Quasi Morphisms for Almost Full Relations

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Introduction to WQOs and AF relations

Well Quasi Orders (WQO)

- Classical defn. for $R : \texttt{rel}_2\, X$ (ie. $X \to X \to \texttt{Prop}$) :
  - R is a Quasi Order (refl., trans.)
  - Almost Full (AF): $\forall f : \mathbb{N} \to X, \exists i < j, R\ f_i\ f_j$
  - any ∞ *sequence* contains a good pair
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- Important in computer science and mathematics
  - termination: terminator rule, Karp-Miller
  - decidability: relevance logic (Kripke)
  - polynomial ideals and Gröbner basis (Hilbert)
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- This AF notion is *constructively* too weak:
  - requires added constructively "acceptable" axioms
  - Count. Ch., bar ind. princ. (Veldman&Bezem 93)
  - Stumps and Brouwer's thesis (Veldman 2004)
  - limited to relations over $\mathbb{N}$

AF relations in Inductive Type Theory

- *About Brouwer's Fan Theorem* (Coquand 2003):
  - intuitive explanation of this constructive weakness
  - Almost Full: $\boxed{\forall f : \mathbb{N} \to X}$...
  - only captures sequences $\mathbb{N} \to X$ given by laws
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- Stronger (constructive) AF notions:
  - do not require added axioms
  - bar ind. predicates (Coquand&Fridlender 93)
  - ind. well-foundedness (Seisenberger 2003)
    - only for decidable relations
  - inductive AF relations (Vytiniostis *et al.* 2012)

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- Veldman's proof of Kruskal in Coq (DLW2015)
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- The project published on opam − coq
- Description: `@GH/DmxLarchey/Coq-Kruskal`

Almost Fullness inductively

Inductive Almost Full relations

- Inductive predicate (Vytiniostis *et al.* 2012)
- For $R : \texttt{rel}_2\, X$, define $\texttt{af}\ R : \texttt{Prop}$ (or $\texttt{Type}$)

$$\frac{\forall x\, y,\ R\ x\ y}{\texttt{af}\ R}\ \langle\texttt{af\_full}\rangle \qquad \frac{\forall a,\ \texttt{af}\ R{\uparrow}a}{\texttt{af}\ R}\ \langle\texttt{af\_lift}\rangle$$

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$$\mathtt{af}\ R\ \to\ \forall f : \mathbb{N} \to X, \exists_{\mathrm{t}} m\, \exists i < j < m, R\ f_i\ f_j$$

- Enough for constructive Ramsey (Dickson's lemma):

$$\mathtt{af}\ R \to \mathtt{af}\ T \to \begin{cases}\mathtt{af}\ (R \cap T)\\ \mathtt{af}\ (R \times T)\end{cases}$$

AF transfer: how to prove af $R \to$ af T

- In the artifact of (Vytiniostis *et al.* 2012)
- $\texttt{af\_mono} : R \subseteq T \to \texttt{af}\ R \to \texttt{af}\ T$
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- Transfers $\texttt{af}\ R \to \texttt{af}\ T$ w/o those limitations
- Using surjective morphisms $f : X \to Y$
  - surjective: $\forall y : Y, \exists_{\mathrm{t}} x : X, y = f\ x$
  - morphism: $\forall x_1\, x_2,\ R\ x_1\ x_2 \to T\ (f\ x_1)\ (f\ x_2)$

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- But what about e.g. $\texttt{af}\ R \to \texttt{af}\ (R{\Downarrow}P)$?
  - surjective on to carrier $\{y \mid P\ y\}$?
  - unless assuming P to be Boolean...

Surjective relational morphisms

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  - $\forall y : Y, \exists_{\mathrm{t}} x : X, f\ x\ y$
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- We get $\boxed{\texttt{af}\ R \to \texttt{af}\ T \text{ under surjective morphisms}}$
- Versatile tool, subsumes `af_mono` and `af_comap`
- Example of direct application:
  - $\texttt{af}\ R \to \texttt{af}\ (R{\Downarrow}P)$ (partial id. map)
  - $\texttt{af}\ R{\uparrow}a \leftrightarrow \texttt{af}\ R{\Downarrow}(\neg R\,a)$ (when $R\,a$ dec.)

Quasi morphisms

Transfers using Quasi morphisms

- Inlined in (Fridlender99) and (Veldman04) proofs
  - a bit specific to this use case
  - but abstracts away the FAN theorem
- For transfers: $\boxed{\texttt{af}\ R \rightarrow \texttt{af}\ T{\uparrow}y_0}$

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- For transfers: $\boxed{\texttt{af}\ R \to \texttt{af}\ T{\uparrow}y_0}$
- An *evaluation* function $ev : X \to Y$
  - $X = analyses$, $Y = evaluations$
  - $E : \texttt{rel}_1\ X$ are *exceptional* analyses
  - finite inverse image: $\forall y, \texttt{fin}\,(ev^{-1}\ y)$
  - $\forall x_1\ x_2,\ R\ x_1\ x_2 \to T\ (ev\ x_1)\ (ev\ x_2) \lor E\ x_1$
  - $\forall y, (ev^{-1}\ y) \subseteq E \to T\ y_0\ y$

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  - $\forall y,\ (ev^{-1}\ y) \subseteq E \to T\ y_0\ y$
- Quasi morphisms can be ext. to relational maps
  - requires several extra (technical) assumptions
  - used in `@GH/DmxLarchey/Kruskal-Veldman`

Quasi morphisms: the decidable case

- This case is easy to understand, but less general
- Assuming $T\ y_0$ and E are decidable:
  - $\forall y : Y,\ T\ y_0\ y \vee_{\mathrm{t}} \neg T y_0 y$
  - $\forall x : X,\ E\, x \vee_{\mathrm{t}} \neg E\, x$

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- Surj. rel. morph. from $R{\Downarrow}(\neg E)$ to $T{\Downarrow}(\neg T\ y_0)$
  - rel. morph.: $\lambda\, x\, y,\ \pi_1(ev\, x) = \pi_1\, y$
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  - surj. by finitary choice over $ev^{-1}\ y$ for E
- But $\mathtt{af}\ R \to \mathtt{af}\ \big(R{\Downarrow}(\neg E)\big)$ (always)
- And $\mathtt{af}\ R{\Downarrow}(\neg T\ y_0) \to \mathtt{af}\ T{\uparrow}y_0$ (by dec.)
- Hence $\boxed{\mathtt{af}\ R \to \mathtt{af}\ T{\uparrow}y_0}$

Quasi morphisms: the general case

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  - This case is not trivial
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Quasi morphisms: the general case

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- The full argument in the artifact
- We just introduce the tools involved:
  - Bar inductive predicates and good lists
  - The FAN theorem for inductive bars
  - A finitary combinatorial principle

FANs as finitary choice sequences

Bar inductive predicates and AF

- $R : \mathtt{rel}_2\, X$ and good R, $P : \mathtt{rel}_1\, (\mathtt{list}\ X)$
- $\mathtt{bar}\ P : \mathtt{list}\ X \to \mathtt{Prop}$ (or Type)

$$\frac{P\ l}{\mathtt{bar}\ P\ l} \qquad \frac{\forall x,\ \mathtt{bar}\ P\ (x :: l)}{\mathtt{bar}\ P\ l} \quad \Bigg| \quad \frac{R\ y\ x \qquad y \in l}{\mathtt{good}\ R\ (x :: l)} \qquad \frac{\mathtt{good}\ R\ l}{\mathtt{good}\ R\ (x :: l)}$$

- $\mathtt{bar}\ P\ l$: P is bound to be met...

$$\mathtt{bar}\ P\ [] \ \to\ \forall f : \mathbb{N} \to X\ \exists_{\mathrm{t}} m, P\ [f_{m-1}; \ldots; f_0]$$

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- equivalences:

$$\mathtt{good}\ R\ [x_1; \ldots ; x_n] \ \leftrightarrow\ \exists\, i\, j, j < i \wedge R\ x_i\ x_j$$
$$\mathtt{bar}\ (\mathtt{good}\ R)\ [x_1; \ldots ; x_n] \ \leftrightarrow\ \mathtt{af}\ (R{\uparrow}x_n{\uparrow}\ldots{\uparrow}x_1)$$

- derive $\boxed{\mathtt{af}\ R \leftrightarrow \mathtt{bar}\ (\mathtt{good}\ R)\ [\,]}$

The FAN theorem for inductive bars

- The product embedding for lists for $R : X \to Y \to \texttt{Prop}$
- $\texttt{Forall}_2\ R : \texttt{list}\ X \to \texttt{list}\ Y \to \texttt{Prop}$

$$\frac{}{\texttt{Forall}_2\ R\ []\ []} \qquad \frac{R\ x\ y \qquad \texttt{Forall}_2\ R\ l\ m}{\texttt{Forall}_2\ R\ (x :: l)\ (y :: m)}$$

- define $\texttt{FAN}\ lw \coloneqq \lambda\, c, \texttt{Forall}_2\ (\cdot \in \cdot)\ c\ lw$
  - collects finitely many choices sequences

$$[c_1; \ldots; c_n] \in \texttt{FAN}\,[w_1; \ldots; w_n] \quad \leftrightarrow \quad c_1 \in w_1, \ldots, c_n \in w_n$$

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- FAN theorem for $P : \texttt{rel}_1\ (\texttt{list}\ X)$ (Fridlender 99)
  - if *monotonic*: $\forall\, x\, l,\ P\ l \to P\ (x :: l)$
  - then $\texttt{bar}\ P\ [] \to \texttt{bar}\ (\lambda\, lw, \texttt{FAN}\ lw \subseteq P)\ []$
- mono. predicates bound to be met uniformly /FAN

Finitary choice principles

- Finite one dimensional choice:
  - for $F, P, Q : \mathtt{rel}_1\ X$
  - if $\mathtt{fin}\, F$ and $F \subseteq P \cup Q$
  - then $F \subseteq P$ or $\exists x, F\,x \wedge Q\,x$

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  - if $\texttt{fin}\ F$ and $F \subseteq P \cup Q$
  - then $F \subseteq P$ or $\exists x, F\ x \wedge Q\ x$
- Finite two dimensional choice:
- for $P : \texttt{rel}_1\ (\texttt{list}\ X)$, $B : \texttt{rel}_1\ X$, and $lw : \texttt{list}\ (\texttt{list}\ X)$
- assuming $\forall c, \texttt{FAN}\ lw\ c \to P\ c \vee \exists x, x \in c \wedge B\ x$
  - any choice sequence satisfies P or meets B
- we have either:
  - $\exists c, \texttt{FAN}\ lw\ c \wedge P\ c$ (P contains a choice sequence)
  - or $\exists w, w \in lw \wedge w \subseteq B$ (B is unavoidable)

Termination using AF relations

From Almost Full to Well Founded

- Induction principle from (Vytiniostis *et al.* 2012)

$$\texttt{af}\,R \;\to\; T^{+} \cap R^{-1} \subseteq \emptyset \;\to\; \texttt{well\_founded}\,T$$

- small examples in *Stop when you are almost full...*

From Almost Full to Well Founded

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- small examples in *Stop when you are almost full...*
- larger example: Karp-Miller (Yamamoto *et al.* 17)
  - deciding coverability for Petri nets
- revisited at `@GH/DmxLarchey/Karp-Miller`
  - decision: a covering or its impossibility
  - refined: Karp-Miller tree with accel. transitions

Bounding search using Almost Fullness

- A constructive König's lemma:
  - for $R : \texttt{rel}_2\, X$ with $\texttt{af}\ R$
  - and $P : \mathbb{N} \to \texttt{rel}_1\, X$ with $\forall n, \texttt{fin}\,(P\ n)$

$$\exists_{\mathrm{t}} m, \forall v : X^m, (\forall i, P\ \underline{i}\ v_i) \to \exists i < j,\ R\ v_i\ v_j$$

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- is m obtained via `bar` P [] and the FAN theorem
- Coq proof here: `@GH/DmxLarchey/Kruskal-FAN`

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- Coq proof here: `@GH/DmxLarchey/Kruskal-FAN`
- used for redundancy avoiding (proof-)search:
  - deciding Implicational Relevance Logic (IJCAR 18)
  - m bounds height of irredundant search branches
  - at `@GH/DmxLarchey/Relevant-decidability`
- Friedman's $\texttt{tree}(n)$ and $\texttt{TREE}(n)$ monsters
  - m guards termination of unbounded linear search
  - Coq code at `@GH/DmxLarchey/Friedman-TREE`