Kripke Models of Boolean BI and Invertible Resources

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The logic of Bunched Implications (BI) is composed of additives $\wedge$, $\vee$ and $\rightarrow$ and multiplicatives $*$ and $-\!\!*$. It differs from Linear Logic because the additive fragment of the logic can be either exactly Intuitionistic Logic (IL) or exactly Classical Logic (CL). Both have a nice Kripke semantics: the "original" BI where the implication $\rightarrow$ is interpreted intuitionistically corresponding to the class of *partially ordered partial monoids* (POPM) and Boolean BI (BBI) corresponds to the class of *partial monoids* (PM), the classical Kripke interpretation of $\rightarrow$ requiring a flat order. Separation and spatial logics are usually models of BBI rather than of BI. The relation between BI and BBI is often misunderstood: as we have recently discovered, it is possible to faithfully embed BI into BBI.

Labelled deduction methods transform the proof-search process into a semantic *constraints* solving problem. For resource logics like BI and BBI, the constraints on resources can be syntactically expressed by relations of the form $m \sqsubseteq n$ (for BI) or $m \sim n$ (for BBI). The *labels* m and n are multisets composed of *atomic labels*. Using these labelled methods, one can exhibit a sub-class of POPM complete for BI. It can be viewed as the set of (least) solutions of *sequences of constraints* of the form $ab \sqsubseteq m, am \sqsubseteq b, m \sqsubseteq b$ or $\epsilon \sqsubseteq m$ where m is an already defined label, a and b are new (atomic) labels and ϵ is the empty label. The solution to these constraints can be represented by a *resource graph* which is a finite structure provided the sequence of constraints is finite.

In the case of BBI, the complete sub-class of PM we obtain is described by *basic sequences of constraints*, which are composed of constraints of the form $ab \sim m, am \sim b$ or $\epsilon \sim m$ where m is already defined and a, b are new. Because of the symmetry of $\sim$, the solution of basic constraints is *not always finite* as was the case for BI. Indeed, the singleton constraint $\{ab \sim \epsilon\}$ has the group $(\mathbb{Z}, +)$ as solution (with $[a \mapsto -1, b \mapsto 1]$). Constraints solving in BBI involves new challenges when compared to BI: we now have to deal with *infinite models* and *invertible resources*.

We study the properties of basic constraints. We distinguish between atomic invertible labels (set I) and atomic non-invertible labels (set N) and by extension, between invertible labels (set $I^\star$) and non-invertible labels (set $N^+I^\star$). We present a general representation theorem for the solutions of constraints based on a matrix $M : N^\star \times N^\star \longrightarrow \mathcal{C}(\mathbb{Z}^I)$ of congruence classes of the $\mathbb{Z}$-module $(\mathbb{Z}^I, +)$. Congruence classes either empty or of the form $x + G$ where $x \in \mathbb{Z}^I$ and G is a subgroup of $\mathbb{Z}^I$. We also describe a constraint normalization procedure that discriminates invertible and non-invertible atomic labels.

From these results we derive properties of basic constraints: *regularity* and *lack of non-invertible squares*. The set of defined labels in $N^\star$ is thus finite (no non-invertible atomic label of N can occur twice) and regularity implies that each non-empty congruence class in the matrix M is of the form $M_{x,y} = \delta_{x,y} + G$ for some subgroup $G \subseteq \mathbb{Z}^I$ common to all the non empty $M_{x,y}$. Even though the solution of basic constraints can be infinite, we nevertheless provide a *finite representation* for it, as a finite matrix of congruence classes of $\mathbb{Z}^I$ and an algorithm to compute this representation from a basic sequence of constraints.

Invertibility has to be provided a concrete interpretation when labels are viewed as resources, which is a usual intuition for the models of BI and BBI. We discuss some possible intuitive interpretations for such invertible resources.