The Computational Content of the Constructive Kruskal Tree Theorem

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May 10, 2017

Abstract

We present a Coq mechanization of an inductive proof of Kruskal's tree theorem on Well Quasi Orders and we discuss the computational content of that theorem.

Well Quasi Orders (WQO) are an important class of quasi orders (reflexive and transitive relations) which moreover satisfy the property of being well:

> A binary relation R over set/type X is *Well* if any infinite sequence $s : \mathbb{N} \to X$ contains a good pair, i.e. $i < j$ such that $R(s_i, s_j)$

But there are numerous classically equivalent characterizations of that property, see [1] for instance. WQO are stable under several constructs as exemplified by Dickson's lemma, the finite sequence theorem, Higman's lemma, Higman's theorem and Kruskal's tree theorem. Nachum Dershowitz decisively used Kruskal's tree theorem in Computer Science to show the termination of *recursive path orderings*. But WQOs can be used to show termination properties in a much larger contexts, see [5] for instance.

The Kruskal tree theorem states that the class of WQOs is stable under the tree homeomorphic embedding:

If $\leq$ is a WQO then `embed_tree_homeo`($\leq$) is a WQO.

One particular case of that theorem is Vazsonyi's conjecture: in every infinite set S of undecorated finite trees, there is a pair $t_1 \neq t_2 \in S$ of trees such that t_1 embeds into t_2. Solving that conjecture was certainly one of the main motivations behind the tree theorem.

There are many classical proofs of the tree theorem, including J.B. Kruskal's original proof. Among them, the most well known is the "short proof" of Crispin Nash-Williams based on the *minimal bad sequence* argument. That proof typically uses the excluded middle and the axiom of choice. It has been implemented in Isabelle/HOL by Christian Sternagel [3].

Contrary to classical proofs, there are few instances of intuitionistic proofs for Kruskal's tree theorem. Some require the assumption that the ground relation is decidable (e.g. [1, 2]). Veldman's [4] is the only published proof that does not require that decidability property, but it requires *Brouwer's thesis.* Moreover, no intuitionistic proof had been mechanized before.

The difficulty behind instuitionistic/constructive proofs of Kruskal's tree theorem is that proofs based on the *minimal bad sequence* argument typically uses the excluded middle and the axiom of choice. According to Veldman [4], Kruskal's original proof was much more intuitionistic in spirit. But it is also much longer. Another important obstacle is the following: the several classically equivalent definitions of the notion of WQO are (for most of them) not intuitionistically equivalent. Hence, the statement of the theorem depends (intuitionistically) on the choice of a particular definition of WQO, mostly of the Well property.

The inductive and type theoretical proof we have developed shows that a suitable intuitionistic formulation of Well is the notion of *Almost Full relation* as defined by Thierry Coquand [5] (there is also an intuitionistically equivalent formulation in terms of *Bar inductive predicates*). Hence, we prove the following inductive Kruskal tree theorem:

If a relation R is almost full then so is `embed_tree_homeo`(R)

From that theorem, we can intuitionistically derive Vazsonyi's conjecture: we keep the full power of Kruskal's theorem in that intuitionistically formulation.

Our proof follows the pattern of Veldman's [4] intuitionistic proof but the intuitionistic set-theoretic context is replaced by inductive type theory. As we use Coquand inductive formulation of almost full relation as a substitute of the well property, the *Brouwer's thesis* axiom used by Veldman is not necessary anymore: our proof is *axiom free*. We discuss the computational content of the almost full predicate and of the intuitionistic Kruskal tree theorem.

References

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