gamma, \mathrm{A} \rightarrow \mathrm{B} : \ell \vdash \Delta}\ \rightarrow_{\mathrm{L}}$$

$$\frac{\mathfrak{m}(\ell, \ell_1) \leqslant \ell_2, \Gamma \vdash \mathrm{A} : \ell_1, \Delta \qquad \mathfrak{m}(\ell, \ell_1) \leqslant \ell_2, \Gamma, \mathrm{B} : \ell_2 \vdash \Delta}{\mathfrak{m}(\ell, \ell_1) \leqslant \ell_2, \Gamma, \mathrm{A} \mathrel{-\!\!*} \mathrm{B} : \ell \vdash \Delta}\ \mathrel{-\!\!*}_{\mathrm{L}}$$

$$\frac{\mathfrak{a}(\ell, \ell_1) \leqslant \ell_2, \Gamma, \mathrm{A} : \ell_1 \vdash \mathrm{B} : \ell_2, \Delta}{\Gamma \vdash \mathrm{A} \rightarrow \mathrm{B} : \ell, \Delta}\ \rightarrow_{\mathrm{R}} \qquad \frac{\mathfrak{m}(\ell, \ell_1) \leqslant \ell_2, \Gamma, \mathrm{A} : \ell_1 \vdash \mathrm{B} : \ell_2, \Delta}{\Gamma \vdash \mathrm{A} \mathrel{-\!\!*} \mathrm{B} : \ell, \Delta}\ \mathrel{-\!\!*}_{\mathrm{R}}$$

$$\frac{\mathfrak{a}(\ell_1, \ell_2) \leqslant \ell, \Gamma, \mathrm{A} : \ell_1, \mathrm{B} : \ell_2 \vdash \Delta}{\Gamma, \mathrm{A} \wedge \mathrm{B} : \ell \vdash \Delta}\ \wedge_{\mathrm{L}} \qquad \frac{\mathfrak{m}(\ell_1, \ell_2) \leqslant \ell, \Gamma, \mathrm{A} : \ell_1, \mathrm{B} : \ell_2 \vdash \Delta}{\Gamma, \mathrm{A} * \mathrm{B} : \ell \vdash \Delta}\ *_{\mathrm{L}}$$

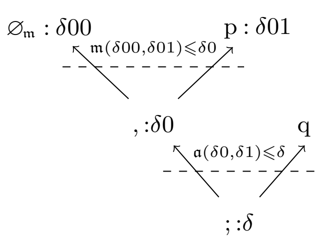
$$\frac{\mathfrak{a}(\ell_1, \ell_2) \leqslant \ell, \Gamma \vdash \mathrm{A} : \ell_1, \Delta \qquad \mathfrak{a}(\ell_1, \ell_2) \leqslant \ell, \Gamma \vdash \mathrm{B} : \ell_2, \Delta}{\mathfrak{a}(\ell_1, \ell_2) \leqslant \ell, \Gamma \vdash \mathrm{A} \wedge \mathrm{B} : \ell, \Delta}\ \wedge_{\mathrm{R}}$$

$$\frac{\mathfrak{m}(\ell_1, \ell_2) \leqslant \ell, \Gamma \vdash \mathrm{A} : \ell_1, \Delta \qquad \mathfrak{m}(\ell_1, \ell_2) \leqslant \ell, \Gamma \vdash \mathrm{B} : \ell_2, \Delta}{\mathfrak{m}(\ell_1, \ell_2) \leqslant \ell, \Gamma \vdash \mathrm{A} * \mathrm{B} : \ell, \Delta}\ *_{\mathrm{R}}$$

$$\frac{\Gamma, \mathrm{A} : \ell \vdash \Delta \qquad \Gamma, \mathrm{B} : \ell \vdash \Delta}{\Gamma, \mathrm{A} \vee \mathrm{B} : \ell \vdash \Delta}\ \vee_{\mathrm{L}} \qquad \frac{\Gamma \vdash \mathrm{A}_{\mathrm{i} \in \{1,2\}} : \ell, \Delta}{\Gamma \vdash \mathrm{A}_1 \vee \mathrm{A}_2 : \ell, \Delta}\ \vee_{\mathrm{R}}^{\mathrm{i}}$$

Side conditions: ℓ_1 and ℓ_2 must be fresh label letters in $*_{\mathrm{L}}$, $\wedge_{\mathrm{L}}$, $\mathrel{-\!\!*}_{\mathrm{R}}$, and $\rightarrow_{\mathrm{R}}$.

Fig. 4. Logical Rules of GBI.



$$\begin{array}{c}
\mathfrak{m}(\delta 00, \delta 01) \leqslant \delta 0, \mathfrak{a}(\delta 0, \delta 1) \leqslant \delta, \mathrm{p} : \delta 01, \mathrm{q} : \delta 1 \vdash \mathrm{r} : \delta \\
\uparrow \\
\mathfrak{a}(\delta 0, \delta 1) \leqslant \delta, (\varnothing_{\mathfrak{m}}\, , \mathrm{p}) : \delta 0, \mathrm{q} : \delta 1 \vdash \mathrm{r} : \delta \\
\uparrow \\
((\varnothing_{\mathfrak{m}}\, , \mathrm{p})\, ; \mathrm{q}) : \delta \vdash \mathrm{r} : \delta
\end{array}$$

Fig. 5. Translation of the LBI-sequent $(\varnothing_{\mathfrak{m}}\, , \mathrm{p})\ ;\ \mathrm{q} \vdash \mathrm{r}$

$$\frac{\Gamma(\mathfrak{a}(\delta s0, \delta s1) \leqslant \delta s, \Theta : \delta s0, \Theta : \delta s1) : \delta \vdash \mathrm{A} : \delta}{\Gamma(\Theta : \delta s) : \delta \vdash \mathrm{A} : \delta}\ \mathrm{C_T}$$

$$\frac{\Gamma(\mathfrak{r}(\delta s0, \delta s1) \leqslant \delta s, \mathrm{r} \leqslant \delta s\bar{x}, \Theta : \delta sx) : \delta \vdash \mathrm{A} : \delta}{\Gamma(\Theta : \delta s) : \delta \vdash \mathrm{A} : \delta}\ \mathrm{Z}_{\mathfrak{r}}^{x+1}$$

Fig. 6. Tree-like Structural Rules of GBI.

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{id}
\UnaryInfC{$-, \mathrm{p} : \ell_4 \vdash \mathrm{p} : \ell_4$}
\AxiomC{}
\RightLabel{id}
\UnaryInfC{$-, \mathrm{q} : \ell_1 \vdash \mathrm{q} : \ell_1$}
\RightLabel{$\mathrm{K_R}$}
\UnaryInfC{$\ell_1 \leqslant \ell_2, -, \mathrm{q} : \ell_1, \mathrm{p} : \ell_4 \vdash \mathrm{q} : \ell_2$}
\RightLabel{$\mathrm{P}^2_{\mathfrak{m}}$}
\UnaryInfC{$-, \mathfrak{m}(\ell_0, \ell_1) \leqslant \ell_2, -, \mathrm{q} : \ell_1, \mathrm{p} : \ell_4 \vdash \mathrm{q} : \ell_2$}
\RightLabel{$-\!\!*_{\mathrm{L}}$}
\BinaryInfC{$\mathfrak{m}(\ell_3, \ell_4) \leqslant \ell_1, -, \mathrm{p} -\!\!* \mathrm{q} : \ell_3, \mathrm{p} : \ell_4 \vdash \mathrm{q} : \ell_2$}
\RightLabel{$*_{\mathrm{L}}$}
\UnaryInfC{$\mathfrak{m}(\ell_0, \ell_1) \leqslant \ell_2, -, (\mathrm{p} -\!\!* \mathrm{q}) * \mathrm{p} : \ell_1 \vdash \mathrm{q} : \ell_2$}
\RightLabel{$-\!\!*_{\mathrm{R}}$}
\UnaryInfC{$\mathrm{m} \leqslant \ell_0 \vdash ((\mathrm{p} -\!\!* \mathrm{q}) * \mathrm{p}) -\!\!* \mathrm{q} : \ell_0$}
\end{prooftree}
$$

Fig. 7. A Proof in GBI.

3.1 Soundness of GBI

Definition 5 (Realization). *Let $\mathcal{K} = (\mathcal{M}, \models, [\,-\,])$ be a Kripke resource model with $\mathcal{M} = (M, \otimes, 1, \oplus, 0, \infty, \sqsubseteq)$. Let $s = \Gamma \vdash \Delta$ be a labelled sequent. A* realization *of s in $\mathcal{K}$ is a total function ρ from the labels of s to M such that:*

- *$\rho(\mathrm{m}) = 1$, $\rho(\mathfrak{m}(\ell_1, \ell_2)) = \rho(\ell_1) \otimes \rho(\ell_2)$,*
- *$\rho(\mathrm{a}) = 0$, $\rho(\varpi) = \infty$, $\rho(\mathfrak{a}(\ell_1, \ell_2)) = \rho(\ell_1) \oplus \rho(\ell_2)$,*
- *for all $\ell_1 \leqslant \ell_2$ in Γ, $\rho(\ell_1) \sqsubseteq \rho(\ell_2)$ in $\mathcal{M}$,*
- *for all $\mathrm{A} : \ell$ in Γ, $\rho(\ell) \models \mathrm{A}$ and for all $\mathrm{A} : \ell$ in Δ, $\rho(\ell) \not\models \mathrm{A}$.*

We say that s is realizable in $\mathcal{K}$ *if there exists a realization of s in $\mathcal{K}$ and that s is* realizable *if it is realizable in some Kripke resource model $\mathcal{K}$.*

Lemma 3. *If in a* GBI*-proof the sequent $s = \Gamma \vdash \Delta$ is an initial sequent,* i.e.*, a leaf sequent that is the conclusion of a zero-premiss rule, then s is not realizable.*

Proof. Suppose that s is realizable, then we have a realization ρ of s in some Kripke resource model $\mathcal{K} = (\mathcal{M}, \models, [\,-\,])$. We proceed by case analysis on the zero-premiss rule r of which s is the conclusion. If r is id then s has the form $\Gamma, \mathrm{A} : \ell \vdash \mathrm{A} : \ell, \Delta$, which implies the contradiction $\rho(\ell) \models \mathrm{A}$ and $\rho(\ell) \not\models \mathrm{A}$. If r is $\top_{\mathfrak{m}\mathrm{L}}$ then s has the form $\mathrm{m} \leqslant \ell, \Gamma \vdash \top_{\mathfrak{m}} : \ell, \Delta$ so that both $\rho(\ell) \not\models \top_{\mathfrak{m}}$ and $1 \sqsubseteq \rho(\ell)$, which is a contradiction since $1 \sqsubseteq \rho(\ell)$ implies $\rho(\ell) \models \top_{\mathfrak{m}}$. Similarly for the case when r is $\top_{\mathfrak{a}\mathrm{L}}$. Finally, if r is $\bot_{\mathrm{R}}$ then s has the form $\Gamma, \varpi \leqslant \ell \vdash \mathrm{A} : \ell, \Delta$ so that $\infty \sqsubseteq \rho(\ell)$ and $\rho(\ell) \not\models \mathrm{A}$, which is a contradiction because by Kripke monotonicity, $\rho(\ell) \models \mathrm{A}$.

Lemma 4. *Every proof-rule in* GBI *preserves realizability.*

Proof. By case analysis of the proof rules of GBI.

Theorem 1 (Soundness). *If a formula* A *is provable in* GBI*, then it is valid in the Kripke resource semantics of* BI.

Proof. Suppose that A is provable in GBI but not valid in the Kripke resource semantics of BI. Then, the sequent $\vdash \mathrm{A} : \mathrm{m}$ is trivially realizable and we have a GBI-proof $\mathcal{P}$ of A. It follows from Lemma 4 that $\mathcal{P}$ contains a branch the sequents of which are all realizable. Since $\mathcal{P}$ is a proof, the branch ends with an initial (axiom) sequent and Lemma 3 implies that this initial sequent is not realizable, which is a contradiction. Therefore, A is valid.

4 From LBI-Proofs to GBI-Proofs

In this section, we introduce the concepts for translating sequents of LBI to sequents of GBI. In order to highlight the relationships between the labels and the tree structure of bunches more easily we use label letters of the form xs where x is a non-greek letter and $s \in \{0, 1\}^*$ is a binary string that encodes the path of the node xs in a tree structure the root of which is x. We thus call x the root of a label letter xs. We use greek letters to range over label letters with the convention that distinct greek letters denote label letters with distinct roots.

Definition 6. *Given a bunch* Γ *and a label letter* δ*, we define* $\mathfrak{L}(\Gamma, \delta)$*, the* translation of Γ *according to* δ*, by induction on the structure of* Γ *as follows:*

- $\mathfrak{L}(\mathrm{A}, \delta) = \{\mathrm{A} : \delta\}$, $\mathfrak{L}(\varnothing_{\mathfrak{a}}, \delta) = \{\mathrm{a} \leqslant \delta\}$, $\mathfrak{L}(\varnothing_{\mathfrak{m}}, \delta) = \{\mathrm{m} \leqslant \delta\}$,
- $\mathfrak{L}((\Delta_0 \,, \Delta_1), \delta) = \mathfrak{L}(\Delta_0, \delta 0) \cup \mathfrak{L}(\Delta_1, \delta 1) \cup \{\mathfrak{m}(\delta_0, \delta_1) \leqslant \delta\}$,
- $\mathfrak{L}((\Delta_0 \,; \Delta_1), \delta) = \mathfrak{L}(\Delta_0, \delta 0) \cup \mathfrak{L}(\Delta_1, \delta 1) \cup \{\mathfrak{a}(\delta_0, \delta_1) \leqslant \delta\}$.

The definition extends to LBI*-sequents as follows:* $\mathfrak{L}(\Gamma \vdash \mathrm{A}, \delta) = \mathfrak{L}(\Gamma, \delta) \vdash \mathrm{A} : \delta$.

We write $\Gamma : \delta$ as a shorthand for $\mathfrak{L}(\Gamma, \delta)$ so that $\mathfrak{L}(\Gamma \vdash \mathrm{A}, \delta) = \Gamma : \delta \vdash \mathrm{A} : \delta$. An illustration of Definition 6 is given in Fig. 5. Let Δ be a sub-bunch of Γ, then for any label letter δ, $\Gamma : \delta$ will contain the multiset $\Delta : \delta s$ for some (possibly empty) binary suffix s, in which case we write $\Gamma(\Delta : \delta s) : \delta$.

Before translating LBI-proofs into GBI-proofs we introduce the notion of *label substitution*, which is a mapping from label letters to atomic labels, written $[\alpha_1 \mapsto \ell_1, \ldots, \alpha_n \mapsto \ell_n]$. Since label letters have the form xs where s is a binary string, we write $\alpha \hookrightarrow \ell$ as a shorthand for $\forall s.\, \alpha s \mapsto \ell s$.

Theorem 2. *If a sequent* $\Gamma \vdash \mathrm{A}$ *is provable in* LBI*, then for any label letter* δ*, the labelled sequent* $\Gamma : \delta \vdash \mathrm{A} : \delta$ *is provable in* GBI.

Proof. The proof is by induction on the height of LBI-proofs, using a case distinction on the last rule R applied. We show that for an arbitrary label letter δ, we can build a GBI-proof of the translation of the conclusion of R from translations of its premises. Several LBI-rules that operate on a bunch Δ that can be nested inside a bunch $\Gamma(\Delta)$ require a careful distinction between their shallow (no actual Γ around Δ) and deep variants. We only consider a few cases, the others being similar.

- Axiom id: $\dfrac{}{\mathrm{A} \vdash \mathrm{A}}\,\mathrm{id}$ is translated to $\dfrac{}{\mathrm{A} : \alpha \vdash \mathrm{A} : \alpha}\,\mathrm{id}$
- Axiom $\top_{\mathfrak{m}}\mathrm{R}$: $\dfrac{}{\varnothing_{\mathfrak{m}} \vdash \top_{\mathfrak{m}}}\,\top_{\mathfrak{m}}\mathrm{R}$ is translated to $\dfrac{}{\mathrm{m} \leqslant \alpha \vdash \top_{\mathfrak{m}} : \alpha}\,\top_{\mathfrak{m}}\mathrm{R}$
- Axiom $\top_{\mathfrak{a}}\mathrm{R}$: This case is similar to $\top_{\mathfrak{m}}\mathrm{R}$
- Case $\mathrel{-\!\!*}_{\mathrm{R}}$: Consider the LBI-proof depicted below on the left-hand side where $\mathcal{D}$ is a proof of $\Gamma \,; \mathrm{A} \vdash \mathrm{B}$, the premiss of $\mathrel{-\!\!*}_{\mathrm{R}}$

$$\dfrac{\dfrac{\mathcal{D}}{\Gamma \,, \mathrm{A} \vdash \mathrm{B}}}{\Gamma \vdash \mathrm{A} \mathrel{-\!\!*} \mathrm{B}}\,\mathrel{-\!\!*}_{\mathrm{R}} \qquad \dfrac{\mathcal{P}}{\mathfrak{m}(\alpha 0, \alpha 1) \leqslant \alpha, \Gamma : \alpha 0, \mathrm{A} : \alpha 1 \vdash \mathrm{B} : \alpha}$$

Given an arbitrary label letter δ, we are required to build a GBI-proof of $\Gamma : \delta \vdash \mathrm{A} \mathrel{-\!\!*} \mathrm{B} : \delta$. By I.H. on $\mathcal{D}$ for some label letter α, we have a proof $\mathcal{P}$ of $(\Gamma\,, \mathrm{A}) : \alpha \vdash \mathrm{B} : \alpha$ depicted above on the right-hand side from which we get

$$
\dfrac{\begin{array}{c} \mathcal{P}[\,\alpha 0 \hookrightarrow \delta\,] \\ \hdashline \mathfrak{m}(\delta, \alpha 1) \leqslant \alpha, \Gamma : \delta, \mathrm{A} : \alpha 1 \vdash \mathrm{B} : \alpha \end{array}}{\Gamma : \delta \vdash \mathrm{A} \mathrel{-\!\!*} \mathrm{B} : \delta}\ {-\!\!*}_{\mathrm{R}}
$$

Let us note that $\alpha 1$ and α are indeed fresh labels in the premiss of ${-\!\!*}_{\mathrm{R}}$ since by convention α and δ have distinct roots.

– Case W (Shallow): By I.H. suppose we have for some α

$$
\dfrac{\begin{array}{c} \mathcal{D} \\ \hdashline \Delta_0 \vdash \mathrm{A} \end{array}}{\Delta_0\,;\Delta_1 \vdash \mathrm{A}}\ \mathrm{W}
\qquad\qquad
\begin{array}{c} \mathcal{P} \\ \hdashline \Delta_0 : \alpha \vdash \mathrm{A} : \alpha \end{array}
$$

We then construct the following proof

$$
\dfrac{\dfrac{\dfrac{\begin{array}{c} \mathcal{P}[\,\alpha \hookrightarrow \delta 0\,] \\ \hdashline \Delta_0 : \delta 0 \vdash \mathrm{A} : \delta 0 \end{array}}{\delta 0 \leqslant \delta, \mathfrak{a}(\delta 0, \delta 1) \leqslant \delta, \Delta_0 : \delta 0, \Delta_1 : \delta 1 \vdash \mathrm{A} : \delta 0}\ \mathrm{W_L}}{\delta 0 \leqslant \delta, \mathfrak{a}(\delta 0, \delta 1) \leqslant \delta, \Delta_0 : \delta 0, \Delta_1 : \delta 1 \vdash \mathrm{A} : \delta}\ \mathrm{K_R}}{\mathfrak{a}(\delta 0, \delta 1) \leqslant \delta, \Delta_0 : \delta 0, \Delta_1 : \delta 1 \vdash \mathrm{A} : \delta}\ \mathrm{P}^1_{\mathfrak{a}}
$$

Let us note that we used $\mathrm{W_L}$ to make the premiss of $\mathrm{K_R}$ exactly match $\mathcal{P}[\,\alpha \hookrightarrow \delta 0\,]$. We can get rid of $\mathrm{W_L}$ in all translation patterns by pasting the missing material to every sequent in the proofs obtained by I.H.

– Case $\mathrm{U}_{\mathfrak{m}}{\downarrow}$ (Deep): Suppose we have a proof

$$
\dfrac{\begin{array}{c} \mathcal{D} \\ \hdashline \Gamma(\varnothing_{\mathfrak{m}}\,, \Delta) \vdash \mathrm{A} \end{array}}{\Gamma(\Delta) \vdash \mathrm{A}}\ \mathrm{U}_{\mathfrak{m}}{\downarrow}
$$

By I.H., for some α, $s \in \{0,1\}^*$ and $x \in \{0,1\}$, we have a proof

$$
\begin{array}{c} \mathcal{P} \\ \hdashline \Gamma(\mathfrak{m}(\alpha s x 0, \alpha s x 1) \leqslant \alpha s x, \mathrm{m} \leqslant \alpha s x 0, \Delta : \alpha s x 1) : \alpha \vdash \mathrm{A} : \alpha \end{array}
$$

We then construct the following proof

$$
\dfrac{\dfrac{\begin{array}{c} \mathcal{P}[\,\alpha s x 0 \mapsto \mathrm{m}\,][\,\alpha s x 1 \hookrightarrow \delta s x\,][\,\alpha \hookrightarrow \delta\,] \\ \hdashline \Gamma(\mathfrak{m}(\mathrm{m}, \delta s x) \leqslant \delta s x, \mathrm{m} \leqslant \mathrm{m}, \Delta : \delta s x) : \delta \vdash \mathrm{A} : \delta \end{array}}{\Gamma(\delta s x \leqslant \delta s x, \mathrm{m} \leqslant \mathrm{m}, \Delta : \delta s x) : \delta \vdash \mathrm{A} : \delta}\ \mathrm{U}^1_{\mathfrak{m}}}{\Gamma(\Delta : \delta s x) : \delta \vdash \mathrm{A} : \delta}\ \mathrm{R}
$$

Using tree-like identity we get an alternative proof

$$
\dfrac{\begin{array}{c} \mathcal{P}[\,\alpha \hookrightarrow \delta\,] \\ \hdashline \Gamma(\mathfrak{m}(\delta s x 0, \delta s x 1) \leqslant \delta s x, \mathrm{m} \leqslant \delta s 0, \Delta : \delta s x 1) : \delta \vdash \mathrm{A} : \delta \end{array}}{\Gamma(\Delta : \delta s x) : \delta \vdash \mathrm{A} : \delta}\ \mathrm{Z}^1_{\mathfrak{m}}
$$

LBI	GBI	LBI	GBI	LBI	GBI	LBI	GBI
id	id	$-\!\!*_{\mathrm{R}}$	$-\!\!*_{\mathrm{R}}$	$\wedge_{\mathrm{L}}$	$\wedge_{\mathrm{L}}$	$\mathrm{W(S)}$	$\mathrm{P}^1_{\mathfrak{a}}\ \ \mathrm{K_R}\ \cancel{\mathrm{W_L}}$
$\top_{\mathfrak{m}\mathrm{R}}$	$\top_{\mathfrak{m}\mathrm{R}}$	$\rightarrow_{\mathrm{R}}$	$\rightarrow_{\mathrm{R}}$	$*_{\mathrm{R}}$	$*_{\mathrm{R}}\ \cancel{\mathrm{W_L}}$	$\mathrm{W(D)}$	$\mathrm{P}^1_{\mathfrak{a}}\ \ \mathrm{C}^{\mathrm{i}}_{\mathrm{t}}\ \ \cancel{\mathrm{W_L}}$
$\top_{\mathfrak{a}\mathrm{R}}$	$\top_{\mathfrak{a}\mathrm{R}}$	$-\!\!*_{\mathrm{L}}$	$-\!\!*_{\mathrm{L}}\ \cancel{\mathrm{W_L}}$	$\wedge_{\mathrm{R}}$	$\wedge_{\mathrm{R}}\ \cancel{\mathrm{W_L}}$	C	$(\mathrm{C_L}\ \mathrm{I}_{\mathfrak{a}})$ or $\mathrm{C_T}$
$\perp_{\mathrm{L}}(\mathrm{S})$	$\perp_{\mathrm{L}}\ \perp_{\mathrm{R}}$	$\rightarrow_{\mathrm{L}}$	$\rightarrow_{\mathrm{L}}\ \cancel{\mathrm{W_L}}$	$\mathrm{E}_{\mathfrak{r}}$	$\mathrm{E}_{\mathfrak{r}}$	$\mathrm{U}_{\mathfrak{r}}{\uparrow}(\mathrm{S})$	$\mathrm{C}^1_{\mathfrak{r}}\ \ \mathrm{P}^2_{\mathfrak{r}}\ \mathrm{K_R}\ \cancel{\mathrm{W_L}}$
$\perp_{\mathrm{L}}(\mathrm{D})$	$\perp_{\mathrm{L}}\ (\mathrm{C}^{\mathrm{i}}_{\mathrm{t}}\ \ \mathrm{P}^{\mathrm{i}}_{\mathrm{t}}\ )^+\ \perp_{\mathrm{R}}$	$\vee_{\mathrm{L}}$	$\vee_{\mathrm{L}}$	$\mathrm{A}_{\mathfrak{r}}{\uparrow}$	$\mathrm{A}^1_{\mathfrak{r}}\ \cancel{\mathrm{W_L}}$	$\mathrm{U}_{\mathfrak{r}}{\uparrow}(\mathrm{D})$	$\mathrm{C}^1_{\mathfrak{r}}\ \ \mathrm{P}^2_{\mathfrak{r}}\ \mathrm{C}^{\mathrm{i}}_{\mathrm{t}}\ \ \cancel{\mathrm{W_L}}$
$\top_{\mathfrak{m}\mathrm{L}}$	$\top_{\mathfrak{m}\mathrm{L}}$	$\vee^{\mathrm{i}}_{\mathrm{R}}$	$\vee^{\mathrm{i}}_{\mathrm{R}}$	$\mathrm{A}_{\mathfrak{r}}{\downarrow}$	$\mathrm{A}^2_{\mathfrak{r}}\ \cancel{\mathrm{W_L}}$	$\mathrm{U}_{\mathfrak{r}}{\downarrow}$	$(\mathrm{R}\ \mathrm{U}^1_{\mathfrak{r}})$ or $\mathrm{Z}^1_{\mathfrak{m}}$
$\top_{\mathfrak{a}\mathrm{L}}$	$\top_{\mathfrak{a}\mathrm{L}}$	$*_{\mathrm{L}}$	$*_{\mathrm{L}}$				

Fig. 8. Translation Patterns with $\mathrm{t}, \mathfrak{r} \in \{\mathfrak{m}, \mathfrak{a}\}$, $\mathrm{i} \in \{1, 2\}$, S=Shallow, D=Deep.

Fig. 8 summarizes the translation patterns from LBI to GBI, where left-to-right reading of the rules means bottom-up application in a proof. We write $\cancel{\mathrm{W_L}}$ to indicate the patterns for which explicit uses of weakening in GBI can be discarded as explained in the proof of Theorem 2. In LBI, the rules $-\!\!*_{\mathrm{L}}$, $\rightarrow_{\mathrm{L}}$, $*_{\mathrm{R}}$ and $\wedge_{\mathrm{R}}$ require context splitting, which is problematic for bottom-up proof-search. Removing weakening from GBI is desirable as context splitting is no longer needed, which also makes the labelled calculus more interesting as its sequents become more than just an isomorphic term-like transcription of bunches. Besides, removing $\mathrm{W_L}$ allows the translation to send all logical rules in LBI directly to their counterpart in GBI. Finally, we also learn from the patterns that $\mathrm{K_R}$ instead of $\mathrm{C}^{\mathrm{i}}_{\mathfrak{r}}$ is what distinguishes the shallow cases from the deep ones, that $\mathrm{I}_{\mathfrak{a}}$ identifies contraction in LBI while R identifies upward identity $\mathrm{U}_{\mathfrak{r}}{\uparrow}$ and that T and $\mathrm{K_L}$ are never used and can thus be removed from GBI without harming its ability to prove any LBI-provable formula.

5 Back from GBI-Proofs to LBI-Proofs

In this section we define the notion of normal GBI-proofs and show how to translate them into LBI-proofs. The main problem is that bunches are binary trees, while label constraints describe graphs that capture the accessibility relations between the worlds of a resource model. We observe that translating a bunch as of Definition 6 results in label constraints encoding a binary tree, which might only be destroyed by the rules $\mathrm{W_L}$, $\mathrm{I}_{\mathfrak{a}}$ and $\mathrm{U}^{\mathrm{i}}_{\mathfrak{r}}$. Using label letters of the form xs, we can formulate (without requiring explicit substitutions) two tree preserving rules $\mathrm{C_T}$ and $\mathrm{Z}^{\mathrm{i}}_{\mathfrak{r}}$ described in Fig. 6. $\mathrm{C_T}$ duplicates the whole subtree Θ rooted at δs into two subtrees rooted at $\delta s0$ and $\delta s1$ (thus renaming all labels in the new subtrees) and inserts a new node δs as the parent of the duplicated subtrees. $\mathrm{Z}^{\mathrm{i}}_{\mathfrak{r}}$ behaves similarly except that one of the new subtrees is linked with the unit r.

From now on, without harming completeness w.r.t. LBI, we restrict GBI to the rules that are actually used in the patterns of Fig. 8 (discarding $\mathrm{W_L}$) and replace $\mathrm{C_L}$ and $\mathrm{U}^{\mathrm{i}}_{\mathfrak{r}}$ with $\mathrm{C_T}$ and $\mathrm{Z}^{\mathrm{i}}_{\mathfrak{r}}$ of Fig. 6. We also slightly modify LBI: we discard the surrounding $\Gamma(-)$ in the axiom $\perp_{\mathrm{L}}$ and extend the weakening rule to "," whenever the bunch to weaken is $\perp$.

Let $\Gamma \vdash \mathrm{A}$ be labelled sequent with label letters in a set of label letters L. For $\mathfrak{r} \in \{\mathfrak{a}, \mathfrak{m}\}$, Γ induces a *subterm relation* $\twoheadrightarrow = (\twoheadrightarrow_{\mathfrak{a}} \cup \twoheadrightarrow_{\mathfrak{m}})$ defined as follows:

$$\ell_0 \twoheadrightarrow_{\mathfrak{r}} \ell_1 \;\text{ iff }\; \ell_1 \in L \text{ and } \exists\, \ell_2(\mathfrak{r}(\ell_1, \ell_2) \leqslant \ell_0 \in \Gamma \text{ or } \mathfrak{r}(\ell_2, \ell_1) \leqslant \ell_0 \in \Gamma).$$

Intuitively, the subterm relation is intended to characterize the links from parent to children nodes when the relation represents a tree.

Γ also induces a *reduction relation* $\rightsquigarrow$ defined as follows:

$$\ell_0 \rightsquigarrow \ell_1 \;\text{ iff }\; \ell_1 \leqslant \ell_0 \in \Gamma, \ell_1 \in \mathcal{L}^0_L, \ell_0 \in L \text{ and } \ell_1 \neq \ell_0.$$

Intuitively, the reduction relation will help us track steps that trigger weakenings in LBI. A label ℓ_0 is irreducible in Γ if Γ has no redex $\ell_0 \rightsquigarrow \ell_1$. A redex $\ell_0 \rightsquigarrow \ell_1$ is minimal if ℓ_1 is irreducible. A reduction of ℓ_0 to ℓ_n in Γ is a path $\ell_0 \rightsquigarrow \ell_1 \ldots \rightsquigarrow \ell_n$ such that for all $0 \leqslant i < n$, $\ell_i \rightsquigarrow \ell_{i+1}$ in Γ. A reduction of ℓ_0 to ℓ_n is minimal if ℓ_n is irreducible. If all minimal reductions of ℓ_0 terminate with the same irreducible label ℓ_n, then ℓ_n is called the normal form of ℓ_0 (in Γ).

A label ℓ' is reachable from a label ℓ in Γ, written $\ell \rightarrowtail \ell'$, if $\ell = \ell'$ or there is a path P from ℓ to ℓ' with no redex pointing outside P, more formally, P is a sequence $\ell_0 \twoheadrightarrow \ell_1 \ldots \twoheadrightarrow \ell_n$ such that $\ell_0 = \ell$, $\ell_n = \ell'$ and for all $0 \leqslant i < n$ and all ℓ'' such that $\ell_i \rightsquigarrow \ell''$, $\ell'' \in P$. If $\mathrm{A} : \ell \in \Gamma$ then A is an ℓ-leaf in Γ. A label constraint $\ell_2 \leqslant \ell_1$ is reachable from ℓ_0 in Γ if ℓ_1 is reachable from ℓ_0 and there is no formula A and no irreducible ℓ' on the path from ℓ_0 to ℓ_1 such $\mathrm{A} : \ell' \in \Gamma$.

Definition 7 (Tree Property). *A labelled sequent* $\Gamma \vdash \Delta$ *has the* tree property *if it satisfies all of the following conditions:*

(T_1) $\Delta = \{\mathrm{A} : \ell\}$ *and* $\mathrm{A} : \ell$ *is called the* root formula *with* root label ℓ*,*
(T_2) *for all* $\mathrm{C} : \ell_0 \in \Gamma \cup \Delta$*,* ℓ_0 *is a label letter,*
(T_3) *for all* $\ell_1 \leqslant \ell_0 \in \Gamma$*,* ℓ_0 *is a label letter and if so is* ℓ_1 *then* $\ell_0 \twoheadrightarrow \ell_1$*,*
(T_4) *for all* $\mathfrak{r}(\ell_1, \ell_2) \leqslant \ell_0 \in \Gamma$*,* ℓ_1 *and* ℓ_2 *are atomic,*
(T_5) *if* $\ell \rightarrowtail \ell_0$ *and* ℓ_0 *is reducible then* ℓ_0 *has a normal form and* Γ *has no* ℓ_0*-leaf,*
(T_6) *if* $\ell \rightarrowtail \ell_0$ *and* ℓ_0 *is irreducible,* Γ *has exactly one* ℓ_0*-leaf,*
(T_7) *the set* $\{\ell_1 \twoheadrightarrow \ell_0 \mid \ell \rightarrowtail \ell_0\}$ *is a tree with root* ℓ *in which all internal nodes have exactly two children linked with* $\twoheadrightarrow_{\mathfrak{r}}$ *arrows of the same* $\mathfrak{r}$ *type.*

A GBI*-proof has the* tree property *iff all of its sequents have the tree property.*

A careful analysis of the translation patterns shows that all LBI-translated GBI-proofs satisfy conditions (T_1) to (T_6). (T_7) might seem very restrictive as it implies that for all sequents s in a GBI-proof and all labels ℓ in s, s contains at most one corresponding label constraint of the form $\mathfrak{r}(\ell_1, \ell_2) \leqslant \ell$. Actually, we can allow sequents in a proof to have more than one label constraint with the same label on its right-hand side as long as we can decide which one has to be used for the subterm relation to represent a tree structure. This can be achieved either by managing label constraints with a stack strategy, always picking the one which has been introduced into the sequent the most recently, or by using a notion of rank corresponding to the depth at which the label constraint has been introduced in (a bottom up reading of) the proof.

A sequent $\Gamma \vdash \mathrm{A} : \ell$ is *terminal* if it admits a proof of height 0. For any GBI proof-rule R, the *principal label* and *principal label constraints* of R are the labels and label constraints explicitly mentionned in the conclusion of R as written in Fig. 3, Fig. 4 or Fig. 6.

Definition 8. *A* GBI-*proof is* normal *if it satisfies the tree property, all of its terminal sequents are initial sequents and in all sequents s that are the conclusion of an instance of a proof-rule R, the principal label and principal label constraints of R in s can be reached from the root label of s.*

Given a finite set B of bunches we define (up to associativity and commutativity of bunches) $\mathcal{B}_{\mathfrak{a}}(B)$ as $\varnothing_{\mathrm{a}}$ if B is empty and $B_1 ; \ldots ; B_n$ with $B_i \in B$ otherwise. Similarly for $\mathcal{B}_{\mathfrak{m}}(B)$ w.r.t. $\varnothing_{\mathrm{m}}$ and ",".

Definition 9. *Given a labelled sequent $\Gamma \vdash \mathrm{A} : \ell$ in a normal* GBI-*proof, its translation to an* LBI-*sequent is defined as $\mathfrak{B}(\Gamma \vdash \mathrm{A} : \ell) = \Gamma@\ell \vdash \mathrm{A}$ where $\Gamma@\ell$ is defined by induction as follows:*

- $\Gamma@\mathrm{m} = \varnothing_{\mathrm{m}}$, $\Gamma@\mathrm{a} = \varnothing_{\mathrm{a}}$, $\Gamma@\varpi = \perp$,
- $\Gamma@\ell = \Gamma@\ell'$ *if for some* ℓ', $\ell \rightsquigarrow \ell'$ *in* Γ,
- *let* $\mathrm{L} = \{ \mathrm{A}_i \mid \mathrm{A}_i : \ell \in \Gamma \}$ *and* $\mathrm{S}_{\mathfrak{r}} = \{ \ell_i \mid \ell \twoheadrightarrow_{\mathfrak{r}} \ell_i \text{ in } \Gamma \}$,

$$\Gamma@\ell = \begin{cases} \mathcal{B}_{\mathfrak{a}}(\mathrm{L}) & \textit{if} \quad \mathrm{L} \neq \varnothing. \\ \mathcal{B}_{\mathfrak{a}}(\mathrm{S}_{\mathfrak{a}}) & \textit{if} \quad \mathrm{L} = \varnothing, \mathrm{S}_{\mathfrak{m}} = \varnothing, \mathrm{S}_{\mathfrak{a}} \neq \varnothing. \\ \mathcal{B}_{\mathfrak{m}}(\mathrm{S}_{\mathfrak{m}}) & \textit{if} \quad \mathrm{L} = \varnothing, \mathrm{S}_{\mathfrak{m}} \neq \varnothing, \mathrm{S}_{\mathfrak{a}} = \varnothing. \end{cases}$$

Theorem 3. *Any normal* GBI-*proof of a formula* A *can be translated into an* LBI-*proof of* $\varnothing_{\mathrm{m}} \vdash \mathrm{A}$.

Proof. The proof is by induction on the height of normal GBI-proofs. We only give a few illustrative cases, the others being similar.

- Base Case id: We show that the normal GBI-proof

$$\frac{}{\Gamma(\mathrm{A} : \ell) \vdash \mathrm{A} : \ell}\,\mathrm{id} \qquad \text{translates to} \qquad \frac{}{\Delta ; \mathrm{A} \vdash \mathrm{A}}\,\mathrm{id}$$

  Since A is a ℓ-leaf in Γ, Γ has no redex for ℓ. Therefore, $\Gamma@\ell$ is by definition a bunch of the form $\mathrm{A}_1 ; \ldots ; \mathrm{A}_n$ where $\mathrm{A} = \mathrm{A}_i$ for some $1 \leq i \leq n$ and $\mathrm{A}_i : \ell \in \Gamma$ for all $1 \leq i \leq n$. Up to associativity and commutativity, $\Gamma@\ell$ can therefore be rewritten as a bunch $\Delta ; \mathrm{A}$.
- Base Case $\top_{\mathrm{m}}\mathrm{R}$: We show that the normal GBI-proof

$$\frac{}{\Gamma(\mathrm{m} \leqslant \ell) \vdash \top_{\mathrm{m}} : \ell}\,\top_{\mathrm{m}}\mathrm{R} \qquad \text{translates to} \qquad \frac{}{\varnothing_{\mathrm{m}} \vdash \top_{\mathrm{m}}}\,\top_{\mathrm{m}}\mathrm{R}$$

  Since ℓ is a label letter, Γ has a redex $\ell \rightsquigarrow \mathrm{m}$. Therefore, Γ cannot have any ℓ-leaf, so that $\Gamma(\mathrm{m} \leqslant \ell)@\ell = \Gamma(\mathrm{m} \leqslant \ell)@\mathrm{m} = \varnothing_{\mathrm{m}}$.
- Case $\top_{\mathrm{m}}\mathrm{L}$: We show that the normal GBI-proof (below)

$$\frac{\begin{array}{c}\mathcal{D} \\ \hdashline s_1 = \Gamma(\mathrm{m} \leqslant \ell) \vdash \mathrm{A} : \ell_0\end{array}}{s_0 = \Gamma(\top_{\mathrm{m}} : \ell) \vdash \mathrm{A} : \ell_0}\,\top_{\mathrm{m}}\mathrm{L} \qquad \text{translates to} \qquad \frac{\begin{array}{c}\mathcal{P} \\ \hdashline \Delta(\varnothing_{\mathrm{m}}) \vdash \mathrm{A}\end{array}}{\Delta(\top_{\mathrm{m}}) \vdash \mathrm{A}}\,\top_{\mathrm{m}}\mathrm{L}$$

By I.H., we have an LBI-proof $\mathcal{P}$ of the sequent $\Gamma(\mathrm{m} \leqslant \ell)@\ell_0 \vdash \mathrm{A}$. Since $\mathcal{D}$ is normal, we have $\ell_0 \rightarrowtail \ell$ in the last two sequents s_0, s_1 so that ℓ is actually treated by the translation of s_1. Then $\Gamma(\mathrm{m} \leqslant \ell)@\ell_0$ is of the form $\Delta(\varnothing_{\mathrm{m}})$. Since $\top_{\mathrm{m}}$ is the only ℓ-leaf in s_0, $\Gamma(\top_{\mathrm{m}} : \ell)@\ell_0$ is $\Delta(\top_{\mathrm{m}})$.

- Case $\mathrm{K_R}$: Suppose we have a normal GBI-proof

$$\dfrac{\begin{array}{c}\mathcal{D}\\ \hdashline s_1 = \Gamma(\ell_1 \leqslant \ell) \vdash \mathrm{A} : \ell_1\end{array}}{s_0 = \Gamma(\ell_1 \leqslant \ell) \vdash \mathrm{A} : \ell}\ \mathrm{K_R}$$

Since ℓ is a label letter, Γ has a redex $\ell \rightsquigarrow \ell_1$. Therefore, Γ has no ℓ-leaf so that $\Gamma@\ell = \Gamma@\ell_1$ by definition. By I.H. we have an LBI-proof of $s_1@\ell_1$, which is also an LBI-proof of $s_0@\ell$.

- Case $-\!\!*_\mathrm{L}$: Suppose we have a normal GBI-proof

$$\dfrac{\begin{array}{c}\mathcal{D}_1\\ \hdashline s_1 = \Gamma(\mathfrak{m}(\ell_1,\ell_2) \leqslant \ell) \vdash \mathrm{B} : \ell_2\end{array} \qquad \begin{array}{c}\mathcal{D}_2\\ \hdashline s_2 = \Gamma(\mathfrak{m}(\ell_1,\ell_2) \leqslant \ell, \mathrm{C} : \ell) \vdash \mathrm{A} : \ell_0\end{array}}{s_0 = \Gamma(\mathfrak{m}(\ell_1,\ell_2) \leqslant \ell, \mathrm{B} -\!\!* \mathrm{C} : \ell_1) \vdash \mathrm{A} : \ell_0}\ -\!\!*_\mathrm{L}$$

Since $\mathfrak{m}(\ell_1,\ell_2) \leqslant \ell$ is reachable from ℓ_0 in s_0, s_0 contains no ℓ-leaf. $s_0@\ell_0$ then has the form $\Delta(\mathrm{B} -\!\!* \mathrm{C}, \Gamma@\ell_2) \vdash \mathrm{A}$ and $s_2@\ell_0$ has the form $\Delta(\mathrm{C}) \vdash \mathrm{A}$ since s_2 has a ℓ-leaf C making $\mathfrak{m}(\ell_1,\ell_2) \leqslant \ell$ unreachable from ℓ_0. By I.H., we have LBI-proofs $\mathcal{P}_1$, $\mathcal{P}_2$ of s_1, s_2 respectively, leading to the LBI-proof

$$\dfrac{\begin{array}{c}\mathcal{P}_1\\ \hdashline \Gamma@\ell_2 \vdash \mathrm{B}\end{array} \qquad \begin{array}{c}\mathcal{P}_2\\ \hdashline \Delta(\mathrm{C}) \vdash \mathrm{A}\end{array}}{\Delta(\mathrm{B} -\!\!* \mathrm{C}, \Gamma@\ell_2) \vdash \mathrm{A}}\ -\!\!*_\mathrm{L}$$

6 Conclusion and Future Work

In this paper we have shown how to translate any LBI-proof into a GBI-proof. We also showed how to translate (normal) GBI-proofs satisfying the tree property back into an LBI-proof. A first perspective is to investigate whether any GBI-proof can be normalized so as to satisfy the tree property. We conjecture that it is indeed the case. A second interesting perspective would be to find an effective (algorithmic) procedure translating TBI-proofs into GBI-proofs since TBI is known to be sound and complete w.r.t. total KRMs. Finally, a third perspective relies on the construction of counter-models in the KRM semantics of BI directly from failed GBI-proof attempts. This direction requires building countermodels from a single-conclusioned calculus in which backtracking is allowed. Those perspectives would help us to show that total Kripke monoidal models with explicit inconsistency are complete w.r.t. the label-free sequent calculus LBI, thus solving a long-lasting open problem.

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