Quasi Morphisms for Almost Full Relations

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Introduction to WQOs and AF relations
Well Quasi Orders (WQO)

- Classical defn. for $R : \text{rel}_2 X$ (ie. $X \rightarrow X \rightarrow \text{Prop}$):
  - $R$ is a Quasi Order (refl., trans.)
  - Almost Full (AF): $\forall f : \mathbb{N} \rightarrow X, \exists i < j, R f_i f_j$
  - any $\infty$ sequence contains a good pair
  - univ. quantified over $\infty$ sequences, as classical wf.
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- Important in computer science and mathematics
  - termination: terminator rule, Karp-Miller
  - decidability: relevance logic (Kripke)
  - polynomial ideals and Gröbner basis (Hilbert)
  - Dickson, Higman, Kruskal, Robertson-Seymour
Well Quasi Orders (WQO)

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- This AF notion is constructively too weak:
  - requires added constructively “acceptable” axioms
  - Count. Ch., bar ind. princ. (Veldman&Bezem 93)
  - Stumps and Brouwer’s thesis (Veldman 2004)
  - limited to relations over $\mathbb{N}$
Quasi Morphisms for AF
D. Larchey-W.

Intro. to WQOs
WQOs
AF in TT
Why morphisms

Inductive AF
Ind. rules
Transfers
Rel. morphisms

Quasi morphisms
Transfers
The easy case
General case

FANs and choice
Bar predicates
FAN theorem
Choice principles

Termination & AF
From AF to WF
Bounding search

AF relations in Inductive Type Theory

About Brouwer’s Fan Theorem (Coquand 2003):

► intuitive explanation of this constructive weakness
► Almost Full: \( \forall f : \mathbb{N} \rightarrow X \)...
► only captures sequences \( \mathbb{N} \rightarrow X \) given by laws
► bar ind. predicates capture arbitrary \( \infty \) sequences
AF relations in Inductive Type Theory

- **About Brouwer’s Fan Theorem** (Coquand 2003):
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  - Almost Full: $\forall f : \mathbb{N} \to X$...
  - only captures sequences $\mathbb{N} \to X$ given by laws
  - bar ind. predicates capture arbitrary $\infty$ sequences

- Stronger (constructive) AF notions:
  - do not require added axioms
  - bar ind. predicates (Coquand&Fridlender 93)
  - ind. well-foundedness (Seisenberger 2003)
    - only for decidable relations
  - inductive AF relations (Vytiniostis et al. 2012)
Why (quasi) morphisms are important

- Veldman’s proof of Kruskal in Coq (DLW2015)
  - Major cleanup and refactoring (2022–24)
  - Morphisms used extensively
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- Surjective relational morphisms
  - Monotonicity, functional maps have drawbacks
  - But rel. morph. versatile tool to transfer AF
- Quasi morphisms
  - Emerged as an abstraction (was inlined)
  - Can be understood independently
  - Factors out FAN and bar inductive predicates
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- The project published on opam — coq
- Description: @GH/DmxLarchey/Coq-Kruskal
Almost Fullness inductively
Inductive Almost Full relations

- Inductive predicate (Vytiniostis et al. 2012)
- For $R : \text{rel}_2 X$, define $\text{af } R : \text{Prop (or Type)}$

\[
\frac{\forall x \ y, \ R \ x \ y}{\text{af } R} \quad \frac{\forall a, \ \text{af } R \uparrow a}{\text{af } R} \quad \langle \text{af\_full} \rangle \quad \langle \text{af\_lift} \rangle
\]
Inductive Almost Full relations

- Inductive predicate (Vytiniostis et al. 2012)
- For \( R : \text{rel}_2 X \), define \( \text{af } R : \text{Prop} \) (or Type)

\[
\forall x, y, R \times y, \quad \text{af } R \quad \text{(af_full)}
\]

\[
\forall a, \text{af } R \uparrow^a, \quad \text{af } R \quad \text{(af_lift)}
\]

- the lifted relation: \((R \uparrow^a) \times y \supseteq R \times y \lor R \, a \, x\)
- any seq. containing \( x \) (\( R \)-above \( a \)) is \( R \uparrow^a \)-good
Inductive Almost Full relations

- Inductive predicate (Vytiniostis et al. 2012)
- For $R : \text{rel}_2 X$, define $af R : \text{Prop}$ (or Type)

$$\forall x \, y, \, R \times y \quad \frac{\text{af } R}{\langle \text{af_full} \rangle} \quad \forall a, \, \text{af } R \uparrow a \quad \frac{\text{af } R}{\langle \text{af_lift} \rangle}$$

- the lifted relation: $(R \uparrow a) \times y \coloneqq R \times y \lor R \, a \, x$
- any seq. containing $x$ ($R$-above $a$) is $R \uparrow a$-good
- any sequence of liftings ultimately renders $R$ full and

$$af \, R \rightarrow \forall f : \mathbb{N} \rightarrow X, \exists_t m \exists i < j < m, \, R \, f_i \, f_j$$
Inductive Almost Full relations

- Inductive predicate (Vytiniostis et al. 2012)
- For $R : \text{rel}_2 X$, define $\text{af } R : \text{Prop}$ (or Type)

\[
\begin{align*}
\forall x\ y, \ R \times y & \quad \text{af } R \quad \langle \text{af\_full} \rangle \\
\forall a, \ \text{af } R \uparrow a & \quad \text{af } R \quad \langle \text{af\_lift} \rangle
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- any seq. containing $x$ ($R$-above $a$) is $R \uparrow a$-good
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\[
\text{af } R \to \forall f : \mathbb{N} \to X, \exists t m \exists i < j < m, R f_i f_j
\]

- Enough for constructive Ramsey (Dickson’s lemma):

\[
\text{af } R \to \text{af } T \to \begin{cases}
\text{af } (R \cap T) \\
\text{af } (R \times T)
\end{cases}
\]
AF transfer: how to prove \( \text{af } R \rightarrow \text{af } T \)

- In the artifact of (Vytiniostis et al. 2012)
- \( \text{af} \_\text{mono} : R \subseteq T \rightarrow \text{af } R \rightarrow \text{af } T \)
  - limited: \( R, T : \text{rel}_2 X \) have same carrier type

\[ \text{af} \_\text{mono} \]:

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- \( \text{af}_\text{comap} : \text{af } R \rightarrow \text{af } (\lambda x_1 x_2, R (f x_1) (f x_2)) \)
  - impose a shape \( R (f \cdot) (f \cdot) \) on goal
AF transfer: how to prove $af \ R \to af \ T$

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- $af\_mono : R \subseteq T \to af \ R \to af \ T$
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  - impose a shape $R (f \cdot) (f \cdot)$ on goal
- Transfers $af \ R \to af \ T$ w/o those limitations
- Using surjective morphisms $f : X \to Y$
  - surjective: $\forall y : Y, \exists x : X, y = f \ x$
  - morphism: $\forall x_1 x_2, R x_1 x_2 \to T (f \ x_1) (f \ x_2)$
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  - surjective: $\forall y : Y, \exists t x : X, y = f x$
  - morphism: $\forall x_1 x_2, R x_1 x_2 \rightarrow T (f x_1) (f x_2)$
- But what about e.g. $af\ R \rightarrow af\ (R\downarrow P)$?
  - surjective on to carrier $\{ y \mid P y \}$?
  - unless assuming $P$ to be Boolean...
Surjective relational morphisms

- A restricted rel. $R\downarrow P$ has carrier type $\{x \mid P x\}$
  - is an important use case
  - $P$ decidable/Boolean is too strong assumption
Surjective relational morphisms

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- But the morphism need not be a function!!
- As a relational map: $f : X \rightarrow Y \rightarrow \text{Prop}$ with
  - $\forall y : Y, \exists_t x : X, f x y$
  - $\forall x_1 x_2 y_1 y_2, f x_1 y_1 \rightarrow f x_2 y_2 \rightarrow R x_1 x_2 \rightarrow T y_1 y_2$
- We get $\text{af } R \rightarrow \text{af } T$ under surjective morphisms
Surjective relational morphisms

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- We get $\text{af } R \rightarrow \text{af } T$ under surjective morphisms
- Versatile tool, subsumes if_mono and if_comap
- Example of direct application:
  - $\text{af } R \rightarrow \text{af } (R \downarrow P)$ (partial id. map)
  - $\text{af } R \uparrow a \leftrightarrow \text{af } R \downarrow (\neg R a)$ (when $R a$ dec.)
Quasi morphisms
Transfers using Quasi morphisms

- Inlined in (Fridlender99) and (Veldman04) proofs
  - a bit specific to this use case
  - but abstracts away the FAN theorem
- For transfers: \( \text{af } R \rightarrow \text{af } T \uparrow y_0 \)
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- For transfers: \[ \text{af } R \rightarrow \text{af } T \uparrow y_0 \]
- An evaluation function \( \text{ev} : X \rightarrow Y \)
  - \( X = \text{analyses} \), \( Y = \text{evaluations} \)
  - \( E : \text{rel} \_1 X \) are exceptional analyses
  - finite inverse image: \( \forall y, \text{fin}(\text{ev}^{-1} y) \)
  - \( \forall x_1 \ x_2, R \ x_1 \ x_2 \rightarrow T (\text{ev} \ x_1) (\text{ev} \ x_2) \lor E \ x_1 \)
  - \( \forall y, (\text{ev}^{-1} y) \subseteq E \rightarrow T \ y_0 \ y \)
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  - $\forall y, (\text{ev}^{-1} y) \subseteq E \rightarrow T \ y_0 y$
- Quasi morphisms can be ext. to relational maps
  - requires several extra (technical) assumptions
  - used in @GH/DmxLarchey/Kruskal-Veldman

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Quasi morphisms: the decidable case

- This case is easy to understand, but less general
- Assuming $T y_0$ and $E$ are decidable:
  - $\forall y : Y, T y_0 y \lor_t \neg T y_0 y$
  - $\forall x : X, E x \lor_t \neg E x$
Quasi morphisms: the decidable case

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  - $\forall y : Y, \ T y_0 y \lor_t \neg T y_0 y$
  - $\forall x : X, \ E x \lor_t \neg E x$
- Surj. rel. morph. from $R \downarrow (\neg E)$ to $T \downarrow (\neg T y_0)$
  - rel. morph.: $\lambda x y, \ \pi_1 (ev x) = \pi_1 y$
  - surj. by finitary choice over $ev^{-1} y$ for $E$
Quasi morphisms: the decidable case

- This case is easy to understand, but less general
- Assuming $T y_0$ and $E$ are decidable:
  - $\forall y : Y, T y_0 y \lor t \neg T y_0 y$
  - $\forall x : X, E x \lor t \neg E x$
- Surj. rel. morph. from $R \downarrow (\neg E)$ to $T \downarrow (\neg T y_0)$
  - rel. morph.: $\lambda x y, \pi_1(ev x) = \pi_1 y$
  - surj. by finitary choice over $ev^{-1} y$ for $E$
- But $\text{af } R \rightarrow \text{af } (R \downarrow (\neg E))$ (always)
- And $R \downarrow (\neg T y_0) \rightarrow \text{af } T \uparrow y_0$ (by dec.)
- Hence $\text{af } R \rightarrow \text{af } T \uparrow y_0$
Quasi morphisms: the general case

- No dec. assumption on $T y_0$ nor on $E$
  - This case is not trivial
- The full argument in the artifact
Quasi morphisms: the general case

- No dec. assumption on $T \nu_0$ nor on $E$
  - This case is not trivial
- The full argument in the artifact
- We just introduce the tools involved:
  - Bar inductive predicates and good lists
  - The FAN theorem for inductive bars
  - A finitary combinatorial principle
FANs as finitary choice sequences
Bar inductive predicates and AF

1. $R : \text{rel}_2 X$ and good $R$, $P : \text{rel}_1 (\text{list } X)$
2. $\text{bar } P : \text{list } X \to \text{Prop (or Type)}$

\[
\begin{align*}
P & \mid \hline \text{bar } P & \mid \hline
\forall x, \text{bar } P (x :: l) & \mid \\
\hline
\text{bar } P \mid
\end{align*}
\]

\[
\begin{align*}
R & y x \quad y \in l \\
\hline
\text{good } R (x :: l) & \\
\hline
\text{good } R l
\end{align*}
\]

Bar $P l$: $P$ is bound to be met...

\[
\text{bar } P [] \rightarrow \forall f : \mathbb{N} \rightarrow X \exists t m, P [f_{m-1}; \ldots; f_0]
\]
Bar inductive predicates and AF

- \( R : \text{rel}_2 X \) and good \( R, P : \text{rel}_1 (\text{list } X) \)
- \( \text{bar } P : \text{list } X \to \text{Prop (or Type)} \)

\[
\frac{P \perp}{\text{bar } P \perp} \quad \frac{R \ y \ x \ y \in l}{\text{good } R (x :: l)}
\]
\[
\forall x, \text{bar } P (x :: l) \quad \text{good } R \perp
\]

- \( \text{bar } P \perp: P \) is bound to be met...

\[
\text{bar } P [] \to \forall f : \mathbb{N} \to X \exists_t m, P [f_{m-1}; \ldots; f_0]
\]

- equivalences:

\[
\text{good } R [x_1; \ldots; x_n] \iff \exists i, j, j < i \land R x_i x_j
\]

\[
\text{bar } (\text{good } R) [x_1; \ldots; x_n] \iff \text{af } (R \uparrow x_n \uparrow \ldots \uparrow x_1)
\]

- derive \( \text{af } R \iff \text{bar } (\text{good } R) [] \)
The FAN theorem for inductive bars

- The product embedding for lists for $R : X \to Y \to \text{Prop}$
- $\forall R : \text{list } X \to \text{list } Y \to \text{Prop}$

\[
\begin{array}{c}
\text{Forall}_2 R [] [] \\
\text{Forall}_2 R (x :: l) (y :: m)
\end{array}
\]

- define $\text{FAN } lw \equiv \lambda c, \text{Forall}_2 (\cdot \in \cdot) c lw$
- collects finitely many choices sequences

\[[c_1; \ldots; c_n] \in \text{FAN } [w_1; \ldots; w_n] \iff c_1 \in w_1, \ldots, c_n \in w_n\]
The FAN theorem for inductive bars

- The product embedding for lists for $R : X \rightarrow Y \rightarrow \text{Prop}$
- Forall$_2 R : \text{list} X \rightarrow \text{list} Y \rightarrow \text{Prop}$

\[
\begin{align*}
\text{Forall}_2 R \; [] \; [] & \quad R \times y \quad \text{Forall}_2 R \; l \; m \\
\text{Forall}_2 R \; (x :: l) \; (y :: m)
\end{align*}
\]

- define FAN $lw \equiv \lambda c, \text{Forall}_2 (\cdot \in \cdot) c \; lw$
  - collects finitely many choices sequences
  \[
  [c_1; \ldots ; c_n] \in \text{FAN} \; [w_1; \ldots ; w_n] \iff c_1 \in w_1, \ldots , c_n \in w_n
  \]
- FAN theorem for $P : \text{rel}_1 (\text{list} X) \; (\text{Fridlender 99})$
  - if monotonic: $\forall x \; l, \; P \; l \rightarrow P \; (x :: l)$
  - then bar $P \; [] \rightarrow \text{bar} \; (\lambda lw, \; \text{FAN} \; lw \subseteq P) \; []$
- mono. predicates bound to be met uniformly /FAN
Finitary choice principles

- Finite one dimensional choice:
  - for $F, P, Q : \text{rel}_1 X$
  - if $\text{fin } F$ and $F \subseteq P \cup Q$
  - then $F \subseteq P$ or $\exists x, F x \land Q x$
Finitary choice principles

- Finite one dimensional choice:
  - for $F, P, Q : \text{rel}_1 X$
  - if $\text{fin } F$ and $F \subseteq P \cup Q$
  - then $F \subseteq P$ or $\exists x, F x \land Q x$

- Finite two dimensional choice:
  - for $P : \text{rel}_1 (\text{list } X)$, $B : \text{rel}_1 X$, and $lw : \text{list (list } X)$
  - assuming $\forall c, \text{FAN } lw c \rightarrow P c \lor \exists x, x \in c \land B x$
    - any choice sequence satisfies $P$ or meets $B$
  - we have either:
    - $\exists c, \text{FAN } lw c \land P c$ ($P$ contains a choice sequence)
    - or $\exists w, w \in lw \land w \subseteq B$ ($B$ is unavoidable)
Termination using AF relations
From Almost Full to Well Founded

- Induction principle from (Vytiniostis et al. 2012)
  \[ \text{af } R \rightarrow T^+ \cap R^{-1} \subseteq \emptyset \rightarrow \text{well-founded } T \]

- small examples in *Stop when you are almost full*...
From Almost Full to Well Founded

- Induction principle from (Vytiniostis et al. 2012)

\[
\text{af } R \rightarrow T^+ \cap R^{-1} \subseteq \emptyset \rightarrow \text{well-founded } T
\]

- small examples in *Stop when you are almost full*...
- larger example: Karp-Miller (Yamamoto et al. 17)
  - deciding coverability for Petri nets
- revisited at @GH/DmxLarchey/Karp-Miller
  - decision: a covering or its impossibility
  - refined: Karp-Miller tree with accel. transitions
Bounding search using Almost Fullness

- A constructive König’s lemma:
  - for $R : \text{rel}_2 X$ with $\text{af} \ R$
  - and $P : \mathbb{N} \to \text{rel}_1 X$ with $\forall n, \text{fin}(P \ n)$
    $$\exists t m, \forall v : X^m, (\forall i, P \ i \ v_i) \to \exists i < j, R \ v_i \ v_j$$
  - $P$ as a finitely branching search space
Bounding search using Almost Fullness

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  - for $R : \text{rel}_2 X$ with $\text{af } R$
  - and $P : \mathbb{N} \rightarrow \text{rel}_1 X$ with $\forall n, \text{fin}(P n)$
    
    $$\exists t m, \forall v : X^m, (\forall i, P i v_i) \rightarrow \exists i < j, R v_i v_j$$

- $P$ as a finitely branching search space
- is $m$ obtained via $\text{bar } P []$ and the FAN theorem
- Coq proof here: @GH/DmxLarchey/Kruskal-FAN
Bounding search using Almost Fullness

- A constructive König’s lemma:
  - for $R : \text{rel}_2 X$ with $\text{af } R$
  - and $P : \mathbb{N} \to \text{rel}_1 X$ with $\forall n, \text{fin}(P_n)$

$$\exists m, \forall x : X^m, (\forall i, P_i x_i) \rightarrow \exists i < j, R x_i x_j$$

- $P$ as a finitely branching search space
- is $m$ obtained via $\text{bar } P []$ and the FAN theorem
- Coq proof here: @GH/DmxLarchey/Kruskal-FAN
- used for redundancy avoiding (proof-)search:
  - deciding Implicational Relevance Logic (IJCAR 18)
  - $m$ bounds height of irredundant search branches
  - at @GH/DmxLarchey/Relevant-decidability
- Friedman’s $\text{tree}(n)$ and $\text{TREE}(n)$ monsters
  - $m$ guards termination of unbounded linear search
  - Coq code at @GH/DmxLarchey/Friedman-TREE