# An Alternative Direct Simulation of Minsky Machines into Classical Bunched Logics via Group Semantics

Dominique Larchey-Wendling

TYPES team, LORIA – CNRS

Nancy, France

MFPS XXVI, Ottawa, Canada, May 2010.

# Boolean/Classical Bunched Logics

- Separation Logic (SL) (Reynolds&O'Hearn 01)
  - finite stacks of cells, $A : \mathsf{Loc} \longrightarrow_{\mathrm{f}} \mathsf{Val}$
  - $C = A \uplus B$ models aggregation of cells into wider structures
- BBI is the logical core of SL
- BBI connectives (Pym 02, $\mathsf{BI} + \{\neg\neg A \rightarrow A\}$)
  - Boolean logic connectives: $\wedge$, $\vee$, $\rightarrow$, $\neg$, ...
  - multiplicative connectives: $*$, $-\!\!*$, $\mathsf{I}$
- more connectives for CBI (Brotherston&Calcagno 09)
  - multiplicative zero $\mathsf{O}$, linear negation $\sim$
  - dualizing operator for Kripke semantics

# Proof theory for BBI/CBI

- Compared to (intuitionistic) BI: much less satisfying situation
  - BI has Bunched sequent calculus (O'Hearn&Pym 99)
  - with cut-elimination from its inception
  - BI is decidable (Galmiche et al. 05)
- Hilbert system s/c for relational BBI (LW.&Galmiche 06, Yang)
- Semantic tableaux s/c for (partial) monoidal BBI
  - (unexpected) embedding of BI into BBI (LW.&Galmiche 09)
- Display calculi s/c for relational CBI (Brother.&Calcagno 09)
- Display calculi s/c for relational BBI (Brotherston 09, 10)

# Kripke semantics of BBI/CBI (i)

- Non-deterministic(/relational) monoid (NDm) $(M, \circ, \epsilon)$
  - $\circ : M \times M \longrightarrow \mathcal{P}(M)$ and $\epsilon \in M$
  - for $X, Y \in \mathcal{P}(M)$, $\boxed{X \circ Y = \{z \mid \exists x \in X, \exists y \in Y, z \in x \circ y\}}$
  - $\epsilon \circ x = \{x\}$ (neutrality), $x \circ y = y \circ x$ (commutativity)
  - $x \circ (y \circ z) = (x \circ y) \circ z$ (associativity)
  - $(\mathcal{P}(M), \circ, \{\epsilon\})$ is a (usual) commutative monoid
  - residuation: $X \multimap Y = \{z \mid z \circ X \subseteq Y\}$
- Non-deterministic groupoid (NDg) $(M, \circ, \epsilon, -, \infty)$
  - $(M, \circ, \epsilon)$ is a NDm
  - $\infty \in x \circ y$ iff $y = -x$ (pseudo inverse)

# Kripke semantics of **BBI/CBI** (ii)

- Boolean (pointwise) Kripke semantics extended by:

$$
\begin{array}{lll}
m \Vdash A * B & \text{iff} & \exists a, b \text{ s.t. } m \in a \circ b \text{ and } a \Vdash A \text{ and } b \Vdash B \\
m \Vdash A \mathrel{-\!\!*} B & \text{iff} & \forall a, b \;\; (b \in a \circ m \text{ and } a \Vdash A) \Rightarrow b \Vdash B \\
m \Vdash \mathsf{I} & \text{iff} & m = \epsilon
\end{array}
$$

- Moreover, for CBI: $m \Vdash \sim A$ iff $-m \nVdash A$ $\quad m \Vdash \mathsf{O}$ iff $m \neq \infty$

- Decision problems:
  - checking a particular model ($m \Vdash A$), Calcagno et al. 01 (SL)
  - validity in a particular interpretation ($\forall m, m \Vdash A$)
  - univ. validity w.r.t. class of models ($\forall \mathcal{M} \forall \Vdash \forall m, m \Vdash A$)

# Classes of models for **BBI/CBI**

- Partial (deterministic) monoids/groupoids (Dm/Dg): $x \circ y \subseteq \{k\}$
- Total (deterministic) monoids/groupoids (Tm/Tg): $x \circ y = \{k\}$
- Groups (G): $\epsilon = \infty$ and $x \circ -x = \{\epsilon\}$
- $\mathrm{G} \subsetneq \mathrm{Tm} \subsetneq \mathrm{Dm} \subsetneq \mathrm{NDm}$ and $\mathrm{G} \subsetneq \mathrm{Tg} \subsetneq \mathrm{Dg} \subsetneq \mathrm{NDg}$
- Separation models are in Dm/Dg (Brotherston&Kanovich 10):
  - RAM-domain model for BBI: $(\mathcal{P}_{\mathrm{f}}(\mathbb{N}), \uplus, \emptyset)$
  - RAM-domain model for CBI: $(\mathcal{P}_{\mathrm{f/c}}(\mathbb{N}), \uplus, \emptyset, \mathbb{N}\backslash(\cdot), \mathbb{N})$
- Universal validity defines different logics:

$\mathsf{BBI}_{\mathrm{ND}} \subsetneq \mathsf{BBI}_{\mathrm{D}} \subsetneq \mathsf{BBI}_{\mathrm{T}} \subsetneq \mathsf{BBI}_{\mathrm{G}}$ $\quad$ $\mathsf{CBI}_{\mathrm{ND}} \subsetneq \mathsf{CBI}_{\mathrm{D}} \subsetneq \mathsf{CBI}_{\mathrm{T}} \subsetneq \mathsf{CBI}_{\mathrm{G}}$

# Undecidability of BBI/CBI

- Minsky machines encoded in fragments of BBI/CBI
- $\mathsf{BBI}_X$ undecidable:
  - for $X \in \{\mathrm{NDm}, \mathrm{Dm}, \mathrm{Tm}\}$, Larchey&Galmiche 10
  - for $X \in \{\mathrm{NDm}, \mathrm{Dm}, \text{sep. models}\}$, Brotherston&Kanovich 10
- $\mathsf{CBI}_X$ undecidable:
  - for $X \in \{\mathrm{NDg}, \mathrm{Dg}, \text{sep. models}\}$, Brotherston&Kanovich 10
- B.&K. 10 encoding needs indivisible units $x \circ y = \epsilon \Rightarrow x = y = \epsilon$
- This paper:
  - a proof/encoding covering all these cases
  - also $\mathsf{BBI}_{\mathrm{G}}/\mathsf{CBI}_{\mathrm{G}}$ undecidable (G has divisible units)

# Overview of the encoding

- $\mathsf{ILL}^0_{\multimap,\&,!} \simeq \mathsf{eBBI}$ is a fragment of BBI/CBI
  - Trivial phase semantics/Kripke semantics
- Two counter Minsky machines in $\mathsf{ILL}^0_{\multimap,\&,!}$:
  - add 1, sub 1, zero test, two positive counters
  - "negative" encoding acceptance $(1, m, n) \to^\star (0, 0, 0)$
  - reachability $(\alpha, m, n) \to^\star (\beta, m', n')$ req. $\oplus$, $\otimes$, Kanovich 95
  - Brotherston&Kanovich 10 uses double linear negation
  - LW.&Galmiche 10 works with $\mathbb{N} \times \mathbb{N}$, unsuitable for $\mathbb{Z} \times \mathbb{Z}$
- Faithfullness of the encoding:
  - via $\mathbb{N} \times \mathbb{N}$ sem. (BBI), via $\mathbb{Z} \times \mathbb{Z}$ sem. (BBI/CBI)
  - RAM-domain model bisimilar to $\mathbb{N} \times \mathbb{N}$

# Kripke vs. Phase semantics for BBI

- Change of notation: $m \Vdash A$ iff $m \in [\![A]\!]$
- The interpretation of multiplicative conjuction $*$

$$\begin{array}{rcl} m \Vdash A * B & \text{iff} & \exists a, b \text{ s.t. } a \circ b = m \text{ and } a \Vdash A \text{ and } b \Vdash B \\ [\![A * B]\!] & = & [\![A]\!] \circ [\![B]\!] \end{array}$$

- Phase semantics for BBI (equiv. to Kripke sem.):

$$\begin{array}{rlrl} [\![\bot]\!] = \emptyset & & [\![A \vee B]\!] = [\![A]\!] \cup [\![B]\!] \\ [\![\top]\!] = M & & [\![A \wedge B]\!] = [\![A]\!] \cap [\![B]\!] \\ [\![\mathsf{I}]\!] = \{\epsilon\} & & [\![A * B]\!] = [\![A]\!] \circ [\![B]\!] \\ [\![\neg A]\!] = M \backslash [\![A]\!] & & [\![A \mathrel{-\!\!*} B]\!] = [\![A]\!] \multimap [\![B]\!] \end{array}$$

## Phase semantics for ILL

- Intuitionistic phase space $(M, \circ, \epsilon, (\cdot)^\diamond, K)$:
  - $(M, \circ, \epsilon)$ in NDm (usually Tm)
  - $(\cdot)^\diamond$ is a closure operator with $A^\diamond \circ B^\diamond \subseteq (A \circ B)^\diamond$ (stability)
  - $K$ sub-monoid of $M$: $\epsilon \in K$ and $K \circ K \subseteq K$
  - $K \subseteq \{\epsilon\}^\diamond \cap \{x \in M \mid x \in (x \circ x)^\diamond\}$
- Phase interpretation of ILL operators:

$$
\begin{array}{rclcrcl}
[\![\bot]\!] & = & \emptyset^\diamond & \quad & [\![A \oplus B]\!] & = & ([\![A]\!] \cup [\![B]\!])^\diamond \\
[\![\top]\!] & = & M & & [\![A \mathbin{\&} B]\!] & = & [\![A]\!] \cap [\![B]\!] \\
[\![1]\!] & = & \{\epsilon\}^\diamond & & [\![A \otimes B]\!] & = & ([\![A]\!] \circ [\![B]\!])^\diamond \\
[\![!\, A]\!] & = & (K \cap [\![A]\!])^\diamond & & [\![A \multimap B]\!] & = & [\![A]\!] \multimap [\![B]\!]
\end{array}
$$

# Trivial phase semantics for ILL

- Intuitionistic phase space $(M, \circ, \epsilon, (\cdot)^\diamond, K)$:
  - $(\cdot)^\diamond$ is the identity closure: $A^\diamond = A$
  - and as a consequence $K = \{\epsilon\}$
- Trivial phase interpretation of ILL operators:

$$
\begin{array}{rlrl}
[\![\bot]\!] &= \emptyset & [\![A \oplus B]\!] &= [\![A]\!] \cup [\![B]\!] \\
[\![\top]\!] &= M & [\![A \mathbin{\&} B]\!] &= [\![A]\!] \cap [\![B]\!] \\
[\![1]\!] &= \{\epsilon\} & [\![A \otimes B]\!] &= [\![A]\!] \circ [\![B]\!] \\
[\![!\, A]\!] &= \{\epsilon\} \cap [\![A]\!] & [\![A \multimap B]\!] &= [\![A]\!] \multimap [\![B]\!]
\end{array}
$$

# ILL vs. BBI phase semantics

| Trivial phase sem. for ILL | Phase sem. for BBI |
|---|---|
| $[\![\bot]\!] = \emptyset$ | $[\![\bot]\!] = \emptyset$ |
| $[\![\top]\!] = M$ | $[\![\top]\!] = M$ |
| $[\![1]\!] = \{\epsilon\}$ | $[\![\mathsf{I}]\!] = \{\epsilon\}$ |
| $[\![!\,A]\!] = \{\epsilon\} \cap [\![A]\!]$ | $[\![\mathsf{I} \wedge A]\!] = \{\epsilon\} \cap [\![A]\!]$ |
| $[\![A \oplus B]\!] = [\![A]\!] \cup [\![B]\!]$ | $[\![A \vee B]\!] = [\![A]\!] \cup [\![B]\!]$ |
| $[\![A \,\&\, B]\!] = [\![A]\!] \cap [\![B]\!]$ | $[\![A \wedge B]\!] = [\![A]\!] \cap [\![B]\!]$ |
| $[\![A \otimes B]\!] = [\![A]\!] \circ [\![B]\!]$ | $[\![A * B]\!] = [\![A]\!] \circ [\![B]\!]$ |
| $[\![A \multimap B]\!] = [\![A]\!] \multimap [\![B]\!]$ | $[\![A \mathrel{-\!\!*} B]\!] = [\![A]\!] \multimap [\![B]\!]$ |

## ILL as a fragment of BBI/CBI

- Embedding ILL (with trivial phase sem.) into BBI/CBI
  - replace $1/\mathsf{I}$, $\oplus/\vee$, $\&/\wedge$, $\otimes/*$, $\multimap/\mathrel{-\!\!*}$
  - replace $!\,A$ by $\mathsf{I} \wedge A$
  - sound and faithful for trivial phase semantics
- ILL undecidable but not complete w.r.t. trivial phase semantics
- The elementary fragment of ILL: $\mathsf{ILL}^0_{\multimap,\&,!}$ (LW.&Galmiche 10)
  - contains only $!$, $\multimap$ and $\&$ (not $\oplus$ or $\otimes$)
  - undecidable (Minsky machines)
  - complete for trivial phase sem.
- Elementary BBI (eBBI) corresponds to $\mathsf{ILL}^0_{\multimap,\&,!}$ (via embedding)

## The elementary fragment $\mathsf{ILL}^0_{\multimap,\&,!}$ of $\mathsf{ILL}$

- Elementary sequents: $!\,\Sigma, g_1, \ldots, g_k \vdash d \quad (g_i, a, b, c, d$ variables)
  - In $\Sigma$: $\boxed{a \multimap (b \multimap c),\ (a \multimap b) \multimap c \text{ or } (a \,\&\, b) \multimap c}$
- $\mathsf{Gill}^0$, goal directed rules for $\mathsf{ILL}^0_{\multimap,\&,!}$:

$$\frac{}{!\,\Sigma, a \vdash a}\ \langle\mathrm{Ax}\rangle \qquad \frac{!\,\Sigma, \Gamma \vdash a \qquad !\,\Sigma, \Delta \vdash b}{!\,\Sigma, \Gamma, \Delta \vdash c}\ a \multimap (b \multimap c) \in \Sigma$$

$$\frac{!\,\Sigma, \Gamma, a \vdash b}{!\,\Sigma, \Gamma \vdash c}\ (a \multimap b) \multimap c \in \Sigma \qquad \frac{!\,\Sigma, \Gamma \vdash a \qquad !\,\Sigma, \Gamma \vdash b}{!\,\Sigma, \Gamma \vdash c}\ (a \,\&\, b) \multimap c \in \Sigma$$

- $\mathsf{Gill}^0$ and trivial phase semantics: s/c w.r.t. $\mathsf{ILL}^0_{\multimap,\&,!}$
  - $\mathsf{Gill}^0$ is strongly sound, hence sound w.r.t. any class of models
  - complete w.r.t. Tm, NDm, Dm (LW.&G. 10), w.r.t. Dg, NDg
  - but completeness unknown w.r.t. Tg and G

## Encoding acceptance of two counters MM

- Build a sequent $!\,\Sigma, \mathsf{a}^x, \mathsf{b}^y \vdash \mathsf{q}_i$
  - $\mathsf{a}$ and $\mathsf{b}$ represents the two counters
  - $\mathsf{q}_0, \ldots \mathsf{q}_l$ represents the $l+1$ positions of the MM
  - instructions encoding in $\Sigma$, $\mathsf{a}$ and $\mathsf{b}$ never in goal position
  - acceptance as (universal) validity:

$$\boxed{(i, x, y) \rightarrow^{\star} (0,0,0) \quad \text{iff} \quad !\,\Sigma, \mathsf{a}^x, \mathsf{b}^y \vdash \mathsf{q}_i \text{ univ. valid}}$$

- Introduce $\mathsf{v}$ s.t. $!\,\Sigma, \mathsf{a}^x, \mathsf{b}^y \vdash \mathsf{v}$ iff $x = 0$ (resp. $\mathsf{u}/y$)
  - $\Sigma$ contains $\mathsf{b} \multimap (\mathsf{v} \multimap \mathsf{v})$ and $(\mathsf{a} \multimap \mathsf{a}) \multimap \mathsf{v}$ (zero test on $x$)
  - $\Sigma$ contains $\mathsf{a} \multimap (\mathsf{u} \multimap \mathsf{u})$ and $(\mathsf{a} \multimap \mathsf{a}) \multimap \mathsf{u}$ (zero test on $y$)
- $!\,\Sigma, \mathsf{a}^0, \mathsf{b}^0 \vdash \mathsf{q}_0$, hence $\Sigma$ contains $(\mathsf{a} \multimap \mathsf{a}) \multimap \mathsf{q}_0$ (ground case)

# Encoding add 1 to a (soundness)

- "add 1" instruction: $\boxed{i : \mathsf{a} := \mathsf{a} + 1 \ ; \ \mathsf{goto}\ j}$
- Operational semantics: $(i, x, y) \to (j, x+1, y) \to^{\star} (0, 0, 0)$
- Recursively built $\mathsf{Gill}^0$ proof to establish univ. validity:

$$\dfrac{\dfrac{\cdots}{!\,\Sigma, \mathsf{a}^x, \mathsf{a}, \mathsf{b}^y \vdash \mathsf{q}_j}}{!\,\Sigma, \mathsf{a}^x, \mathsf{b}^y \vdash \mathsf{q}_i}\ (\mathsf{a} \multimap \mathsf{q}_j) \multimap \mathsf{q}_i \in \Sigma$$

# Encoding sub 1/zero test on a (soundness) (i)

- "sub 1/zero t.": $i$ : if a $= 0$ then goto $j$ else a := a $- 1$ ; goto $k$
- Case $x = 0$
- Operational semantics: $(i, 0, y) \rightarrow (j, 0, y) \rightarrow^\star (0, 0, 0)$
- Corresponding $\mathsf{Gill}^0$ proof:

$$\dfrac{\dfrac{\text{z.t. on } x}{!\,\Sigma, \mathsf{b}^y \vdash \mathsf{v}} \quad \dfrac{\ldots}{!\,\Sigma, \mathsf{b}^y \vdash \mathsf{q}_j}}{!\,\Sigma, \mathsf{b}^y \vdash \mathsf{q}_i}\,(\mathsf{v} \,\&\, \mathsf{q}_j) \multimap \mathsf{q}_i \in \Sigma$$

# Encoding sub 1/zero test on a (soundness) (ii)

- "sub 1/zero t.": $i$ : if $\mathsf{a} = 0$ then goto $j$ else $\mathsf{a} := \mathsf{a} - 1$ ; goto $k$
- Case $x + 1 > 0$
- Operational semantics: $(i, x+1, y) \rightarrow (k, x, y) \rightarrow^{\star} (0, 0, 0)$
- Corresponding $\mathsf{Gill}^0$ proof:

$$\dfrac{\dfrac{}{!\,\Sigma, \mathsf{a} \vdash \mathsf{a}}\langle\mathrm{Ax}\rangle \qquad \dfrac{\cdots}{!\,\Sigma, \mathsf{a}^x, \mathsf{b}^y \vdash \mathsf{q}_k}}{!\,\Sigma, \mathsf{a}, \mathsf{a}^x, \mathsf{b}^y \vdash \mathsf{q}_i}\ \mathsf{a} \multimap (\mathsf{q}_k \multimap \mathsf{q}_i) \in \Sigma$$

## Summary of the encoding and soundness

- Start with $\Sigma = \left\{ \begin{array}{l} \mathsf{a} \multimap (\mathsf{u} \multimap \mathsf{u}), \mathsf{b} \multimap (\mathsf{v} \multimap \mathsf{v}), \\ (\mathsf{a} \multimap \mathsf{a}) \multimap \mathsf{u}, (\mathsf{a} \multimap \mathsf{a}) \multimap \mathsf{v}, (\mathsf{a} \multimap \mathsf{a}) \multimap \mathsf{q}_0 \end{array} \right\}$
- For instruction $i$ : $\mathsf{a} := \mathsf{a} + 1$ ; goto $j$
  - add $\left\{ (\mathsf{a} \multimap \mathsf{q}_j) \multimap \mathsf{q}_i \right\}$ to $\Sigma$
- For instruction $i$ : if $\mathsf{a} = 0$ then goto $j$ else $\mathsf{a} := \mathsf{a} - 1$ ; goto $k$
  - add $\left\{ (\mathsf{v} \,\&\, \mathsf{q}_j) \multimap \mathsf{q}_i, \mathsf{a} \multimap (\mathsf{q}_k \multimap \mathsf{q}_i) \right\}$ to $\Sigma$
- Soundness theorem:

  if $(i, x, y) \rightarrow^\star (0, 0, 0)$ then $!\,\Sigma, \mathsf{a}^x, \mathsf{b}^y \vdash \mathsf{q}_i$ has a $\mathsf{Gill}^0$-proof

- as a consequence, $!\,\Sigma, \mathsf{a}^x, \mathsf{b}^y \vdash \mathsf{q}_i$ is univ. valid

# Completeness of the encoding (summary)

- Let us suppose $!\,\Sigma, \mathsf{a}^x, \mathsf{b}^y \vdash \mathsf{q}_i$ is univ. valid, $\Sigma = \sigma_1, \ldots, \sigma_r$
- By trivial phase interpretation in $\mathbb{N} \times \mathbb{N}$ (class Tm)

$$[\![\mathsf{a}]\!] = \{(1,0)\} \quad [\![\mathsf{b}]\!] = \{(0,1)\} \quad [\![\mathsf{u}]\!] = \mathbb{N} \times \{0\} \quad [\![\mathsf{v}]\!] = \{0\} \times \mathbb{N}$$

$$[\![\mathsf{q}_i]\!] = \{(x,y) \in \mathbb{N} \times \mathbb{N} \mid (i,x,y) \to^\star (0,0,0)\}$$

- We will show $\boxed{(0,0) \in [\![\sigma_i]\!]}$ for any $i$ (completeness Lemma)
- By universal validity of $!\,\Sigma, \mathsf{a}^x, \mathsf{b}^y \vdash \mathsf{q}_i$, we derive:

$$[\![!\,\sigma_1]\!] \circ \cdots \circ [\![!\,\sigma_r]\!] \circ [\![\mathsf{a}]\!] \circ \cdots \circ [\![\mathsf{a}]\!] \circ [\![\mathsf{b}]\!] \circ \cdots \circ [\![\mathsf{b}]\!] \subseteq [\![\mathsf{q}_i]\!]$$

- Hence $\{(0,0)\} \circ \cdots \circ \{(0,0)\} \circ \{(x,0)\} \circ \{(0,y)\} \subseteq [\![\mathsf{q}_i]\!]$
- Thus $(x,y) \in [\![\mathsf{q}_i]\!]$, and as a consequence $(i,x,y) \to^\star (0,0,0)$

# Inside the proof of the Completeness Lemma

- Case of instruction $i$ : $\mathsf{a} := \mathsf{a} + 1$ ; goto $j$
- $\Sigma$ contains $(\mathsf{a} \multimap \mathsf{q}_j) \multimap \mathsf{q}_i$
- Completeness Lemma condition: $(0,0) \in [\![(\mathsf{a} \multimap \mathsf{q}_j) \multimap \mathsf{q}_i]\!]$
- Interpreted by $\quad [\![\mathsf{a}]\!] \multimap [\![\mathsf{q}_j]\!] \subseteq [\![\mathsf{q}_i]\!]$
- Translates into $\quad \forall x, y \quad (x, y) + (1, 0) \in [\![\mathsf{q}_j]\!] \Rightarrow (x, y) \in [\![\mathsf{q}_i]\!]$
- Thus $\quad \forall x, y \quad (j, x + 1, y) \rightarrow^\star (0, 0, 0) \Rightarrow (i, x, y) \rightarrow^\star (0, 0, 0)$
- This is exactly the operational semantics of “add 1 to $\mathsf{a}$”

# Consequences of the encoding of MM

- We obtain an encoding suitable for classes NDm, Dm and Tm
  - $\mathbb{N} \times \mathbb{N} \in \mathrm{Tm} \subseteq \mathrm{Dm} \subseteq \mathrm{NDm}$
  - obtain for undecidability of $\mathsf{ILL}^0_{\multimap,\&,!}$, $\mathsf{BBI}_{\mathrm{ND}}$, $\mathsf{BBI}_{\mathrm{D}}$ and $\mathsf{BBI}_{\mathrm{T}}$
  - but not for $\mathsf{BBI}_{\mathrm{G}}$ or $\mathsf{CBI}_X$
- What about an interpretation in $\mathbb{Z} \times \mathbb{Z}$ (class G) ?
- Why not consider $\mathbb{N} \times \mathbb{N} \subset \mathbb{Z} \times \mathbb{Z}$
  - with the same trivial phase interpretation as before ?
  - does this interpretation satisfy the completeness Lemma ?
  - i.e. $(0,0) \in [\![\sigma]\!]$ for any $\sigma \in \Sigma$

## Completeness Lemma (revisited for $\mathbb{Z} \times \mathbb{Z}$)

- Case of instruction $i$ : $\mathsf{a} := \mathsf{a} + 1$ ; goto $j$
- $\Sigma$ contains $(\mathsf{a} \multimap \mathsf{q}_j) \multimap \mathsf{q}_i$
- Completeness Lemma condition: $(0,0) \in [\![(\mathsf{a} \multimap \mathsf{q}_j) \multimap \mathsf{q}_i]\!]$
- Interpreted by $\quad [\![\mathsf{a}]\!] \multimap [\![\mathsf{q}_j]\!] \subseteq [\![\mathsf{q}_i]\!]$
- i.e. $\forall x, y \quad (j, x+1, y) \rightarrow^\star (0,0,0) \Rightarrow (i, x, y) \rightarrow^\star (0,0,0)$
- this is not the operational semantics of "add 1 to $\mathsf{a}$"
  - there is a problem when $x = -1$
- Solution: change condition into: for any $x, y$

$$\big((x,y) \in \mathbb{N} \times \mathbb{N} \text{ and } (j, x+1, y) \rightarrow^\star (0,0,0)\big) \Rightarrow (i, x, y) \rightarrow^\star (0,0,0)$$

# Completeness Lemma (revisited) (cont.)

- Introduction of a variable $\mathsf{k}$ interpreted by $[\![\mathsf{k}]\!] = \mathbb{N} \times \mathbb{N}$
- $(x, y) \in \mathbb{N} \times \mathbb{N}$ thus becomes $(x, y) \in [\![\mathsf{k}]\!]$
- Op. sem. of "add 1 to $\mathsf{a}$" encoded as $(0, 0) \in [\![(\mathsf{k} \,\&\, (\mathsf{a} \multimap \mathsf{q}_j)) \multimap \mathsf{q}_i]\!]$
- Formula $(\mathsf{k} \,\&\, (\mathsf{a} \multimap \mathsf{q}_j)) \multimap \mathsf{q}_i$ not in $\mathsf{ILL}^0_{\multimap,\&,!}$ replaced by two:

$$\left\{(\mathsf{k} \,\&\, \mathsf{q}_j^{\mathsf{a}}) \multimap \mathsf{q}_i, (\mathsf{a} \multimap \mathsf{q}_j) \multimap \mathsf{q}_j^{\mathsf{a}}\right\}$$

- with $[\![\mathsf{q}_j^{\mathsf{a}}]\!] = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid (j, x + 1, y) \rightarrow^\star (0, 0, 0)\}$
- Also add the three following formulae to $\Sigma$:

$$\left\{\mathsf{a} \multimap (\mathsf{k} \multimap \mathsf{k}), \mathsf{b} \multimap (\mathsf{k} \multimap \mathsf{k}), (\mathsf{a} \multimap \mathsf{a}) \multimap \mathsf{k}\right\}$$

## Conclusion and perspectives

- Modified encoding suitable for class G and thus, all classes.
  - $\mathbb{Z} \times \mathbb{Z} \in \mathrm{G} \subseteq X$, for any $X \in \{\mathrm{Tm}, \mathrm{Dm}, \mathrm{NDm}, \mathrm{Tg}, \mathrm{Dg}, \mathrm{NDg}\}$
  - obtain for undecidability of $\mathsf{ILL}^0_{\multimap,\&,!}$, $\mathsf{BBI}_X$, $\mathsf{CBI}_X$ for any $X$
- What about decidability of BBI/CBI restricted to $\mathbb{N}/\mathbb{Z}$ ?
  - 1-counter MM are decidable (Bouajjani et al. 99)
- What about an interpretation in the RAM-domain model ?
  - $(\mathcal{P}_{\mathrm{f}}(\mathbb{N}), \uplus, \emptyset)$ belongs to the class of separation models
  - $\mathcal{P}_{\mathrm{f}}(\mathbb{N})$ bisimulates $\mathbb{N} \times \mathbb{N}$
  - $\mathcal{P}_{\mathrm{f/c}}(\mathbb{N})$ bisimulates $\mathbb{N} \times \mathbb{N} \cup \{\infty\}$

# Bisimulation vs. Kripke/phase semantics of **BBI**

- $(M, \circ, \epsilon)$ and $(N, \bullet, \mathsf{e})$ two ND monoids
- Bisimulation relation $\sim\, \subseteq M \times N$:
  - checks $\sim\, \subseteq \{(\epsilon, \mathsf{e})\} \cup M\backslash\{\epsilon\} \times N\backslash\{\mathsf{e}\}$ and

$$m \sim m' \Rightarrow \begin{cases} \forall a \circ b \ni m \; \exists a' \bullet b' \ni m' \;\; a \sim a' \text{ and } b \sim b' \\ \forall a' \bullet b' \ni m' \; \exists a \circ b \ni m \;\; a \sim a' \text{ and } b \sim b' \\ \forall b \in a \circ m \; \exists b' \in a' \bullet m' \;\; a \sim a' \text{ and } b \sim b' \\ \forall b' \in a' \bullet m' \; \exists b \in a \circ m \;\; a \sim a' \text{ and } b \sim b' \end{cases}$$

- if $m \sim m'$ then for any $F$ of BBI, $\boxed{m \in [\![F]\!] \text{ iff } m' \in [\![F]\!]'}$

# Bisimulating $\mathbb{N} \times \mathbb{N}$ in $\mathcal{P}_\mathrm{f}(\mathbb{N})$

- $(\mathcal{P}_\mathrm{f}(\mathbb{N}), \uplus, \emptyset)$ and $(\mathbb{N} \times \mathbb{N}, +, (0,0))$ are two ND monoids
- Let $\mathbb{N} = \mathbb{E} \uplus \mathbb{O}$ (e.g. even/odd numbers)
- For $X \in \mathcal{P}_\mathrm{f}(\mathbb{N})$, let $\varphi(X) = (\mathrm{card}(X \cap \mathbb{E}), \mathrm{card}(X \cap \mathbb{O}))$
- $\varphi : \mathcal{P}_\mathrm{f}(\mathbb{N}) \longrightarrow \mathbb{N} \times \mathbb{N}$ is a projection (surjective)
- $\varphi \subseteq \mathcal{P}_\mathrm{f}(\mathbb{N}) \times (\mathbb{N} \times \mathbb{N})$ is a bisimulation
- Use $\varphi$ to transform the $\mathbb{N} \times \mathbb{N}$ model into a $\mathcal{P}_\mathrm{f}(\mathbb{N})$ model
  - simply define $[\![\mathsf{x}]\!]' = \varphi^{-1}\big([\![\mathsf{x}]\!]\big)$
- Also $\mathcal{P}_{\mathrm{f}/\mathrm{c}}(\mathbb{N})$ bisimilar to $\mathbb{N} \times \mathbb{N} \cup \{\infty\}$ (RAM-domain for CBI)