Labelled Tableaux for Linear Time Bunched Implication Logic

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Context and Goals

Logic of Bunched Implications (BI)

- **sharing and separation of resources**

  [Pym-O'Hearn 1999]

  [Galmiche-Mery-Pym 2005]

An extension of BI

- with **temporal modalities** from LTL
- to capture properties about **resource evolution** over time
- with a **monoid-based** resource semantics
- and a proof system allowing **counter-model** construction

Related Works

Modal extensions of BI and Boolean BI (BBI)

- **capture epistemic aspects of resource management**

  [Courtault-Galmiche 2018]

  [Courtault-vanDitmarsch-Galmiche 2019]

Temporal extension of BI

- temporal BI logic (tBI) [Kamide 2013]
- Grothendieck semantics of BI + bounded timelines (LTL)
- Bunched sequent calculus LBI + temporal operators $\Box$, $\Diamond$, $\circ$

Overview of the Talk

- Introduction
- Linear Time Bunched Implication Logic
- Expressivity of LTBI
- Tableau Calculus
- Soundness
- Counter-Model Construction
- Completeness Issues
- Conclusion & Future Work

Syntax of LTBI

Let **P** be a countable set of propositional letters.

The set **F** of LTBI formulas is given by the following grammar:

$$\begin{array}{rll} \mathrm{A} ::= & \mathbf{P} & \\ & \mid \bot \mid \mathrm{A} \wedge \mathrm{A} \mid \mathrm{A} \vee \mathrm{A} \mid \mathrm{A} \to \mathrm{A} & \text{(additives)} \\ & \mid \mathrm{I} \mid \mathrm{A} * \mathrm{A} \mid \mathrm{A} \mathrel{-\!\!*} \mathrm{A} & \text{(multiplicatives)} \\ & \mid \Box \mathrm{A} \mid \Diamond \mathrm{A} \mid \circ \mathrm{A} & \text{(temporal)} \end{array}$$

As usual $\neg \mathrm{A} = \mathrm{A} \to \bot$ and $\top = \bot \to \bot$

LTBI Frames

Structures $\mathcal{R} = (\mathbf{R}, \star, \epsilon, \leqslant^{\mathfrak{r}}, \pi, \mathbf{S}, \leqslant^{\mathfrak{s}}, s_0)$ where

- $(\mathbf{R}, \star, \epsilon, \leqslant^{\mathfrak{r}}, \pi)$ is a **commutative monoid** such that

  $\forall\, r \in \mathbf{R}.\, r \leqslant^{\mathfrak{r}} \pi$ and $r \star \pi = \pi$

  $\forall\, r, r', r'' \in \mathbf{R}.\, r \leqslant^{\mathfrak{r}} r'$ implies $r \star r'' \leqslant^{\mathfrak{r}} r' \star r''$

  Elements of $\mathbf{R}$ called **resources**

- $(\mathbf{S}, \leqslant^{\mathfrak{s}}, s_0)$ is a **discrete timeline**, i.e.

  a subset of $(\mathbb{N}, \leqslant)$ with least element s_0

  Elements of $\mathbf{S}$ called **states**

LTBI Models

Valuations: partial functions $[\cdot] : \mathbf{P} \to \wp(\mathbf{R} \times \mathbf{S})$ such that

$$(\mathcal{M}_{\mathrm{K}})\ \forall\, \mathrm{p} \in \mathbf{P}. \forall\, s \in \mathbf{S}. \forall\, r, r' \in \mathbf{R}.$$

$$\text{if } r \leqslant^{\mathfrak{r}} r' \text{ and } (r, s) \in [\mathrm{p}] \text{ then } (r', s) \in [\mathrm{p}]$$

$$(\mathcal{M}_{\pi})\ \forall\, \mathrm{p} \in \mathbf{P}. \forall\, s \in \mathbf{S}. (\pi, s) \in [\mathrm{p}]$$

Models: triples $\mathcal{M} = (\mathcal{R}, [\cdot], \Vdash)$ where

- $\mathcal{M}$ is an LTBI-frame
- $[\cdot]$ is an LTBI-valuation
- $\Vdash \subseteq \mathbf{R} \times \mathbf{S} \times \mathbf{F}$ smallest **forcing relation** such that

LTBI Forcing Relation

$(r, s) \Vdash \mathrm{p}$ iff $(r, s) \in [\mathrm{p}]$

$(r, s) \Vdash \bot$ iff $\pi \leqslant^{\mathfrak{r}} r$
$(r, s) \Vdash \mathrm{A} \vee \mathrm{B}$ iff $(r, s) \Vdash \mathrm{A}$ or $(r, s) \Vdash \mathrm{B}$
$(r, s) \Vdash \mathrm{A} \wedge \mathrm{B}$ iff $(r, s) \Vdash \mathrm{A}$ and $(r, s) \Vdash \mathrm{B}$
$(r, s) \Vdash \mathrm{A} \rightarrow \mathrm{B}$ iff $\forall\, r'. r \leqslant^{\mathfrak{r}} r'$ and $(r', s) \Vdash \mathrm{A}$ imply $(r', s) \Vdash \mathrm{B}$

$(r, s) \Vdash \mathrm{I}$ iff $\epsilon \leqslant^{\mathfrak{r}} r$
$(r, s) \Vdash \mathrm{A} * \mathrm{B}$ iff $\exists\, r', r''. r' \star r'' \leqslant^{\mathfrak{r}} r, (r', s) \Vdash \mathrm{A}$ and $(r'', s) \Vdash \mathrm{B}$
$(r, s) \Vdash \mathrm{A} -\!\!* \mathrm{B}$ iff $\forall\, r', r''.(r', s) \Vdash \mathrm{A}$ and $r' \star r \leqslant^{\mathfrak{r}} r''$ imply $(r'', s) \Vdash \mathrm{B}$

LTBI Forcing Relation

$(r, s) \Vdash \Box \mathrm{A}$ iff $\forall\, s'.$if $s \leqslant^{\mathfrak{s}} s'$ then $(r, s') \Vdash \mathrm{A}$

$(r, s) \Vdash \Diamond \mathrm{A}$ iff $\exists\, s'. s \leqslant^{\mathfrak{s}} s'$ and $(r, s') \Vdash \mathrm{A}$

$(r, s) \Vdash \circ \mathrm{A}$ iff $\exists\, s'. s' = \mathfrak{n}(s)$ and $(r, s') \Vdash \mathrm{A}$

where $\mathfrak{n}$ is the **next** function induced on **S** by $\leqslant^{\mathfrak{s}}$, i.e.

$$\mathfrak{n}(s) \text{ least element of } \{\, s' \mid s' \in \mathbf{S} \text{ and } s <^{\mathfrak{s}} s' \,\}$$

$\Longrightarrow (r, s) \nVdash \circ \mathrm{A}$ when s last point of a bounded timeline

LTBI Validity

Let A be a formula

- A **satisfied** in a model $\mathcal{M}$ ($\mathcal{M} \vDash \mathrm{A}$) iff $(\epsilon, s) \Vdash \mathrm{A}$ for all $s \in \mathbf{S}$
- A **valid** ($\vDash \mathrm{A}$) iff $\mathcal{M} \vDash \mathrm{A}$ for all models $\mathcal{M}$

Let B be a formula

- A **entails** B ($\mathrm{A} \vDash \mathrm{B}$) iff

$$\mathcal{M} \vDash \mathrm{A} \text{ implies } \mathcal{M} \vDash \mathrm{B} \text{ for all models } \mathcal{M}$$

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Prices of Three Goods

Set of goods $G = \{ g_1, g_2, g_3 \}$

good	**2023**	**2024**	**2025**
g_1	2 000 €	2 100 €	2 200 €
g_2	300 €	250 €	350 €
g_3	1 700 €	1 800 €	1 500 €

Pricing function $pr : G \times \mathbf{S} \to \mathbb{N}$

Timeline $(\mathbf{S} = [2023 - 2025], \leqslant^{\mathfrak{s}}, 2023)$

Resource monoid $(\mathbf{R} = \mathbb{N} \cup \{ \infty \}, +, 0, \leqslant^{\mathfrak{r}}, \infty)$

Prices of Three Goods

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Pricing function $pr : G \times \mathbf{S} \to \mathbb{N}$

Affordability of a set of goods gs forall $(r, s) \in \mathbf{R} \times \mathbf{S}$

$$(r, s) \Vdash Af(gs) \ \text{ iff } \ pr(gs, s) \overset{\text{def}}{=} \textstyle\sum_{g \in gs} pr(g, s) \leqslant r$$

Prices of Three Goods

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$$(r, s) \Vdash Af(gs) \quad \text{iff} \quad \ast_{g \in gs} Af(g)$$

Prices of Three Goods

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Pricing function $pr : G \times \mathbf{S} \to \mathbb{N}$

Affordability of a set of goods gs forall $(r, s) \in \mathbf{R} \times \mathbf{S}$

$$(r, s) \Vdash Af(g, g') \text{ iff } (r, s) \Vdash Af(g) * Af(g')$$

Prices of Three Goods

$(3\,000, 2023) \Vdash Af(g_1) \wedge (Af(g_2) * Af(g_3))$

good	**2023**	**2024**	**2025**
g_1	2 000 €	2 100 €	2 200 €
g_2	300 €	250 €	350 €
g_3	1 700 €	1 800 €	1 500 €

In 2023 we can use 3 000 € to buy g_1 and we can split 3 000 € into two disjoint amounts, one to buy g_2, the other to buy g_3

Prices of Three Goods

$$(3\,000, 2023) \Vdash \Box Af(g_2) * (\Diamond Af(g_3) \wedge (Af(g_1) * \circ Af(g_2)))$$

good	**2023**	**2024**	**2025**
g_1	2 000 €	2 100 €	2 200 €
g_2	300 €	250 €	350 €
g_3	1 700 €	1 800 €	1 500 €

In 2023, we can split 3 000 € into two disjoint amounts r_1 and r_2. r_1 keeps g_2 affordable every year between 2023 and 2025. r_2 allows two options. Ensure the affordability of g_3 once between 2023 and 2025. Split r_2 into two amounts r_2', r_2'', r_2' making g_3 affordable in 2023, r_2'' making g_2 affordable one year later.

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Labels and Constraints

- **Resource labels** L_r: given by grammar $X ::= \gamma_r \mid X \circ X$

  $\gamma_r = \{\, \epsilon_{\mathrm{L}}, \mathrm{c}_1, \mathrm{c}_2, \ldots \,\}$ is a countable set of constants

  label composition $\circ$ is associative, commutative, with unit ϵ_{L}
- **Resource constraints**: $\mathrm{x} \leqslant^{\mathfrak{r}}_{\mathrm{L}} \mathrm{y}$ $(\mathrm{x}, \mathrm{y} \in L_r)$
- **State labels** L_s: given by grammar $X ::= \gamma_s \mid \eta X$

  $\gamma_s = \{\, \gamma_0, \gamma_1, \gamma_2, \ldots \,\}$ is a countable set of instants

  η is the successor symbol
- **State constraints**: $\tau \leqslant^{\mathfrak{s}}_{\mathrm{L}} \upsilon$, $\tau <^{\mathfrak{s}}_{\mathrm{L}} \upsilon$, $\tau =^{\mathfrak{s}}_{\mathrm{L}} \upsilon$, $\tau \neq^{\mathfrak{s}}_{\mathrm{L}} \upsilon$ $(\tau, \upsilon \in L_s)$

Closure of Constraints

Let C_r and C_s be sets of resource and state constraints

- Closure $C_r^\bullet$ smallest set containing C_r and closed under

$$\frac{\mathrm{x} \leqslant^{\mathfrak{r}}_{\mathrm{L}} \mathrm{y} \qquad \mathrm{y} \leqslant^{\mathfrak{r}}_{\mathrm{L}} \mathrm{z}}{\mathrm{x} \leqslant^{\mathfrak{r}}_{\mathrm{L}} \mathrm{z}} \qquad \frac{\mathrm{x} \leqslant^{\mathfrak{r}}_{\mathrm{L}} \mathrm{y}}{\mathrm{x} \leqslant^{\mathfrak{r}}_{\mathrm{L}} \mathrm{x}} \qquad \frac{\mathrm{x} \leqslant^{\mathfrak{r}}_{\mathrm{L}} \mathrm{y}}{\mathrm{y} \leqslant^{\mathfrak{r}}_{\mathrm{L}} \mathrm{y}}$$

$$\frac{\mathrm{x}\,\mathrm{y} \leqslant^{\mathfrak{r}}_{\mathrm{L}} \mathrm{x}\,\mathrm{y}}{\mathrm{x} \leqslant^{\mathfrak{r}}_{\mathrm{L}} \mathrm{x}} \qquad \frac{\mathrm{z}\,\mathrm{y} \leqslant^{\mathfrak{r}}_{\mathrm{L}} \mathrm{z}\,\mathrm{y} \qquad \mathrm{x} \leqslant^{\mathfrak{r}}_{\mathrm{L}} \mathrm{y}}{\mathrm{z}\,\mathrm{x} \leqslant^{\mathfrak{r}}_{\mathrm{L}} \mathrm{z}\,\mathrm{y}}$$

- Closure $C_s^\bullet$ smallest containing C_s such that

$\leqslant^{\mathfrak{s}}_{\mathrm{L}}, <^{\mathfrak{s}}_{\mathrm{L}}, =^{\mathfrak{s}}_{\mathrm{L}}, \neq^{\mathfrak{s}}_{\mathrm{L}}, \eta$ reflect $\leqslant, <, =, \neq, \mathfrak{n}$ in $\mathbb{N}$

Tableau Construction

Labelled Formula

- quadruple $(\mathbb{S}, \mathrm{A}, \mathrm{x}, \tau) \in \{\, \mathbb{T}, \mathbb{F} \,\} \times \mathbf{F} \times L_r \times L_s$
- **denoted** $\mathbb{S}\ \mathrm{A} : (\mathrm{x}, \tau)$

Tableau for a formula A

- Start with root node $\mathbb{F}\ \mathrm{A} : (\epsilon_{\mathrm{L}}, \gamma_0)$, apply tableau rules
- Two kinds of constraints

  assertions: introduce new facts in a branch

  requirements: must belong to closure of assertions

Rules of $\mathsf{T}_{\mathsf{LTBI}}$ Tableau Calculus

$$\frac{\mathbb{T}\ A \wedge B : (x, \tau)}{\begin{array}{c}\mathbb{T}\ A : (x, \tau)\\ \mathbb{T}\ B : (x, \tau)\end{array}}
\qquad
\begin{array}{c}\mathbb{F}\ A \wedge B : (x, \tau)\\ \begin{array}{c|c}\mathbb{F}\ A : (x, \tau) & \mathbb{F}\ B : (x, \tau)\end{array}\end{array}$$

$$\frac{\mathbb{F}\ A \vee B : (x, \tau)}{\begin{array}{c}\mathbb{F}\ A : (x, \tau)\\ \mathbb{F}\ B : (x, \tau)\end{array}}
\qquad
\begin{array}{c}\mathbb{T}\ A \vee B : (x, \tau)\\ \begin{array}{c|c}\mathbb{T}\ A : (x, \tau) & \mathbb{T}\ B : (x, \tau)\end{array}\end{array}$$

$$\frac{\mathbb{F}\ A \rightarrow B : (x, \tau)}{\begin{array}{c}\mathbb{A}\ x \leqslant^{\mathfrak{r}}_{L} a\\ \mathbb{T}\ A : (a, \tau)\\ \mathbb{F}\ B : (a, \tau)\end{array}}
\qquad
\begin{array}{c}\mathbb{T}\ A \rightarrow B : (x, \tau)\\ \begin{array}{c|c}\mathbb{R}\ x \leqslant^{\mathfrak{r}}_{L} y & \mathbb{R}\ x \leqslant^{\mathfrak{r}}_{L} y\\ \mathbb{F}\ A : (y, \tau) & \mathbb{T}\ B : (y, \tau)\end{array}\end{array}$$

Rules of $\mathrm{T}_{\mathrm{LTBI}}$ Tableau Calculus

$$\frac{\mathbb{T}\ A * B : (x, \tau)}{\begin{array}{l}\mathbb{A}\ ab \leqslant^{\mathfrak{r}}_{L} x\\ \mathbb{T}\ A : (a, \tau)\\ \mathbb{T}\ B : (b, \tau)\end{array}}
\qquad
\begin{array}{c}\mathbb{F}\ A * B : (x, \tau)\\ \begin{array}{l|l}\mathbb{R}\ y\,z \leqslant^{\mathfrak{r}}_{L} x & \mathbb{R}\ y\,z \leqslant^{\mathfrak{r}}_{L} x\\ \mathbb{F}\ A : (y, \tau) & \mathbb{F}\ B : (z, \tau)\end{array}\end{array}$$

$$\frac{\mathbb{T}\ I : (x, \tau)}{\mathbb{A}\ \epsilon_{L} \leqslant^{\mathfrak{r}}_{L} x}$$

$$\frac{\mathbb{F}\ A -\!\!* B : (x, \tau)}{\begin{array}{l}\mathbb{A}\ x\,a \leqslant^{\mathfrak{r}}_{L} b\\ \mathbb{T}\ A : (a, \tau)\\ \mathbb{F}\ B : (b, \tau)\end{array}}
\qquad
\begin{array}{c}\mathbb{T}\ A -\!\!* B : (x, \tau)\\ \begin{array}{l|l}\mathbb{R}\ x\,y \leqslant^{\mathfrak{r}}_{L} z & \mathbb{R}\ x\,y \leqslant^{\mathfrak{r}}_{L} z\\ \mathbb{F}\ A : (y, \tau) & \mathbb{T}\ B : (z, \tau)\end{array}\end{array}$$

Rules of $\mathrm{T}_{\mathrm{LTBI}}$ Tableau Calculus

$$\frac{\mathbb{T}\ \circ \mathrm{A} : (\mathrm{x}, \tau)}{\mathbb{T}\ \mathrm{A} : (\mathrm{x}, \eta\tau)} \qquad \frac{\mathbb{T}\ \Box \mathrm{A} : (\mathrm{x}, \tau)}{\begin{array}{l} \mathbb{R}\ \tau \leqslant^{\mathfrak{s}}_{\mathrm{L}} \alpha \\ \mathbb{T}\ \mathrm{A} : (\mathrm{x}, \alpha) \end{array}} \qquad \frac{\mathbb{F}\ \Box \mathrm{A} : (\mathrm{x}, \tau)}{\begin{array}{l} \mathbb{A}\ \tau \leqslant^{\mathfrak{s}}_{\mathrm{L}} \upsilon \\ \mathbb{F}\ \mathrm{A} : (\mathrm{x}, \upsilon) \end{array}}$$

$$\frac{\mathbb{F}\ \circ \mathrm{A} : (\mathrm{x}, \tau)}{\mathbb{F}\ \mathrm{A} : (\mathrm{x}, \eta\tau)} \qquad \frac{\mathbb{T}\ \Diamond \mathrm{A} : (\mathrm{x}, \tau)}{\begin{array}{l} \mathbb{A}\ \tau \leqslant^{\mathfrak{s}}_{\mathrm{L}} \alpha \\ \mathbb{T}\ \mathrm{A} : (\mathrm{x}, \alpha) \end{array}} \qquad \frac{\mathbb{F}\ \Diamond \mathrm{A} : (\mathrm{x}, \tau)}{\begin{array}{l} \mathbb{R}\ \tau \leqslant^{\mathfrak{s}}_{\mathrm{L}} \upsilon \\ \mathbb{F}\ \mathrm{A} : (\mathrm{x}, \upsilon) \end{array}}$$

$$\begin{array}{c} \mathrm{CD} \\ \begin{array}{l|l} \mathbb{R}\ \tau \leqslant^{\mathfrak{s}}_{\mathrm{L}} \upsilon & \mathbb{R}\ \tau \leqslant^{\mathfrak{s}}_{\mathrm{L}} \upsilon \\ \mathbb{A}\ \tau <^{\mathfrak{s}}_{\mathrm{L}} \upsilon & \mathbb{A}\ \tau =^{\mathfrak{s}}_{\mathrm{L}} \upsilon \end{array} \end{array} \qquad \begin{array}{c} \mathrm{LR} \\ \begin{array}{l|l} \mathbb{R}\ \tau \leqslant^{\mathfrak{s}}_{\mathrm{L}} \upsilon & \mathbb{R}\ \tau \leqslant^{\mathfrak{s}}_{\mathrm{L}} \upsilon \\ \mathbb{R}\ \tau \leqslant^{\mathfrak{s}}_{\mathrm{L}} \zeta & \mathbb{R}\ \tau \leqslant^{\mathfrak{s}}_{\mathrm{L}} \zeta \\ \mathbb{A}\ \upsilon \leqslant^{\mathfrak{s}}_{\mathrm{L}} \zeta & \mathbb{A}\ \zeta \leqslant^{\mathfrak{s}}_{\mathrm{L}} \upsilon \end{array} \end{array} \qquad \frac{\mathbb{S}\ \mathrm{A} : (\mathrm{c}, \tau)}{\begin{array}{l} \mathbb{R}\ \tau =^{\mathfrak{s}}_{\mathrm{L}} \upsilon \\ \mathbb{S}\ \mathrm{A} : (\mathrm{c}, \upsilon) \end{array}}$$

Example of Tableau Construction

$$\mathbb{F}\ \Diamond \mathrm{A} \wedge \Diamond \mathrm{B} \to \Diamond(\mathrm{A} \wedge \Diamond \mathrm{B}) \vee \Diamond(\mathrm{B} \wedge \Diamond \mathrm{A}) : (\epsilon_{\mathrm{L}} , \gamma_0)_{[1]}$$

1

Example of Tableau Construction

$\mathbb{F}\ \Diamond A \wedge \Diamond B \rightarrow \Diamond(A \wedge \Diamond B) \vee \Diamond(B \wedge \Diamond A) : (\epsilon_L, \gamma_0)$[1]

1

$\mathbb{A}\ \epsilon_L \leqslant^{\mathfrak{r}}_L c_1$

$\mathbb{T}\ \Diamond A \wedge \Diamond B : (c_1, \gamma_0)$[2]

$\mathbb{F}\ \Diamond(A \wedge \Diamond B) \vee \Diamond(B \wedge \Diamond A) : (c_1, \gamma_0)$

2

Example of Tableau Construction

$\mathbb{F}\ \Diamond A \wedge \Diamond B \rightarrow \Diamond(A \wedge \Diamond B) \vee \Diamond(B \wedge \Diamond A) : (\epsilon_L, \gamma_0)_{[1]}$

1 ---

$\mathbb{A}\ \epsilon_L \leqslant^{\mathfrak{r}}_{L} c_1$

$\mathbb{T}\ \Diamond A \wedge \Diamond B : (c_1, \gamma_0)_{[2]}$

$\mathbb{F}\ \Diamond(A \wedge \Diamond B) \vee \Diamond(B \wedge \Diamond A) : (c_1, \gamma_0)$

2 ---

$\mathbb{T}\ \Diamond A : (c_1, \gamma_0)_{[3]}$

$\mathbb{T}\ \Diamond B : (c_1, \gamma_0)$

3 ---

Example of Tableau Construction

$\mathbb{F}\ \Diamond A \wedge \Diamond B \rightarrow \Diamond(A \wedge \Diamond B) \vee \Diamond(B \wedge \Diamond A) : (\epsilon_L, \gamma_0)_{[1]}$

1 ---

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$\mathbb{T}\ \Diamond A \wedge \Diamond B : (c_1, \gamma_0)_{[2]}$

$\mathbb{F}\ \Diamond(A \wedge \Diamond B) \vee \Diamond(B \wedge \Diamond A) : (c_1, \gamma_0)$

2 ---

$\mathbb{T}\ \Diamond A : (c_1, \gamma_0)_{[3]}$

$\mathbb{T}\ \Diamond B : (c_1, \gamma_0)_{[4]}$

3 ---

$\mathbb{A}\ \gamma_0 \leqslant^{\mathfrak{s}}_{L} \gamma_1$

$\mathbb{T}\ A : (c_1, \gamma_1)$

4 ---

Example of Tableau Construction

$\mathbb{F}\ \Diamond A \wedge \Diamond B \rightarrow \Diamond(A \wedge \Diamond B) \vee \Diamond(B \wedge \Diamond A) : (\epsilon_L, \gamma_0)_{[1]}$

1

$\mathbb{A}\ \epsilon_L \leqslant^{\mathfrak{r}}_{L} c_1$

$\mathbb{T}\ \Diamond A \wedge \Diamond B : (c_1, \gamma_0)_{[2]}$

$\mathbb{F}\ \Diamond(A \wedge \Diamond B) \vee \Diamond(B \wedge \Diamond A) : (c_1, \gamma_0)_{[5]}$

2

$\mathbb{T}\ \Diamond A : (c_1, \gamma_0)_{[3]}$

$\mathbb{T}\ \Diamond B : (c_1, \gamma_0)_{[4]}$

3

$\mathbb{A}\ \gamma_0 \leqslant^{\mathfrak{s}}_{L} \gamma_1$

$\mathbb{T}\ A : (c_1, \gamma_1)$

4

$\mathbb{A}\ \gamma_0 \leqslant^{\mathfrak{s}}_{L} \gamma_2$

$\mathbb{T}\ B : (c_1, \gamma_2)$

5

Example of Tableau Construction

$\mathbb{F}\ \Diamond A \wedge \Diamond B \rightarrow \Diamond(A \wedge \Diamond B) \vee \Diamond(B \wedge \Diamond A) : (\epsilon_{L}, \gamma_0)_{[1]}$

1

$\mathbb{A}\ \epsilon_{L} \leqslant^{\mathfrak{r}}_{L} c_1$

$\mathbb{T}\ \Diamond A \wedge \Diamond B : (c_1, \gamma_0)_{[2]}$

$\mathbb{F}\ \Diamond(A \wedge \Diamond B) \vee \Diamond(B \wedge \Diamond A) : (c_1, \gamma_0)_{[5]}$

2

$\mathbb{T}\ \Diamond A : (c_1, \gamma_0)_{[3]}$

$\mathbb{T}\ \Diamond B : (c_1, \gamma_0)_{[4]}$

3

$\mathbb{A}\ \gamma_0 \leqslant^{\mathfrak{s}}_{L} \gamma_1$

$\mathbb{T}\ A : (c_1, \gamma_1)$

4

$\mathbb{A}\ \gamma_0 \leqslant^{\mathfrak{s}}_{L} \gamma_2$

$\mathbb{T}\ B : (c_1, \gamma_2)$

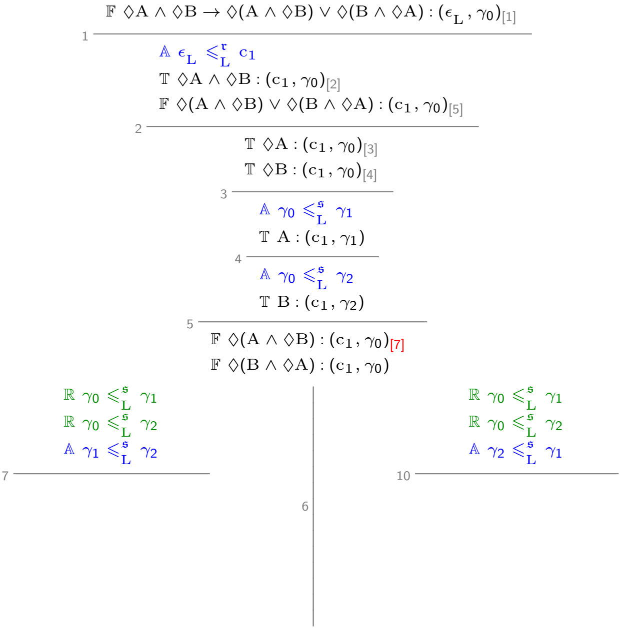
5

$\mathbb{F}\ \Diamond(A \wedge \Diamond B) : (c_1, \gamma_0)$

$\mathbb{F}\ \Diamond(B \wedge \Diamond A) : (c_1, \gamma_0)$

6

Example of Tableau Construction



$\mathbb{F}\ \Diamond A \wedge \Diamond B \to \Diamond(A \wedge \Diamond B) \vee \Diamond(B \wedge \Diamond A) : (\epsilon_L , \gamma_0)_{[1]}$

1

$\mathbb{A}\ \epsilon_L \leqslant^{\mathfrak{r}}_{L} c_1$

$\mathbb{T}\ \Diamond A \wedge \Diamond B : (c_1 , \gamma_0)_{[2]}$

$\mathbb{F}\ \Diamond(A \wedge \Diamond B) \vee \Diamond(B \wedge \Diamond A) : (c_1 , \gamma_0)_{[5]}$

2

$\mathbb{T}\ \Diamond A : (c_1 , \gamma_0)_{[3]}$

$\mathbb{T}\ \Diamond B : (c_1 , \gamma_0)_{[4]}$

3

$\mathbb{A}\ \gamma_0 \leqslant^{\mathfrak{s}}_{L} \gamma_1$

$\mathbb{T}\ A : (c_1 , \gamma_1)$

4

$\mathbb{A}\ \gamma_0 \leqslant^{\mathfrak{s}}_{L} \gamma_2$

$\mathbb{T}\ B : (c_1 , \gamma_2)$

5

$\mathbb{F}\ \Diamond(A \wedge \Diamond B) : (c_1 , \gamma_0)_{[7]}$

$\mathbb{F}\ \Diamond(B \wedge \Diamond A) : (c_1 , \gamma_0)$

$\mathbb{R}\ \gamma_0 \leqslant^{\mathfrak{s}}_{L} \gamma_1$

$\mathbb{R}\ \gamma_0 \leqslant^{\mathfrak{s}}_{L} \gamma_2$

$\mathbb{A}\ \gamma_1 \leqslant^{\mathfrak{s}}_{L} \gamma_2$

7

6

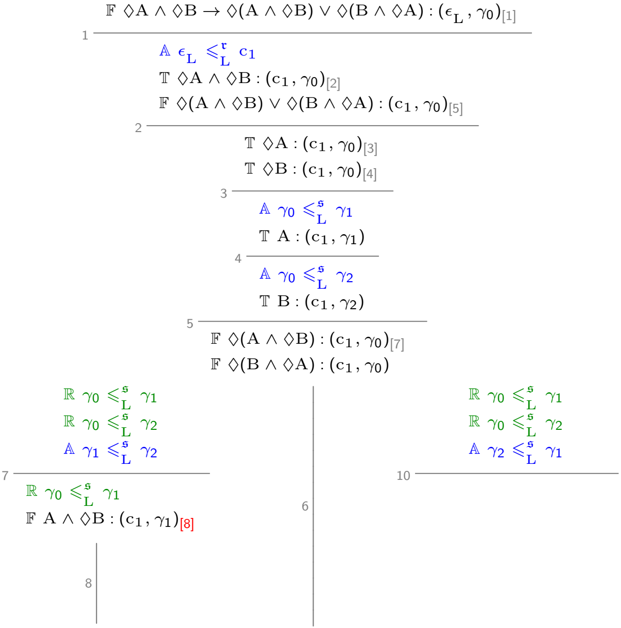
$\mathbb{R}\ \gamma_0 \leqslant^{\mathfrak{s}}_{L} \gamma_1$

$\mathbb{R}\ \gamma_0 \leqslant^{\mathfrak{s}}_{L} \gamma_2$

$\mathbb{A}\ \gamma_2 \leqslant^{\mathfrak{s}}_{L} \gamma_1$

10

Example of Tableau Construction



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$\mathbb{F}\ \Diamond(A \wedge \Diamond B) \vee \Diamond(B \wedge \Diamond A) : (c_1, \gamma_0)_{[5]}$

2

$\mathbb{T}\ \Diamond A : (c_1, \gamma_0)_{[3]}$

$\mathbb{T}\ \Diamond B : (c_1, \gamma_0)_{[4]}$

3

$\mathbb{A}\ \gamma_0 \leqslant^{\mathfrak{s}}_{L} \gamma_1$

$\mathbb{T}\ A : (c_1, \gamma_1)$

4

$\mathbb{A}\ \gamma_0 \leqslant^{\mathfrak{s}}_{L} \gamma_2$

$\mathbb{T}\ B : (c_1, \gamma_2)$

5

$\mathbb{F}\ \Diamond(A \wedge \Diamond B) : (c_1, \gamma_0)_{[7]}$

$\mathbb{F}\ \Diamond(B \wedge \Diamond A) : (c_1, \gamma_0)$

$\mathbb{R}\ \gamma_0 \leqslant^{\mathfrak{s}}_{L} \gamma_1$

$\mathbb{R}\ \gamma_0 \leqslant^{\mathfrak{s}}_{L} \gamma_2$

$\mathbb{A}\ \gamma_1 \leqslant^{\mathfrak{s}}_{L} \gamma_2$

7

$\mathbb{R}\ \gamma_0 \leqslant^{\mathfrak{s}}_{L} \gamma_1$

$\mathbb{F}\ A \wedge \Diamond B : (c_1, \gamma_1)_{[8]}$

8

6

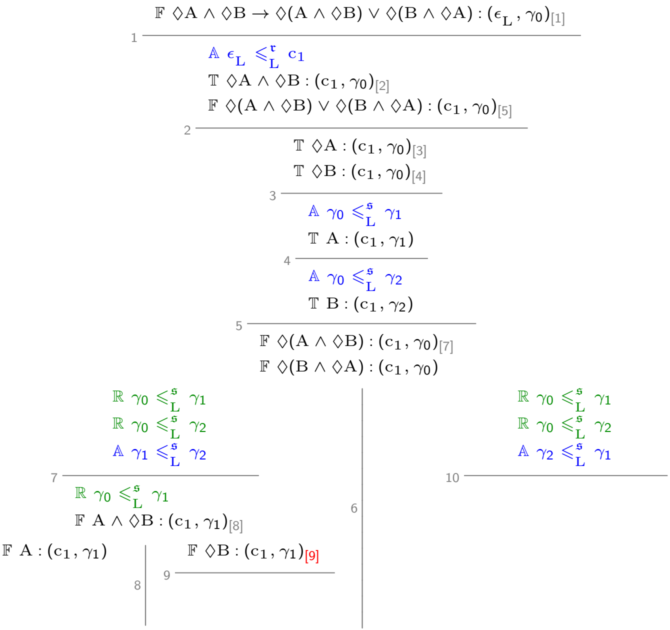
$\mathbb{R}\ \gamma_0 \leqslant^{\mathfrak{s}}_{L} \gamma_1$

$\mathbb{R}\ \gamma_0 \leqslant^{\mathfrak{s}}_{L} \gamma_2$

$\mathbb{A}\ \gamma_2 \leqslant^{\mathfrak{s}}_{L} \gamma_1$

10

Example of Tableau Construction



$\mathbb{F}\ \Diamond A \wedge \Diamond B \rightarrow \Diamond(A \wedge \Diamond B) \vee \Diamond(B \wedge \Diamond A) : (\epsilon_L , \gamma_0)_{[1]}$

1

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$\mathbb{F}\ A : (c_1 , \gamma_1)$

8

$\mathbb{F}\ \Diamond B : (c_1 , \gamma_1)_{[9]}$

9

6

$\mathbb{R}\ \gamma_0 \leqslant^{\mathfrak{s}}_{L} \gamma_1$

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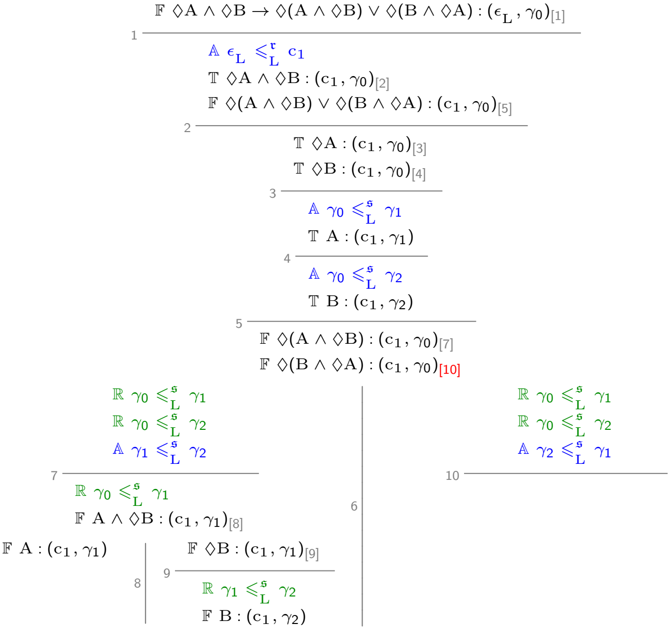
10

Example of Tableau Construction



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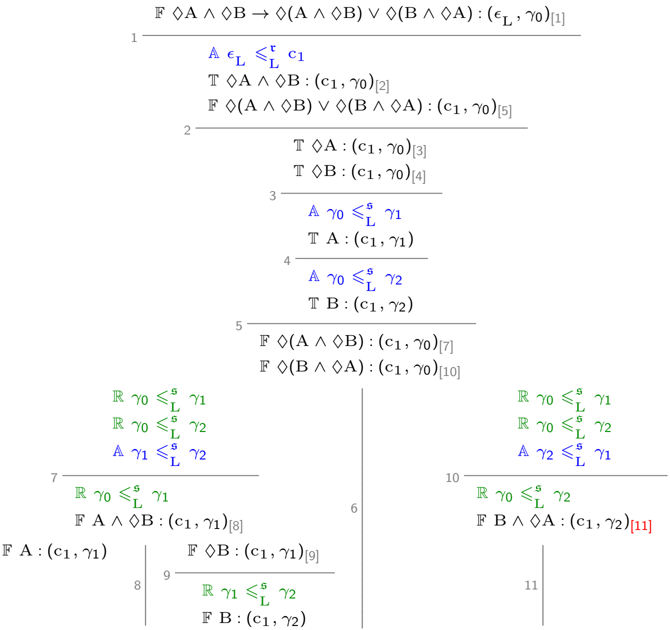
Example of Tableau Construction



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6

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$\mathbb{R}\ \gamma_0 \leqslant^{\mathfrak{s}}_{L} \gamma_2$

$\mathbb{A}\ \gamma_2 \leqslant^{\mathfrak{s}}_{L} \gamma_1$

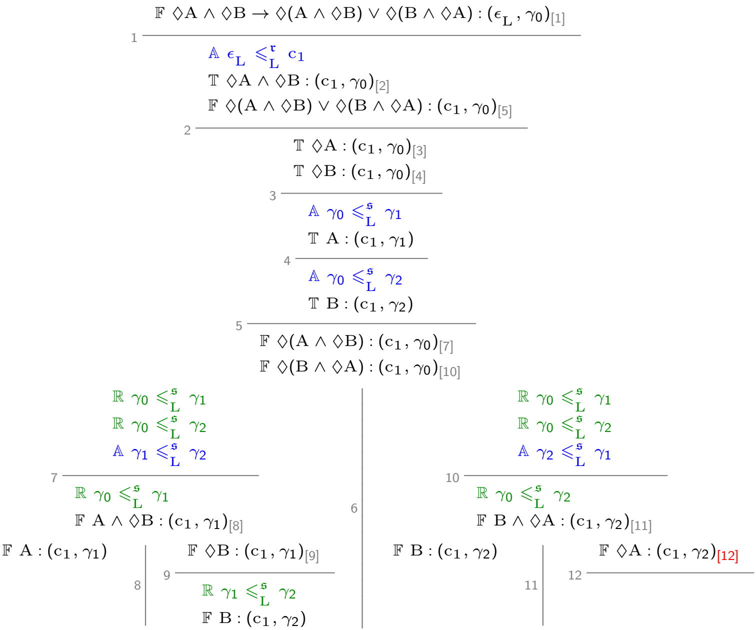
10

$\mathbb{R}\ \gamma_0 \leqslant^{\mathfrak{s}}_{L} \gamma_2$

$\mathbb{F}\ B \wedge \Diamond A : (c_1, \gamma_2)_{[11]}$

11

Example of Tableau Construction



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$\mathbb{F}\ A : (c_1, \gamma_1)$

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$\mathbb{R}\ \gamma_0 \leqslant^{\mathfrak{s}}_{L} \gamma_2$

$\mathbb{A}\ \gamma_2 \leqslant^{\mathfrak{s}}_{L} \gamma_1$

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$\mathbb{F}\ B \wedge \Diamond A : (c_1, \gamma_2)_{[11]}$

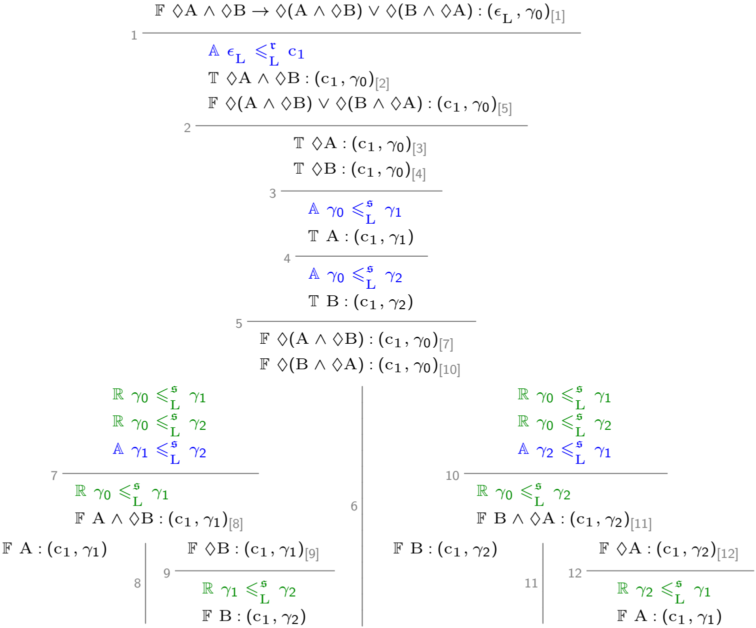
$\mathbb{F}\ B : (c_1, \gamma_2)$

11

$\mathbb{F}\ \Diamond A : (c_1, \gamma_2)_{[12]}$

12

Example of Tableau Construction



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$\mathbb{F}\ B \wedge \Diamond A : (c_1, \gamma_2)_{[11]}$

11

$\mathbb{F}\ B : (c_1, \gamma_2)$

$\mathbb{F}\ \Diamond A : (c_1, \gamma_2)_{[12]}$

12

$\mathbb{R}\ \gamma_2 \leqslant^{\mathfrak{s}}_{L} \gamma_1$

$\mathbb{F}\ A : (c_1, \gamma_1)$

$\mathsf{T}_{\mathsf{LTBI}}$-Tableau Proof

Constrained Temporal Set of Statements (CTSS)

- triple $\langle \mathcal{F}, C_r, C_s \rangle$ where $\mathcal{F}$ set of labelled formulas and C_r, C_s sets of resource and state constraints satisfying
- $\forall\ \mathbb{S}\ \mathrm{A} : (\mathrm{x}, \tau) \in \mathcal{F}.\ \ \mathrm{x} \leqslant^{\mathfrak{r}}_{\mathrm{L}} \mathrm{x} \in C_r$ and $\tau \leqslant^{\mathfrak{s}}_{\mathrm{L}} \tau \in C_s$

Inconsistent Label

Let $\langle \mathcal{F}, C_r, C_s \rangle$ be a CTSS

Label (x, τ) **inconsistent** if there exist two resource labels y and z such that $\mathrm{y} \circ \mathrm{z} \leqslant^{\mathfrak{r}}_{\mathrm{L}} \mathrm{x} \in C_r^{\bullet}$ and $\mathbb{T} \perp : (\mathrm{y}, \tau) \in \mathcal{F}$

$\mathsf{T}_{\mathsf{LTBI}}$-Tableau Proof

CTSS $\langle \mathcal{F}, C_r, C_s \rangle$ **closed** if one of these conditions hold

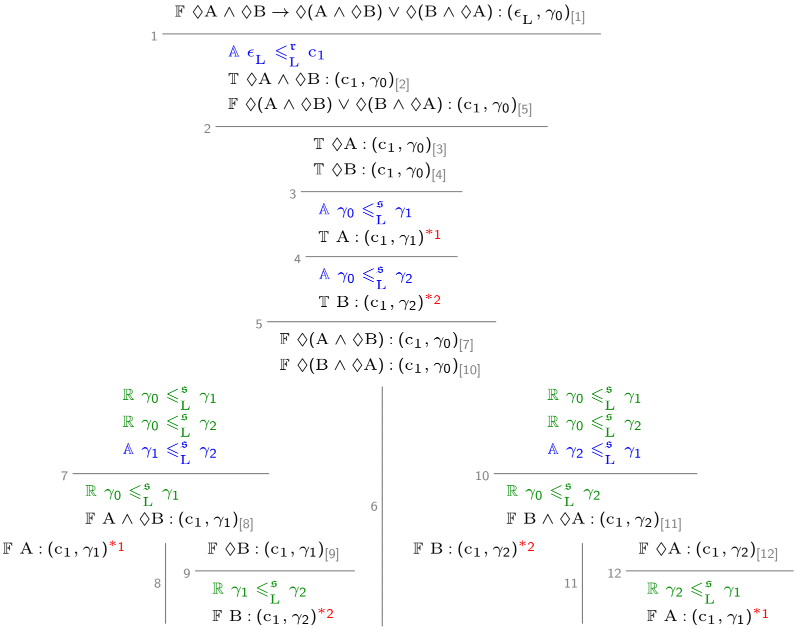
1. $\mathbb{T}\ \mathrm{A} : (\mathrm{x}, \tau), \mathbb{F}\ \mathrm{A} : (\mathrm{y}, \upsilon) \in \mathcal{F}$, $\mathrm{x} \leqslant^{\mathfrak{r}}_{\mathrm{L}} \mathrm{y} \in C_r^{\bullet}$ and $\tau =^{\mathfrak{s}}_{\mathrm{L}} \upsilon \in C_s^{\bullet}$
2. $\mathbb{F}\ \mathrm{I} : (\mathrm{x}, \tau) \in \mathcal{F}$ and $\epsilon_{\mathrm{L}} \leqslant^{\mathfrak{r}}_{\mathrm{L}} \mathrm{x} \in C_r^{\bullet}$
3. $\mathbb{F}\ \top : (\mathrm{x}, \tau) \in \mathcal{F}$
4. $\mathbb{F}\ \mathrm{A} : (\mathrm{x}, \tau) \in \mathcal{F}$ and (x, τ) is inconsistent
5. $\tau =^{\mathfrak{s}}_{\mathrm{L}} \upsilon \in C_s^{\bullet}$ and $\tau \neq^{\mathfrak{s}}_{\mathrm{L}} \upsilon \in C_s^{\bullet}$

Branch is **closed** if its corresponding CTSS closed

Tableau **closed** when all branches closed

$\mathsf{T}_{\mathsf{LTBI}}$-**proof for** A: closed $\mathsf{T}_{\mathsf{LTBI}}$ tableau for A

Example of a Closed Tableau



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1

$\mathbb{A}\ \epsilon_L \leqslant^{\mathfrak{r}}_{L} c_1$

$\mathbb{T}\ \Diamond A \wedge \Diamond B : (c_1, \gamma_0)_{[2]}$

$\mathbb{F}\ \Diamond(A \wedge \Diamond B) \vee \Diamond(B \wedge \Diamond A) : (c_1, \gamma_0)_{[5]}$

2

$\mathbb{T}\ \Diamond A : (c_1, \gamma_0)_{[3]}$

$\mathbb{T}\ \Diamond B : (c_1, \gamma_0)_{[4]}$

3

$\mathbb{A}\ \gamma_0 \leqslant^{\mathfrak{s}}_{L} \gamma_1$

$\mathbb{T}\ A : (c_1, \gamma_1)^{*1}$

4

$\mathbb{A}\ \gamma_0 \leqslant^{\mathfrak{s}}_{L} \gamma_2$

$\mathbb{T}\ B : (c_1, \gamma_2)^{*2}$

5

$\mathbb{F}\ \Diamond(A \wedge \Diamond B) : (c_1, \gamma_0)_{[7]}$

$\mathbb{F}\ \Diamond(B \wedge \Diamond A) : (c_1, \gamma_0)_{[10]}$

6

$\mathbb{R}\ \gamma_0 \leqslant^{\mathfrak{s}}_{L} \gamma_1$

$\mathbb{R}\ \gamma_0 \leqslant^{\mathfrak{s}}_{L} \gamma_2$

$\mathbb{A}\ \gamma_1 \leqslant^{\mathfrak{s}}_{L} \gamma_2$

7

$\mathbb{R}\ \gamma_0 \leqslant^{\mathfrak{s}}_{L} \gamma_1$

$\mathbb{F}\ A \wedge \Diamond B : (c_1, \gamma_1)_{[8]}$

8

$\mathbb{F}\ A : (c_1, \gamma_1)^{*1}$

$\mathbb{F}\ \Diamond B : (c_1, \gamma_1)_{[9]}$

9

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$\mathbb{F}\ B : (c_1, \gamma_2)^{*2}$

$\mathbb{R}\ \gamma_0 \leqslant^{\mathfrak{s}}_{L} \gamma_1$

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$\mathbb{A}\ \gamma_2 \leqslant^{\mathfrak{s}}_{L} \gamma_1$

10

$\mathbb{R}\ \gamma_0 \leqslant^{\mathfrak{s}}_{L} \gamma_2$

$\mathbb{F}\ B \wedge \Diamond A : (c_1, \gamma_2)_{[11]}$

11

$\mathbb{F}\ B : (c_1, \gamma_2)^{*2}$

$\mathbb{F}\ \Diamond A : (c_1, \gamma_2)_{[12]}$

12

$\mathbb{R}\ \gamma_2 \leqslant^{\mathfrak{s}}_{L} \gamma_1$

$\mathbb{F}\ A : (c_1, \gamma_1)^{*1}$

Overview of the Talk

Realizability

Realization of a CTSS $\langle \mathcal{F}, C_r, C_s \rangle$: triple $(\mathcal{M}, [.]_r, [.]_s)$

- $\mathcal{M}$ is an LTBI-model $(\mathbf{R}, \star, \epsilon, \leqslant^{\mathfrak{r}}, \pi, \mathbf{S}, \leqslant^{\mathfrak{s}}, s_0)$
- $[.]_r : D_r(C_r^\bullet) \to \mathbf{R}$ is a $\leqslant^{\mathfrak{r}}$-homomorphism s.t. $[\epsilon_{\mathrm{L}}]_r = \epsilon$
- $[.]_s : D_s(C_s^\bullet) \to \mathbf{S}$ is a $\leqslant^{\mathfrak{s}}$-preserving morphism s.t. $[\eta\tau]_s = \mathfrak{n}[\tau]_s$
- if $\mathbb{T}\ \mathrm{A} : (\mathrm{x}, \tau) \in \mathcal{F}$, then $([\mathrm{x}]_r, [\tau]_s) \Vdash \mathrm{A}$
- if $\mathbb{F}\ \mathrm{A} : (\mathrm{x}, \tau) \in \mathcal{F}$, then $([\mathrm{x}]_r, [\tau]_s) \nVdash \mathrm{A}$

Realizable branch: when associated **CTSS has a realization**

Realizable tableau: contains at least **one realizable branch**

Soundness Proof

Lemma

Let $(\mathcal{M}, [.]_r, [.]_s)$ be a realization of a CTSS $\langle \mathcal{F}, C_r, C_s \rangle$

- for all $\mathrm{x} \leqslant^{\mathfrak{r}}_{\mathrm{L}} \mathrm{y} \in C_r^{\bullet}$, $[\mathrm{x}]_r \leqslant^{\mathfrak{r}} [\mathrm{y}]_r$
- for all $\tau \, \mathrm{R}^{\mathfrak{s}}_{\mathrm{L}} \, \upsilon \in C_s^{\bullet}$, $[\tau]_s \, \mathrm{R}^{\mathfrak{s}} \, [\upsilon]_s$

Lemma

If a $\mathsf{T}_{\mathsf{LTBI}}$ tableau is closed then it is not realizable

Lemma

All $\mathsf{T}_{\mathsf{LTBI}}$ rules preserve realizability

Theorem

If there exists a $\mathsf{T}_{\mathsf{LTBI}}$ proof for A, then A is valid

Overview of the Talk

- Introduction
- Linear Time Bunched Implication Logic
- Expressivity of LTBI
- Tableau Calculus
- Soundness
- **Counter-Model Construction**
- Completeness Issues
- Conclusion & Future Work

Saturation Procedure

Completed branch: branch $\mathcal{B}$ **completed** iff

- all of its labelled formulas are **fulfilled** and
- all possible expansions of CD and LR have been applied

Let $\langle \mathcal{F}, C_s, C_r \rangle$ be the CTSS associated $\mathcal{B}$

Fulfillment: $\mathbb{S}\ \mathrm{C} : (\mathrm{x}, \tau)$ **fulfilled**, denoted $\mathcal{B} \models \mathbb{S}\ \mathrm{C} : (\mathrm{x}, \tau)$, iff

$\mathcal{B}$ contains **all possible expansions** of $\mathbb{S}\ \mathrm{C} : (\mathrm{x}, \tau)$ in $\mathcal{B}$

Saturation:

while $\mathcal{B}$ not closed and expandable
expand formulas with a fair strategy

Counter-Model Construction

Let $\mathcal{B}$ be a open and completed branch

Resource composition: defined on $D_r(C_r^\bullet) \cup \{\,\pi\,\}$ as

$$\begin{cases} \mathrm{x} \star \mathrm{y} = \mathrm{xy} \text{ if } \mathrm{xy} \in D_r(C_r^\bullet) \\ \mathrm{x} \star \epsilon_{\mathrm{L}} = \mathrm{x} \\ \mathrm{x} \star \pi = \pi \end{cases}$$

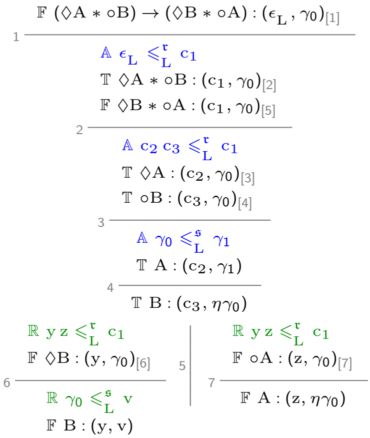
Resource ordering: induced by closure of resource assertions

$$\leqslant^{\mathfrak{r}} = \; C_r^\bullet \cup \{\, \mathrm{x} \leqslant \pi \mid \mathrm{x} \in D_r(C_r^\bullet)\,\}$$

Timeline: induced by closure of state assertions

Forcing relation: induced by formulas with sign $\mathbb{T}$

Example of a Non-closed Tableaux



$$\mathbb{F}\ (\Diamond A * \circ B) \to (\Diamond B * \circ A) : (\epsilon_L, \gamma_0)_{[1]}$$

1

$$\mathbb{A}\ \epsilon_L \leqslant^{\mathfrak{r}}_{L} c_1$$
$$\mathbb{T}\ \Diamond A * \circ B : (c_1, \gamma_0)_{[2]}$$
$$\mathbb{F}\ \Diamond B * \circ A : (c_1, \gamma_0)_{[5]}$$

2

$$\mathbb{A}\ c_2\, c_3 \leqslant^{\mathfrak{r}}_{L} c_1$$
$$\mathbb{T}\ \Diamond A : (c_2, \gamma_0)_{[3]}$$
$$\mathbb{T}\ \circ B : (c_3, \gamma_0)_{[4]}$$

3

$$\mathbb{A}\ \gamma_0 \leqslant^{\mathfrak{s}}_{L} \gamma_1$$
$$\mathbb{T}\ A : (c_2, \gamma_1)$$

4

$$\mathbb{T}\ B : (c_3, \eta\gamma_0)$$

5

$$\mathbb{R}\ y\, z \leqslant^{\mathfrak{r}}_{L} c_1$$
$$\mathbb{F}\ \Diamond B : (y, \gamma_0)_{[6]}$$

6

$$\mathbb{R}\ \gamma_0 \leqslant^{\mathfrak{s}}_{L} v$$
$$\mathbb{F}\ B : (y, v)$$

$$\mathbb{R}\ y\, z \leqslant^{\mathfrak{r}}_{L} c_1$$
$$\mathbb{F}\ \circ A : (z, \gamma_0)_{[7]}$$

7

$$\mathbb{F}\ A : (z, \eta\gamma_0)$$

The requirement $\mathbb{R}\ y\, z \leqslant^{\mathfrak{r}}_{L} c_1$ can only be satisfied in two cases:

(1) $y = c_3,\ z = c_2$ and (2) $y = c_2,\ z = c_3$

Right branch cannot be closed in both cases

Counter-Model Example

Resources: $\{\, \epsilon_{\mathrm{L}} \leqslant^{\mathfrak{r}}_{\mathrm{L}} \mathrm{c}_1, \mathrm{c}_2\mathrm{c}_3 \leqslant^{\mathfrak{r}}_{\mathrm{L}} \mathrm{c}_1 \,\}^{\bullet} + \pi$ greatest element

Timeline: $[\gamma_0]_s = 0 \quad [\eta\gamma_0]_s = 1 \quad [\gamma_1]_s = 2$

Forcing:
$$\begin{cases} [\mathrm{A}] = \{\, (\pi, 0), (\pi, 1), (\pi, 2), (\mathrm{c}_2, 2) \,\} \\ [\mathrm{B}] = \{\, (\pi, 0), (\pi, 1), (\pi, 2), (\mathrm{c}_3, 2) \,\} \end{cases}$$

1. $(\mathrm{c}_2, 2) \Vdash \mathrm{A} \Longrightarrow (\mathrm{c}_2, 0) \Vdash \Diamond\mathrm{A}$
2. $(\mathrm{c}_3, 1) \Vdash \mathrm{B} \Longrightarrow (\mathrm{c}_3, 0) \Vdash \circ\mathrm{B}$
3. $1 \,\&\, 2 \Longrightarrow (\mathrm{c}_2\mathrm{c}_3, 0) \Vdash \Diamond\mathrm{A} * \circ\mathrm{B} \Longrightarrow (\mathrm{c}_1, 0) \Vdash \Diamond\mathrm{A} * \circ\mathrm{B}$
4. A only true at $(\mathrm{c}_2, 2)$ & no state 3
   $\Longrightarrow \forall(\mathrm{x}, \tau).\, (\mathrm{x}, \tau) \not\Vdash \circ\mathrm{A} \Longrightarrow (\mathrm{c}_1, 0) \not\Vdash \Diamond\mathrm{B} * \circ\mathrm{A}$

Overview of the Talk

The Dense Timeline

$$1\;\frac{\mathbb{F}\ ((\Diamond \mathrm{A} \to \mathrm{B}) \to \mathrm{C}) \to \mathrm{C} : (\epsilon_{\mathrm{L}}\,, \gamma_0)_{[1]}}{}$$

The Dense Timeline

$$\mathbb{F}\ ((\Diamond A \to B) \to C) \to C : (\epsilon_L, \gamma_0)_{[1]}$$

1

$$\mathbb{A}\ \epsilon_L \leqslant^{\mathfrak{r}}_{L} c_1$$

$$\mathbb{T}\ (\Diamond A \to B) \to C : (c_1, \gamma_0)_{[2\quad]}$$

$$\mathbb{F}\ C : (c_1, \gamma_0)^{*1}$$

2

The Dense Timeline

$\mathbb{F}\ ((\Diamond A \to B) \to C) \to C : (\epsilon_L, \gamma_0)_{[1]}$

1

$\mathbb{A}\ \epsilon_L \leqslant^{\mathfrak{r}}_L c_1$

$\mathbb{T}\ (\Diamond A \to B) \to C : (c_1, \gamma_0)_{[2\quad]}$

$\mathbb{F}\ C : (c_1, \gamma_0)^{*1}$

$\mathbb{R}\ c_1 \leqslant^{\mathfrak{r}}_L c_1$

$\mathbb{F}\ \Diamond A \to B : (c_1, \gamma_0)_{[3]}$

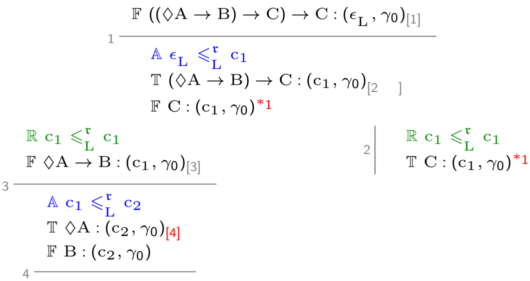
3

2

$\mathbb{R}\ c_1 \leqslant^{\mathfrak{r}}_L c_1$

$\mathbb{T}\ C : (c_1, \gamma_0)^{*1}$

The Dense Timeline



$\mathbb{F}\ ((\Diamond A \to B) \to C) \to C : (\epsilon_L, \gamma_0)_{[1]}$

1

$\mathbb{A}\ \epsilon_L \leqslant^{\mathfrak{r}}_L c_1$

$\mathbb{T}\ (\Diamond A \to B) \to C : (c_1, \gamma_0)_{[2\quad]}$

$\mathbb{F}\ C : (c_1, \gamma_0)^{*1}$

$\mathbb{R}\ c_1 \leqslant^{\mathfrak{r}}_L c_1$

$\mathbb{F}\ \Diamond A \to B : (c_1, \gamma_0)_{[3]}$

3

$\mathbb{A}\ c_1 \leqslant^{\mathfrak{r}}_L c_2$

$\mathbb{T}\ \Diamond A : (c_2, \gamma_0)_{[4]}$

$\mathbb{F}\ B : (c_2, \gamma_0)$

4

2

$\mathbb{R}\ c_1 \leqslant^{\mathfrak{r}}_L c_1$

$\mathbb{T}\ C : (c_1, \gamma_0)^{*1}$

The Dense Timeline

$\mathbb{F}\ ((\Diamond A \to B) \to C) \to C : (\epsilon_L, \gamma_0)_{[1]}$

1

$\mathbb{A}\ \epsilon_L \leqslant^{\mathfrak{r}}_L c_1$
$\mathbb{T}\ (\Diamond A \to B) \to C : (c_1, \gamma_0)_{[2,2']}$
$\mathbb{F}\ C : (c_1, \gamma_0)^{*1}$

$\mathbb{R}\ c_1 \leqslant^{\mathfrak{r}}_L c_1$
$\mathbb{F}\ \Diamond A \to B : (c_1, \gamma_0)_{[3]}$

3

$\mathbb{A}\ c_1 \leqslant^{\mathfrak{r}}_L c_2$
$\mathbb{T}\ \Diamond A : (c_2, \gamma_0)_{[4]}$
$\mathbb{F}\ B : (c_2, \gamma_0)$

4

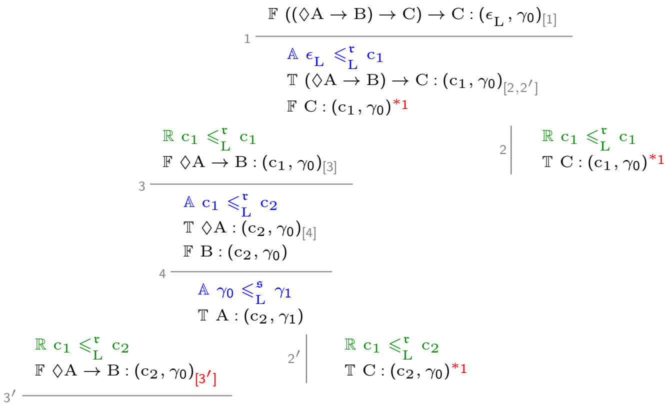
$\mathbb{A}\ \gamma_0 \leqslant^{\mathfrak{s}}_L \gamma_1$
$\mathbb{T}\ A : (c_2, \gamma_1)$

2′

2

$\mathbb{R}\ c_1 \leqslant^{\mathfrak{r}}_L c_1$
$\mathbb{T}\ C : (c_1, \gamma_0)^{*1}$

The Dense Timeline



$\mathbb{F}\ ((\Diamond A \to B) \to C) \to C : (\epsilon_L , \gamma_0)_{[1]}$

1

$\mathbb{A}\ \epsilon_L \leqslant^{\mathfrak{r}}_L c_1$

$\mathbb{T}\ (\Diamond A \to B) \to C : (c_1 , \gamma_0)_{[2,2']}$

$\mathbb{F}\ C : (c_1 , \gamma_0)^{*1}$

$\mathbb{R}\ c_1 \leqslant^{\mathfrak{r}}_L c_1$

$\mathbb{F}\ \Diamond A \to B : (c_1 , \gamma_0)_{[3]}$

2

$\mathbb{R}\ c_1 \leqslant^{\mathfrak{r}}_L c_1$

$\mathbb{T}\ C : (c_1 , \gamma_0)^{*1}$

3

$\mathbb{A}\ c_1 \leqslant^{\mathfrak{r}}_L c_2$

$\mathbb{T}\ \Diamond A : (c_2 , \gamma_0)_{[4]}$

$\mathbb{F}\ B : (c_2 , \gamma_0)$

4

$\mathbb{A}\ \gamma_0 \leqslant^{\mathfrak{s}}_L \gamma_1$

$\mathbb{T}\ A : (c_2 , \gamma_1)$

$\mathbb{R}\ c_1 \leqslant^{\mathfrak{r}}_L c_2$

$\mathbb{F}\ \Diamond A \to B : (c_2 , \gamma_0)_{[3']}$

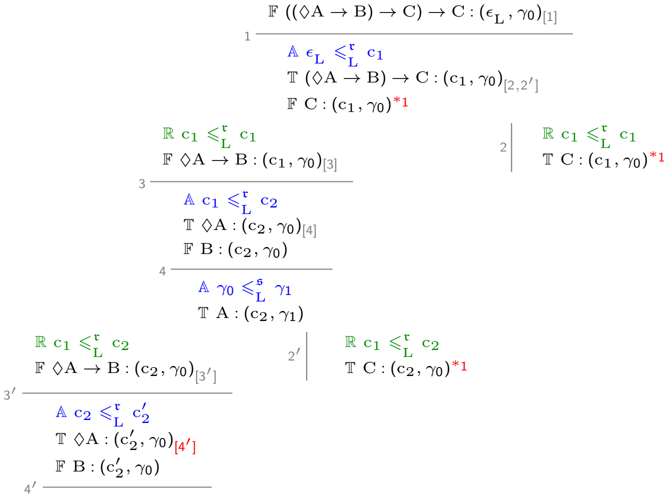
3′

2′

$\mathbb{R}\ c_1 \leqslant^{\mathfrak{r}}_L c_2$

$\mathbb{T}\ C : (c_2 , \gamma_0)^{*1}$

The Dense Timeline



$\mathbb{F}\ ((\Diamond A \to B) \to C) \to C : (\epsilon_L, \gamma_0)_{[1]}$

1

$\mathbb{A}\ \epsilon_L \leqslant^{\mathfrak{r}}_L c_1$

$\mathbb{T}\ (\Diamond A \to B) \to C : (c_1, \gamma_0)_{[2,2']}$

$\mathbb{F}\ C : (c_1, \gamma_0)^{*1}$

2

$\mathbb{R}\ c_1 \leqslant^{\mathfrak{r}}_L c_1$

$\mathbb{F}\ \Diamond A \to B : (c_1, \gamma_0)_{[3]}$

3

$\mathbb{A}\ c_1 \leqslant^{\mathfrak{r}}_L c_2$

$\mathbb{T}\ \Diamond A : (c_2, \gamma_0)_{[4]}$

$\mathbb{F}\ B : (c_2, \gamma_0)$

4

$\mathbb{A}\ \gamma_0 \leqslant^{\mathfrak{s}}_L \gamma_1$

$\mathbb{T}\ A : (c_2, \gamma_1)$

2′

$\mathbb{R}\ c_1 \leqslant^{\mathfrak{r}}_L c_2$

$\mathbb{F}\ \Diamond A \to B : (c_2, \gamma_0)_{[3']}$

3′

$\mathbb{A}\ c_2 \leqslant^{\mathfrak{r}}_L c_2'$

$\mathbb{T}\ \Diamond A : (c_2', \gamma_0)_{[4']}$

$\mathbb{F}\ B : (c_2', \gamma_0)$

4′

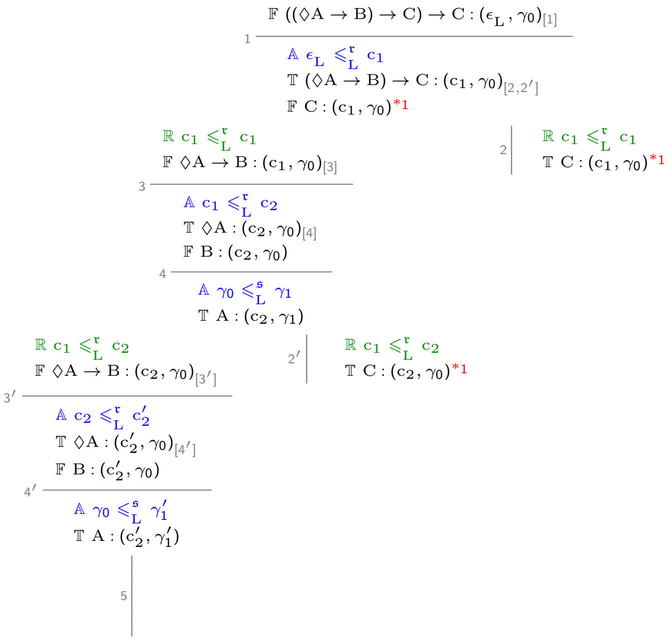
$\mathbb{R}\ c_1 \leqslant^{\mathfrak{r}}_L c_2$

$\mathbb{T}\ C : (c_2, \gamma_0)^{*1}$

$\mathbb{R}\ c_1 \leqslant^{\mathfrak{r}}_L c_1$

$\mathbb{T}\ C : (c_1, \gamma_0)^{*1}$

The Dense Timeline



$\mathbb{F}\ ((\Diamond A \to B) \to C) \to C : (\epsilon_L, \gamma_0)_{[1]}$

1

$\mathbb{A}\ \epsilon_L \leqslant^{\mathfrak{r}}_L c_1$

$\mathbb{T}\ (\Diamond A \to B) \to C : (c_1, \gamma_0)_{[2,2']}$

$\mathbb{F}\ C : (c_1, \gamma_0)^{*1}$

2

$\mathbb{R}\ c_1 \leqslant^{\mathfrak{r}}_L c_1$

$\mathbb{F}\ \Diamond A \to B : (c_1, \gamma_0)_{[3]}$

$\mathbb{R}\ c_1 \leqslant^{\mathfrak{r}}_L c_1$

$\mathbb{T}\ C : (c_1, \gamma_0)^{*1}$

3

$\mathbb{A}\ c_1 \leqslant^{\mathfrak{r}}_L c_2$

$\mathbb{T}\ \Diamond A : (c_2, \gamma_0)_{[4]}$

$\mathbb{F}\ B : (c_2, \gamma_0)$

4

$\mathbb{A}\ \gamma_0 \leqslant^{\mathfrak{s}}_L \gamma_1$

$\mathbb{T}\ A : (c_2, \gamma_1)$

2′

$\mathbb{R}\ c_1 \leqslant^{\mathfrak{r}}_L c_2$

$\mathbb{F}\ \Diamond A \to B : (c_2, \gamma_0)_{[3']}$

$\mathbb{R}\ c_1 \leqslant^{\mathfrak{r}}_L c_2$

$\mathbb{T}\ C : (c_2, \gamma_0)^{*1}$

3′

$\mathbb{A}\ c_2 \leqslant^{\mathfrak{r}}_L c_2'$

$\mathbb{T}\ \Diamond A : (c_2', \gamma_0)_{[4']}$

$\mathbb{F}\ B : (c_2', \gamma_0)$

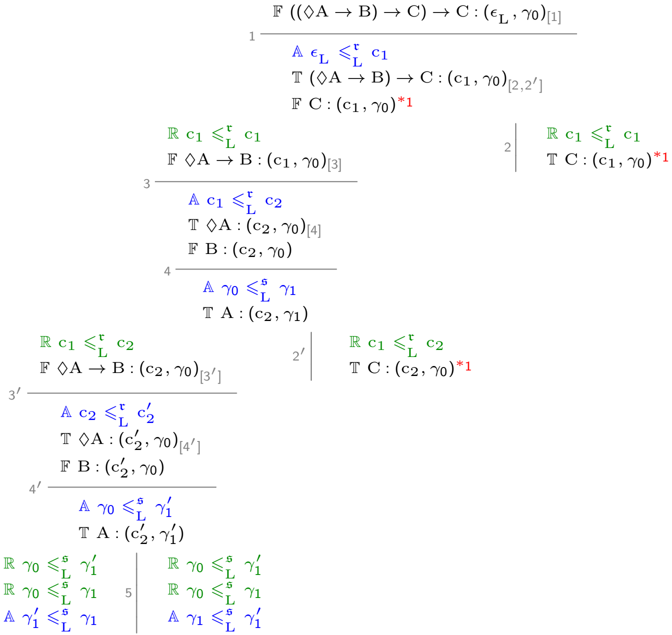
4′

$\mathbb{A}\ \gamma_0 \leqslant^{\mathfrak{s}}_L \gamma_1'$

$\mathbb{T}\ A : (c_2', \gamma_1')$

5

The Dense Timeline



$\mathbb{F}\ ((\Diamond A \to B) \to C) \to C : (\epsilon_L , \gamma_0)_{[1]}$

1

$\mathbb{A}\ \epsilon_L \leqslant^{\mathfrak{r}}_L c_1$

$\mathbb{T}\ (\Diamond A \to B) \to C : (c_1 , \gamma_0)_{[2,2']}$

$\mathbb{F}\ C : (c_1 , \gamma_0)^{*1}$

$\mathbb{R}\ c_1 \leqslant^{\mathfrak{r}}_L c_1$

$\mathbb{F}\ \Diamond A \to B : (c_1 , \gamma_0)_{[3]}$

3

$\mathbb{A}\ c_1 \leqslant^{\mathfrak{r}}_L c_2$

$\mathbb{T}\ \Diamond A : (c_2 , \gamma_0)_{[4]}$

$\mathbb{F}\ B : (c_2 , \gamma_0)$

4

$\mathbb{A}\ \gamma_0 \leqslant^{\mathfrak{s}}_L \gamma_1$

$\mathbb{T}\ A : (c_2 , \gamma_1)$

$\mathbb{R}\ c_1 \leqslant^{\mathfrak{r}}_L c_2$

$\mathbb{F}\ \Diamond A \to B : (c_2 , \gamma_0)_{[3']}$

3′

$\mathbb{A}\ c_2 \leqslant^{\mathfrak{r}}_L c_2'$

$\mathbb{T}\ \Diamond A : (c_2' , \gamma_0)_{[4']}$

$\mathbb{F}\ B : (c_2' , \gamma_0)$

4′

$\mathbb{A}\ \gamma_0 \leqslant^{\mathfrak{s}}_L \gamma_1'$

$\mathbb{T}\ A : (c_2' , \gamma_1')$

$\mathbb{R}\ \gamma_0 \leqslant^{\mathfrak{s}}_L \gamma_1'$

$\mathbb{R}\ \gamma_0 \leqslant^{\mathfrak{s}}_L \gamma_1$

$\mathbb{A}\ \gamma_1' \leqslant^{\mathfrak{s}}_L \gamma_1$

5

$\mathbb{R}\ \gamma_0 \leqslant^{\mathfrak{s}}_L \gamma_1'$

$\mathbb{R}\ \gamma_0 \leqslant^{\mathfrak{s}}_L \gamma_1$

$\mathbb{A}\ \gamma_1 \leqslant^{\mathfrak{s}}_L \gamma_1'$

2′

$\mathbb{R}\ c_1 \leqslant^{\mathfrak{r}}_L c_2$

$\mathbb{T}\ C : (c_2 , \gamma_0)^{*1}$

2

$\mathbb{R}\ c_1 \leqslant^{\mathfrak{r}}_L c_1$

$\mathbb{T}\ C : (c_1 , \gamma_0)^{*1}$

The Dense Timeline

Problem: **interaction between resource and state labels**

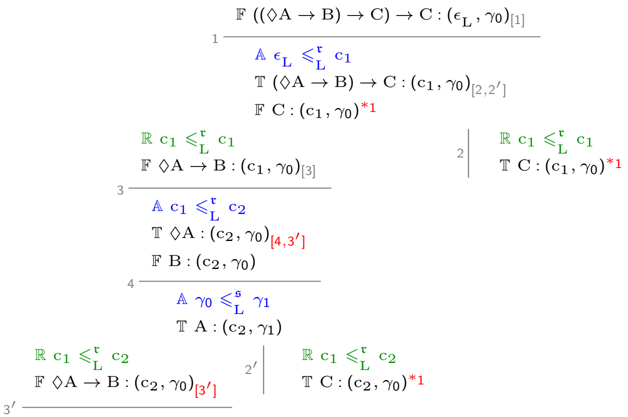
generating an infinite chain of state labels $\gamma_0 <^{\mathfrak{s}} \gamma_1^{i} <^{\mathfrak{s}} \gamma_1$
induced by an infinite chain of resource labels c_2^{i}

Solution: use **liberalized rules** as in BI [Galmiche & Mery, 2005]

- **Reuse constants** instead of generating fresh ones
- Allow $\mathbb{F}\ \mathrm{A} \to \mathrm{B} : (\mathrm{x}, \tau)$ to expand to $\mathbb{T}\ \mathrm{A} : (\mathrm{x}, \tau), \mathbb{F}\ \mathrm{B} : (\mathrm{x}, \tau)$ whenever the branch already contains $\mathbb{T}\ \mathrm{A} : (\mathrm{y}, \tau)$ and the requirement $\mathbb{R}\ \mathrm{y} \leqslant^{\mathfrak{r}}_{\mathrm{L}} \mathrm{x}$ holds

$\Longrightarrow$ leftmost branch completed after Step [3′]

The Dense Timeline (Revisited)



$\mathbb{F}\ ((\Diamond A \to B) \to C) \to C : (\epsilon_L, \gamma_0)_{[1]}$

1

$\mathbb{A}\ \epsilon_L \leqslant^{\mathfrak{r}}_L c_1$

$\mathbb{T}\ (\Diamond A \to B) \to C : (c_1, \gamma_0)_{[2,2']}$

$\mathbb{F}\ C : (c_1, \gamma_0)^{*1}$

$\mathbb{R}\ c_1 \leqslant^{\mathfrak{r}}_L c_1$

$\mathbb{F}\ \Diamond A \to B : (c_1, \gamma_0)_{[3]}$

3

$\mathbb{A}\ c_1 \leqslant^{\mathfrak{r}}_L c_2$

$\mathbb{T}\ \Diamond A : (c_2, \gamma_0)_{[4,3']}$

$\mathbb{F}\ B : (c_2, \gamma_0)$

4

$\mathbb{A}\ \gamma_0 \leqslant^{\mathfrak{s}}_L \gamma_1$

$\mathbb{T}\ A : (c_2, \gamma_1)$

$\mathbb{R}\ c_1 \leqslant^{\mathfrak{r}}_L c_2$

$\mathbb{F}\ \Diamond A \to B : (c_2, \gamma_0)_{[3']}$

3′

2′

$\mathbb{R}\ c_1 \leqslant^{\mathfrak{r}}_L c_2$

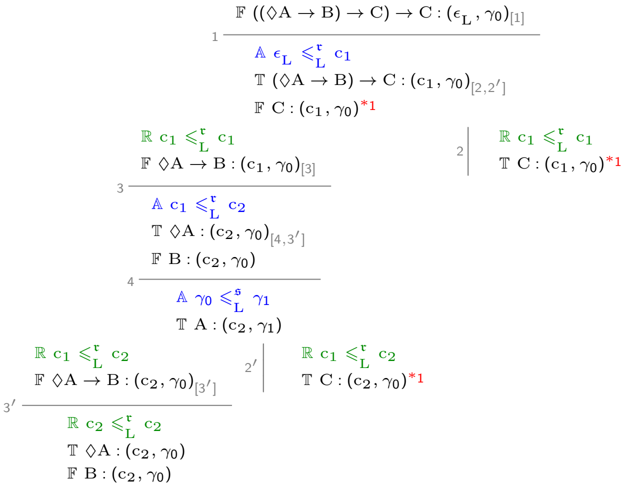
$\mathbb{T}\ C : (c_2, \gamma_0)^{*1}$

2

$\mathbb{R}\ c_1 \leqslant^{\mathfrak{r}}_L c_1$

$\mathbb{T}\ C : (c_1, \gamma_0)^{*1}$

The Dense Timeline (Revisited)



$\mathbb{F}\ ((\Diamond A \to B) \to C) \to C : (\epsilon_L , \gamma_0)_{[1]}$

1

$\mathbb{A}\ \epsilon_L \leqslant^{\mathfrak{r}}_L c_1$

$\mathbb{T}\ (\Diamond A \to B) \to C : (c_1 , \gamma_0)_{[2,2']}$

$\mathbb{F}\ C : (c_1 , \gamma_0)^{*1}$

$\mathbb{R}\ c_1 \leqslant^{\mathfrak{r}}_L c_1$

$\mathbb{F}\ \Diamond A \to B : (c_1 , \gamma_0)_{[3]}$

3

$\mathbb{A}\ c_1 \leqslant^{\mathfrak{r}}_L c_2$

$\mathbb{T}\ \Diamond A : (c_2 , \gamma_0)_{[4,3']}$

$\mathbb{F}\ B : (c_2 , \gamma_0)$

4

$\mathbb{A}\ \gamma_0 \leqslant^{\mathfrak{s}}_L \gamma_1$

$\mathbb{T}\ A : (c_2 , \gamma_1)$

$\mathbb{R}\ c_1 \leqslant^{\mathfrak{r}}_L c_2$

$\mathbb{F}\ \Diamond A \to B : (c_2 , \gamma_0)_{[3']}$

3′

$\mathbb{R}\ c_2 \leqslant^{\mathfrak{r}}_L c_2$

$\mathbb{T}\ \Diamond A : (c_2 , \gamma_0)$

$\mathbb{F}\ B : (c_2 , \gamma_0)$

2′

$\mathbb{R}\ c_1 \leqslant^{\mathfrak{r}}_L c_2$

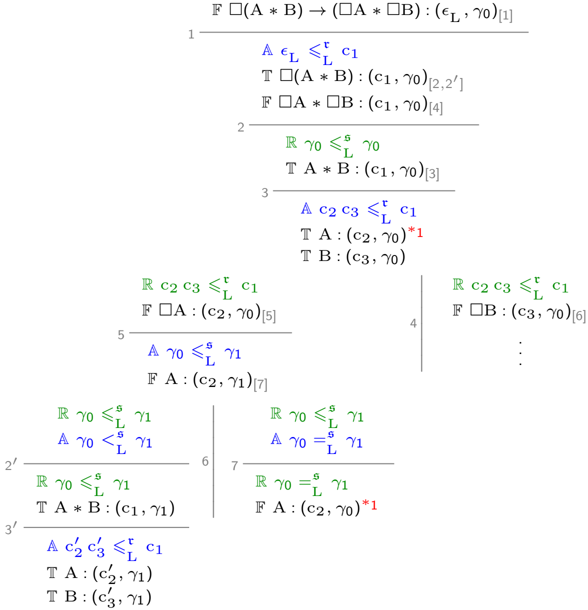
$\mathbb{T}\ C : (c_2 , \gamma_0)^{*1}$

2

$\mathbb{R}\ c_1 \leqslant^{\mathfrak{r}}_L c_1$

$\mathbb{T}\ C : (c_1 , \gamma_0)^{*1}$

Unsoundness of the Liberalized Rules



$\mathbb{F}\ \Box(A * B) \to (\Box A * \Box B) : (\epsilon_L, \gamma_0)_{[1]}$

1

$\mathbb{A}\ \epsilon_L \leqslant^{\mathfrak{r}}_{L} c_1$

$\mathbb{T}\ \Box(A * B) : (c_1, \gamma_0)_{[2,2']}$

$\mathbb{F}\ \Box A * \Box B : (c_1, \gamma_0)_{[4]}$

2

$\mathbb{R}\ \gamma_0 \leqslant^{\mathfrak{s}}_{L} \gamma_0$

$\mathbb{T}\ A * B : (c_1, \gamma_0)_{[3]}$

3

$\mathbb{A}\ c_2\ c_3 \leqslant^{\mathfrak{r}}_{L} c_1$

$\mathbb{T}\ A : (c_2, \gamma_0)^{*1}$

$\mathbb{T}\ B : (c_3, \gamma_0)$

4

$\mathbb{R}\ c_2\ c_3 \leqslant^{\mathfrak{r}}_{L} c_1$

$\mathbb{F}\ \Box A : (c_2, \gamma_0)_{[5]}$

5

$\mathbb{A}\ \gamma_0 \leqslant^{\mathfrak{s}}_{L} \gamma_1$

$\mathbb{F}\ A : (c_2, \gamma_1)_{[7]}$

6

$\mathbb{R}\ \gamma_0 \leqslant^{\mathfrak{s}}_{L} \gamma_1$

$\mathbb{A}\ \gamma_0 <^{\mathfrak{s}}_{L} \gamma_1$

2′

$\mathbb{R}\ \gamma_0 \leqslant^{\mathfrak{s}}_{L} \gamma_1$

$\mathbb{T}\ A * B : (c_1, \gamma_1)$

3′

$\mathbb{A}\ c_2'\ c_3' \leqslant^{\mathfrak{r}}_{L} c_1$

$\mathbb{T}\ A : (c_2', \gamma_1)$

$\mathbb{T}\ B : (c_3', \gamma_1)$

$\mathbb{R}\ \gamma_0 \leqslant^{\mathfrak{s}}_{L} \gamma_1$

$\mathbb{A}\ \gamma_0 =^{\mathfrak{s}}_{L} \gamma_1$

7

$\mathbb{R}\ \gamma_0 =^{\mathfrak{s}}_{L} \gamma_1$

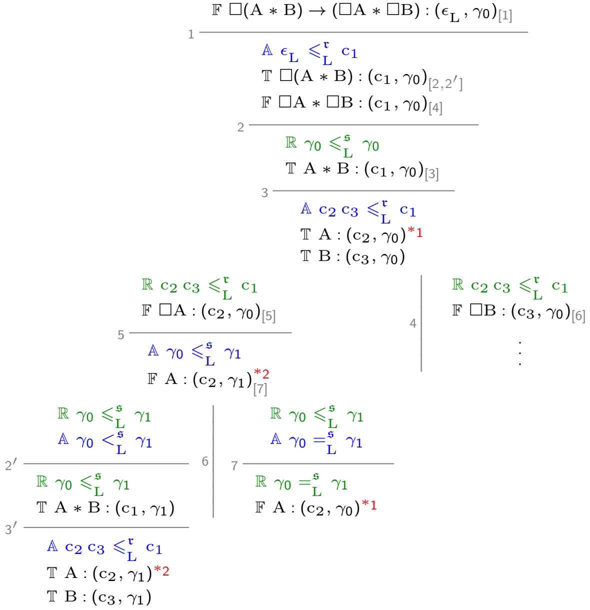
$\mathbb{F}\ A : (c_2, \gamma_0)^{*1}$

$\mathbb{R}\ c_2\ c_3 \leqslant^{\mathfrak{r}}_{L} c_1$

$\mathbb{F}\ \Box B : (c_3, \gamma_0)_{[6]}$

⋮

Unsoundness of the Liberalized Rules



$\mathbb{F}\ \Box(A * B) \rightarrow (\Box A * \Box B) : (\epsilon_L, \gamma_0)_{[1]}$

1

$\mathbb{A}\ \epsilon_L \leqslant^{\mathfrak{r}}_{L} c_1$

$\mathbb{T}\ \Box(A * B) : (c_1, \gamma_0)_{[2,2']}$

$\mathbb{F}\ \Box A * \Box B : (c_1, \gamma_0)_{[4]}$

2

$\mathbb{R}\ \gamma_0 \leqslant^{\mathfrak{s}}_{L} \gamma_0$

$\mathbb{T}\ A * B : (c_1, \gamma_0)_{[3]}$

3

$\mathbb{A}\ c_2\, c_3 \leqslant^{\mathfrak{r}}_{L} c_1$

$\mathbb{T}\ A : (c_2, \gamma_0)^{*1}$

$\mathbb{T}\ B : (c_3, \gamma_0)$

4

$\mathbb{R}\ c_2\, c_3 \leqslant^{\mathfrak{r}}_{L} c_1$

$\mathbb{F}\ \Box A : (c_2, \gamma_0)_{[5]}$

5

$\mathbb{A}\ \gamma_0 \leqslant^{\mathfrak{s}}_{L} \gamma_1$

$\mathbb{F}\ A : (c_2, \gamma_1)^{*2}_{[7]}$

6

$\mathbb{R}\ \gamma_0 \leqslant^{\mathfrak{s}}_{L} \gamma_1$

$\mathbb{A}\ \gamma_0 <^{\mathfrak{s}}_{L} \gamma_1$

2′

$\mathbb{R}\ \gamma_0 \leqslant^{\mathfrak{s}}_{L} \gamma_1$

$\mathbb{T}\ A * B : (c_1, \gamma_1)$

3′

$\mathbb{A}\ c_2\, c_3 \leqslant^{\mathfrak{r}}_{L} c_1$

$\mathbb{T}\ A : (c_2, \gamma_1)^{*2}$

$\mathbb{T}\ B : (c_3, \gamma_1)$

$\mathbb{R}\ \gamma_0 \leqslant^{\mathfrak{s}}_{L} \gamma_1$

$\mathbb{A}\ \gamma_0 =^{\mathfrak{s}}_{L} \gamma_1$

7

$\mathbb{R}\ \gamma_0 =^{\mathfrak{s}}_{L} \gamma_1$

$\mathbb{F}\ A : (c_2, \gamma_0)^{*1}$

$\mathbb{R}\ c_2\, c_3 \leqslant^{\mathfrak{r}}_{L} c_1$

$\mathbb{F}\ \Box B : (c_3, \gamma_0)_{[6]}$

$\vdots$

Unsoundness of the Liberalized Rules

Problem

- After Step [4]: tableau splits in two similar branches
- Repeating Steps [2] through [6]

  leftmost branch of the tableau **grows infinitely**
- Under the liberalized version of $\mathbb{T}*$ in Step [3′]

  reuse c_2 and c_3 instead of new c_2' and c_3'

  $\Longrightarrow$ **branch closed**: $\mathbb{T}\ A : (c_2, \gamma_1)$ [3′] and $\mathbb{F}\ A : (c_2, \gamma_1)$ [5]
- The same holds for the other branch

$\Longrightarrow$ **closed** T_{LTBI} **tableau for non-valid formula**

A Surprising Negative Result

Completeness of BI w.r.t. Kripke Resource Monoids: **unknown**

- 20 year old open problem
- easy and natural clause for disjunction

Beth (Topological) Resource Monoids for BI: **complete**

- addition of a lattice structure to the monoids
- more complicated clause for disjunction

  $(r, s) \Vdash \mathrm{A} \vee \mathrm{B}$ iff $\exists\, r', r''.\ r' \cap r'' \leqslant^{\mathfrak{r}} r$, $r' \Vdash \mathrm{A}$ and $r'' \Vdash \mathrm{B}$
- soundness of the liberalized rules with disjunction

Unsoundness of LTBI **liberalized rules without disjunction !**

Restriction to Bounded LTL

Solution: assume timelime of length n

$$\mathbf{S} = \mathbf{S}_n = \{\, i < n \mid i \in \mathbb{N} \,\}$$

Fixpoint rules: only explicit successors

$$i < n \qquad \begin{array}{c|l} \multicolumn{2}{c}{\mathbb{T}\ \Diamond \mathrm{A} : (\mathrm{x}, \eta^i \gamma_0)} \\ \mathbb{T}\ \mathrm{A} : (\mathrm{x}, \eta^i \gamma_0) & \mathbb{F}\ \mathrm{A} : (\mathrm{x}, \eta^i \gamma_0) \\ & \mathbb{T}\ \Diamond \mathrm{A} : (\mathrm{x}, \eta^{i+1} \gamma_0) \end{array} \qquad i = n \quad \frac{\mathbb{T}\ \Diamond \mathrm{A} : (\mathrm{x}, \eta^i \gamma_0)}{\mathbb{T}\ \mathrm{A} : (\mathrm{x}, \eta^i \gamma_0)}$$

$$i < n \qquad \begin{array}{c|l} \multicolumn{2}{c}{\mathbb{F}\ \Box \mathrm{A} : (\mathrm{x}, \eta^i \gamma_0)} \\ \mathbb{F}\ \mathrm{A} : (\mathrm{x}, \eta^i \gamma_0) & \mathbb{T}\ \mathrm{A} : (\mathrm{x}, \eta^i \gamma_0) \\ & \mathbb{F}\ \Box \mathrm{A} : (\mathrm{x}, \eta^{i+1} \gamma_0) \end{array} \qquad i = n \quad \frac{\mathbb{F}\ \Box \mathrm{A} : (\mathrm{x}, \eta^i \gamma_0)}{\mathbb{F}\ \mathrm{A} : (\mathrm{x}, \eta^i \gamma_0)}$$

$\Longrightarrow$ saturation terminates $\Longrightarrow$ completeness

Overview of the Talk

Introduction

Linear Time Bunched Implication Logic

Expressivity of LTBI

Tableau Calculus

Soundness

Counter-Model Construction

Completeness Issues

Conclusion & Future Work

Conclusion

- A new logic: Linear Time BI (LTBI)
- Syntax and semantics of LTBI
- Labelled tableaux calculus for LTBI
- Soundness w.r.t. Kripke semantics
- Completeness for bounded timelines
- Completeness issues in the general case

Future Work

- Closure conditions for cyclic tableaux (completeness)
- Local resource monoids (fix the liberalized rules)
- Study of decidability of (some fragments of) LTBI
- Comparison with LTL tableaux with graphs
- Study of branching time inside LTBI

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Thank you for your attention