Bowyer-Watson algorithm, from the Euclidean plane to hyperbolic surfaces.

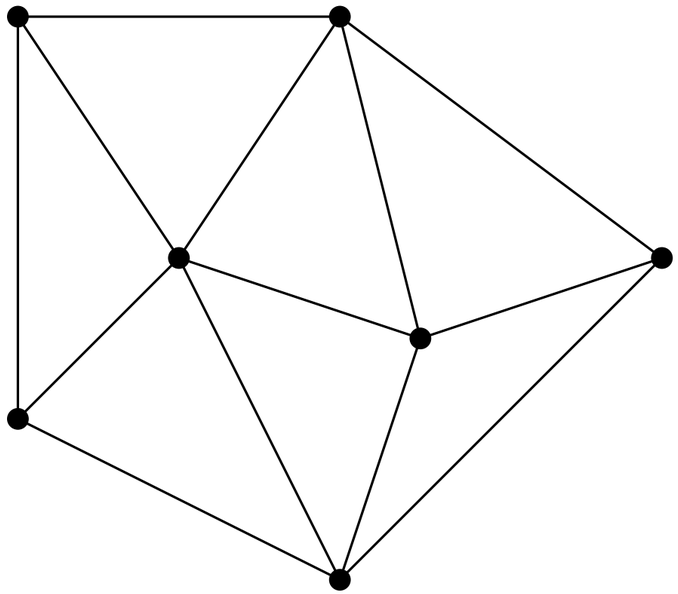
Vincent Despré, Dorian Perrot, Marc Pouget

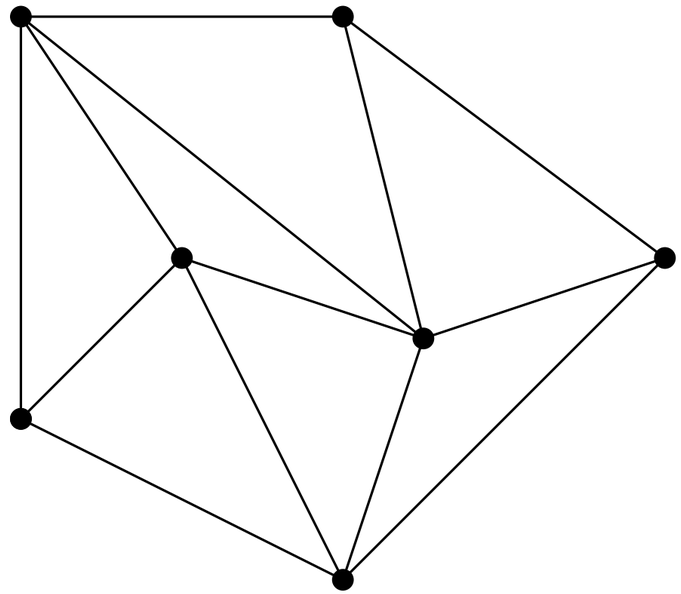
IRIMAS -Mulhouse- 04/2026

- Delaunay triangulation in Euclidean plane
  - BW-algorithm on a torus ($g = 1$)
  - Hyperbolic plane
  - BW-algorithm on the Bolza surface ($g = 2$)
  - BW-algorithm on hyperbolic surface ($g \geq 2$)

Delaunay triangulation

Def : a triangulation in which every circumscribed disk of triangle does not contain any of triangulation's vertices





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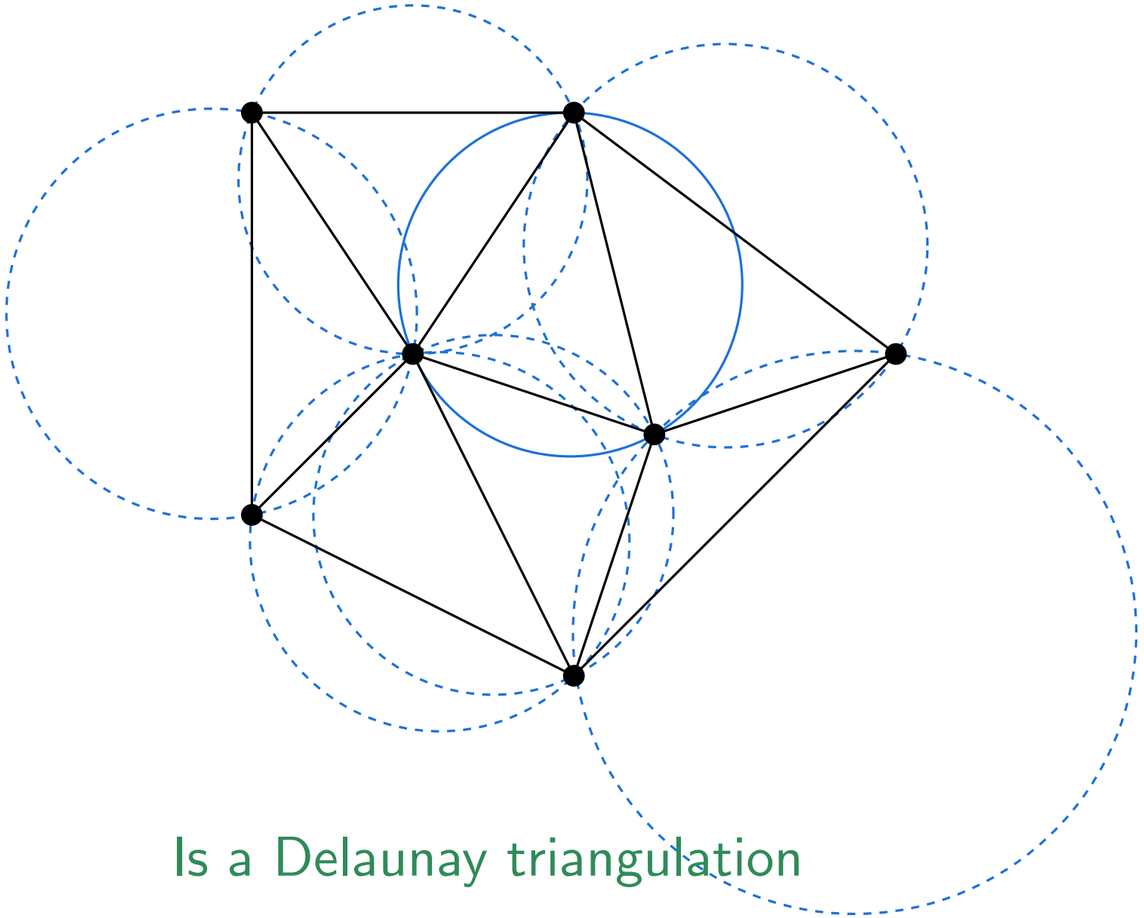
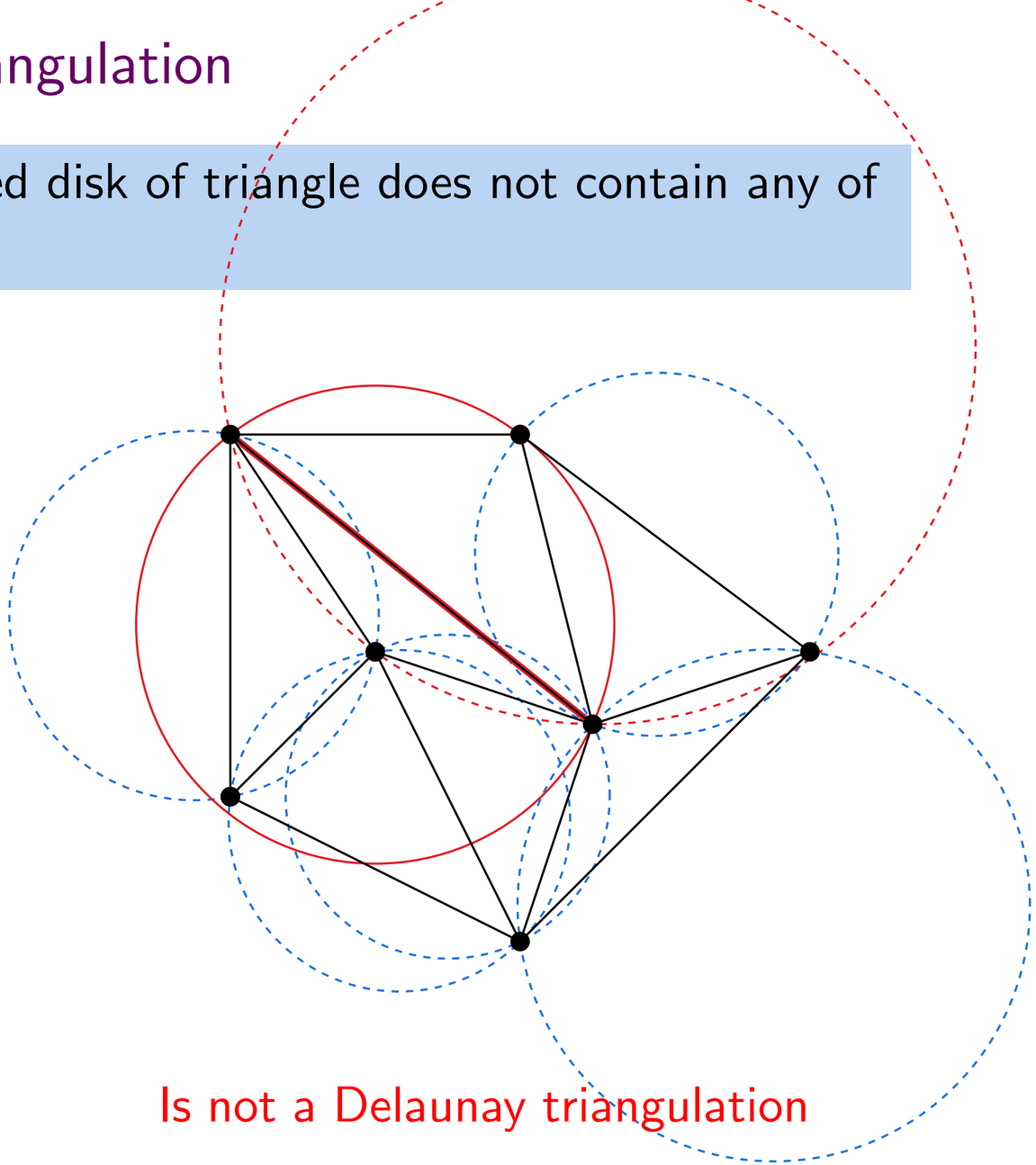


Is a Delaunay triangulation



Is not a Delaunay triangulation

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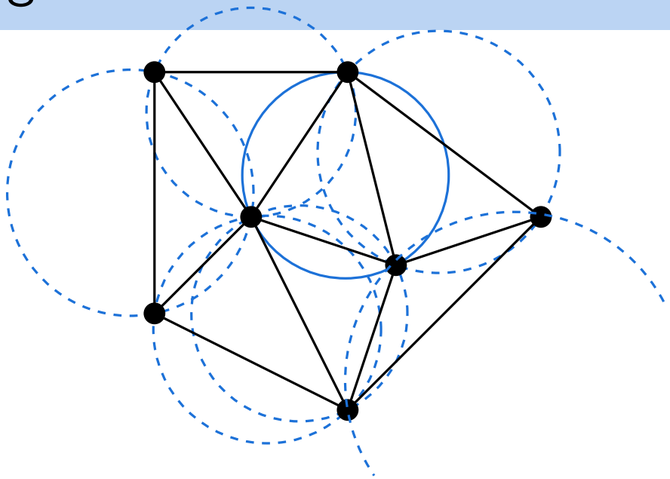

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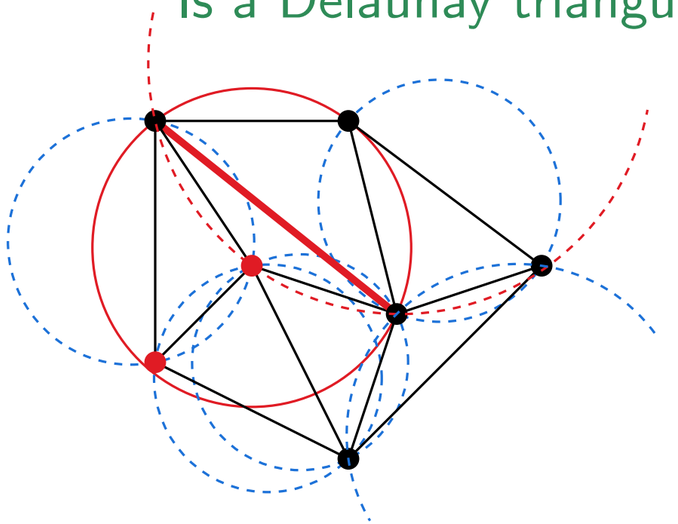


Is not a Delaunay triangulation

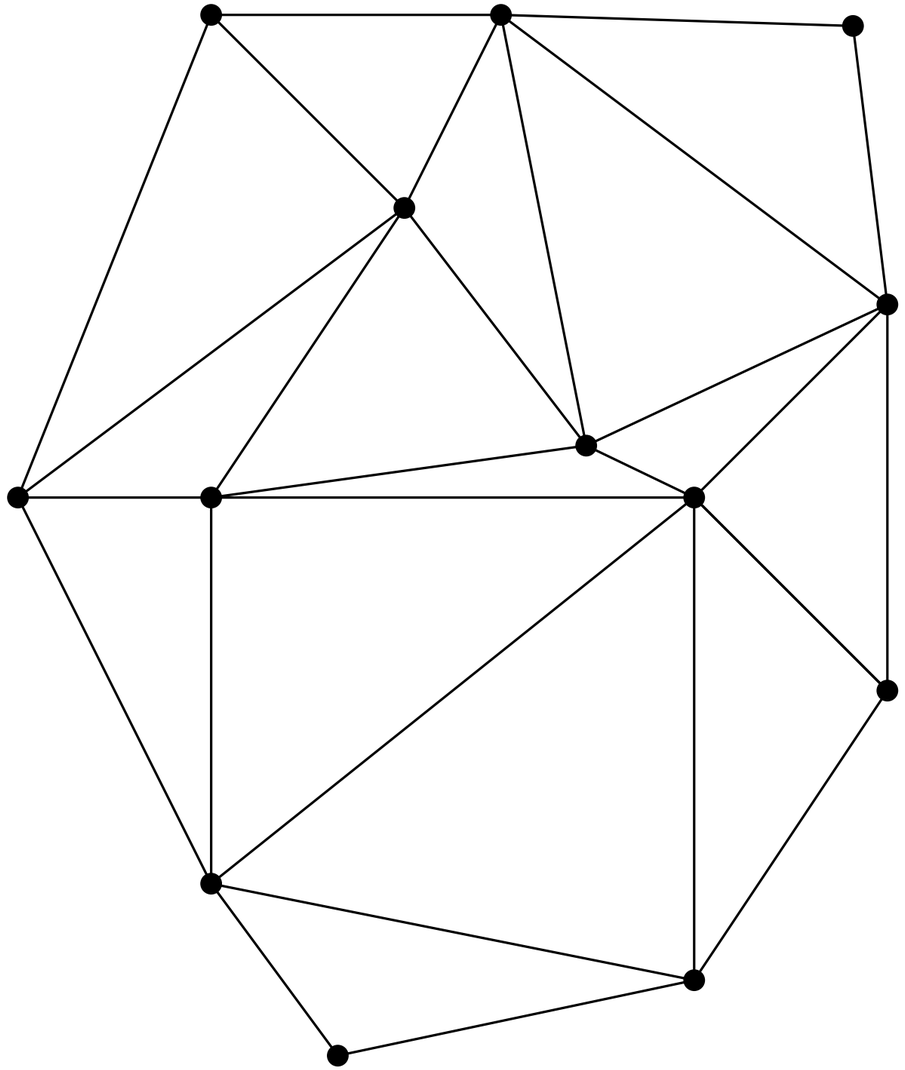
→ Flip algorithm in Euclidean plane [La71].
  → Generalised to a hyperbolic surface [DST24].

→ Divide and conquer [D87]

→ Bowyer-Watson algorithm. [Bo81].
  → Extended to Bolza surface (most symetric surface of genus 2) [IT17]
  ↓
  Generalization of this algorithm to all hyperbolic surfaces.[DDP26]

Flip algorithm

⟶ "improving triangulation" algorithm that computes a Delaunay triangulation



1) Start from a triangulation T

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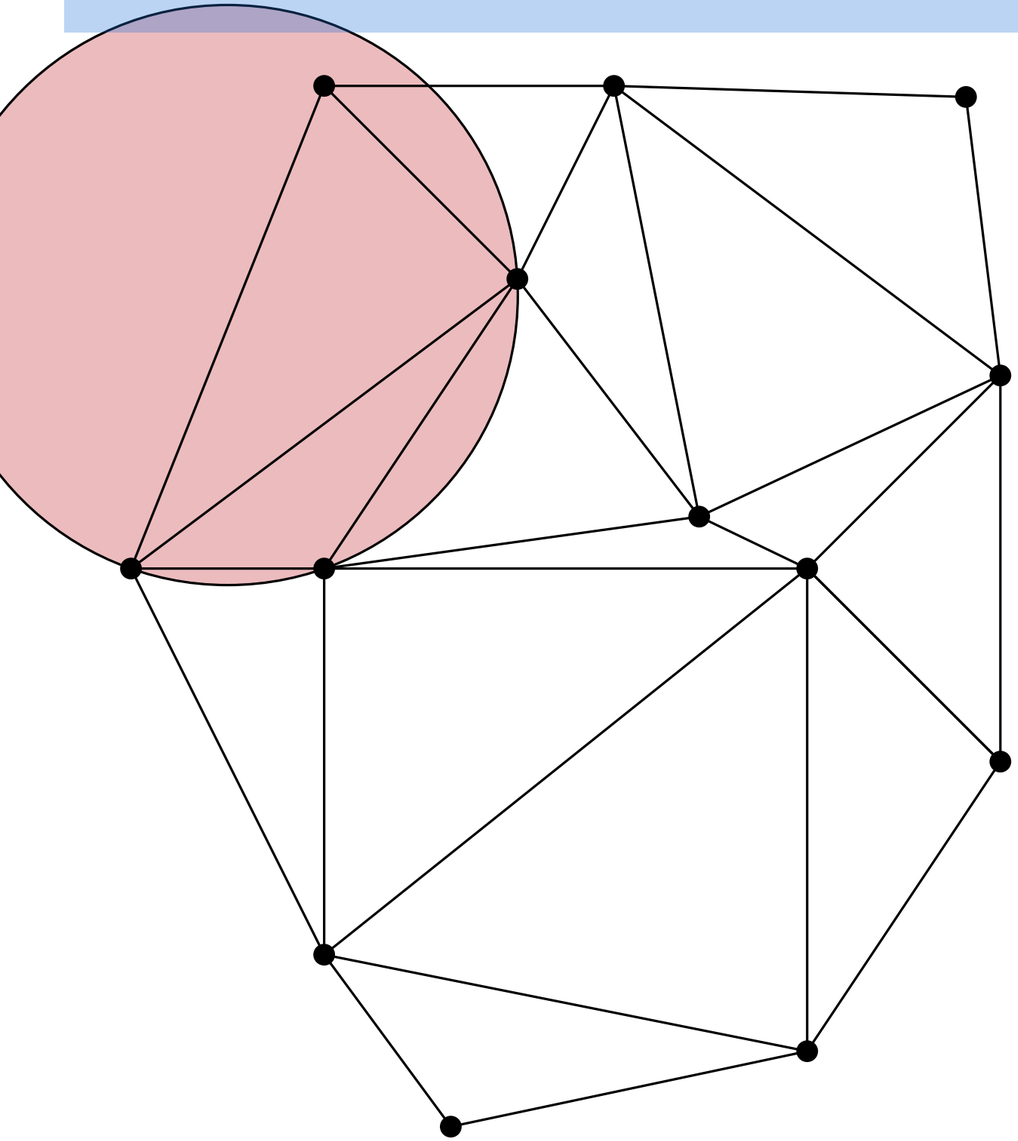

1) Start from a triangulation T

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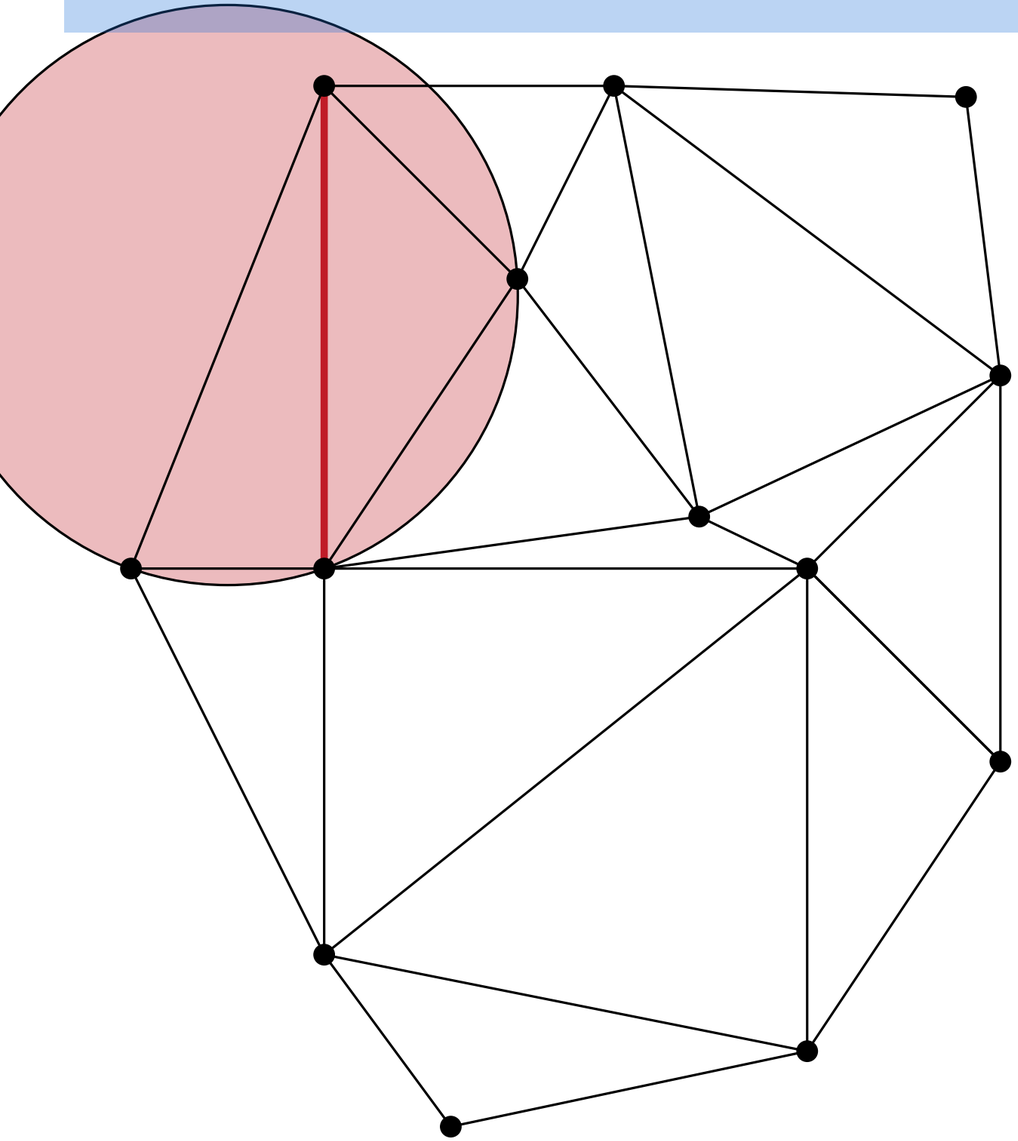

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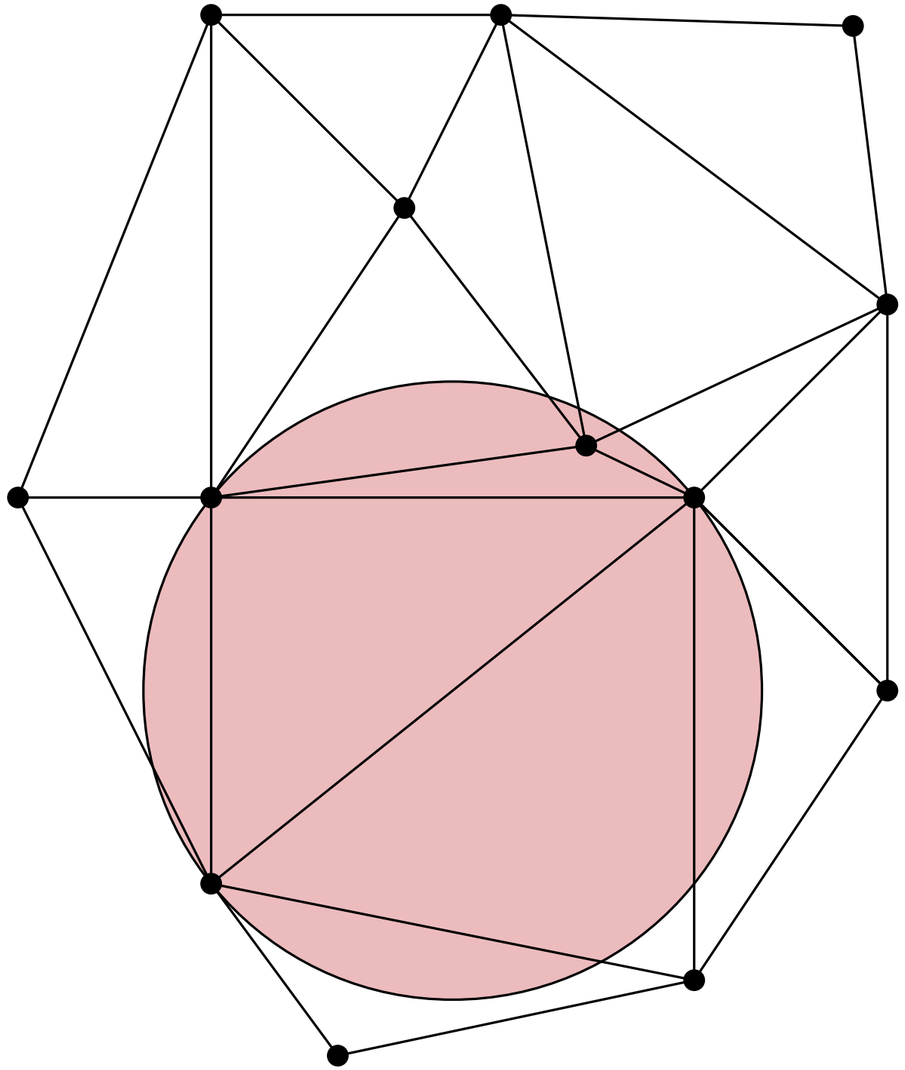

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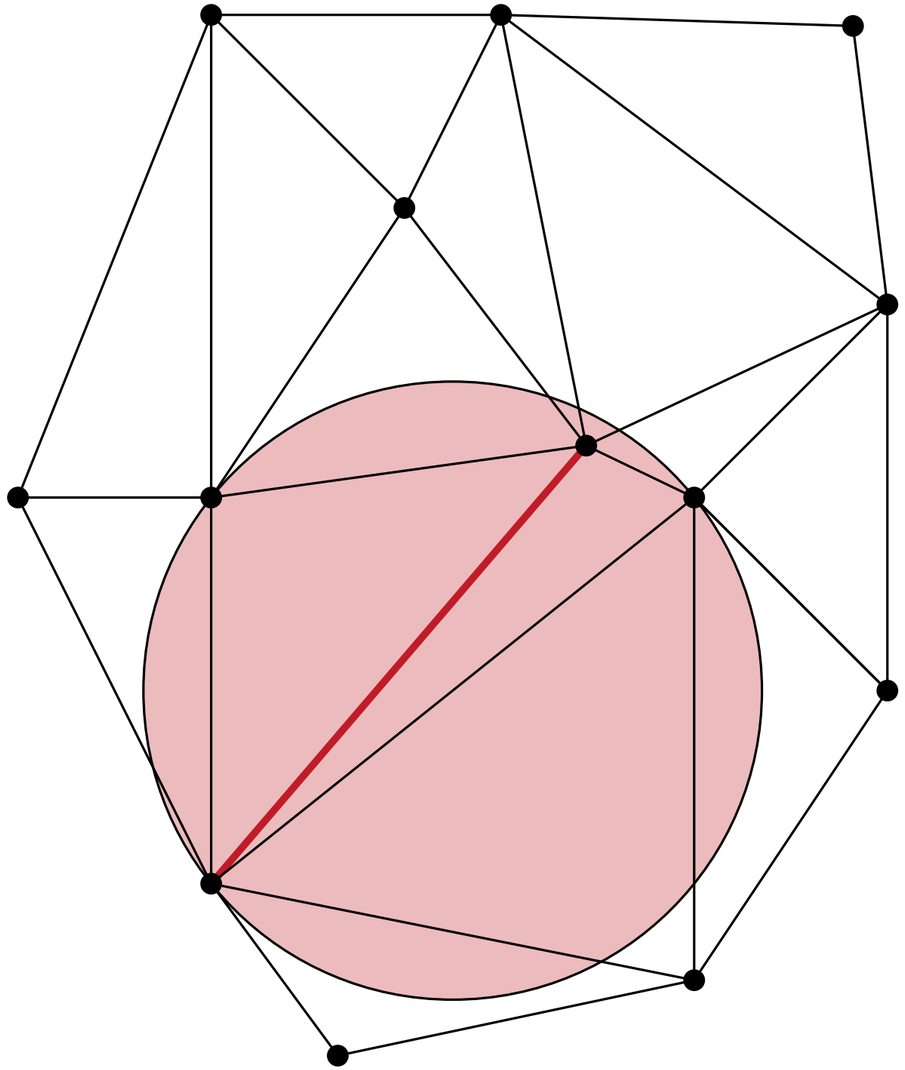

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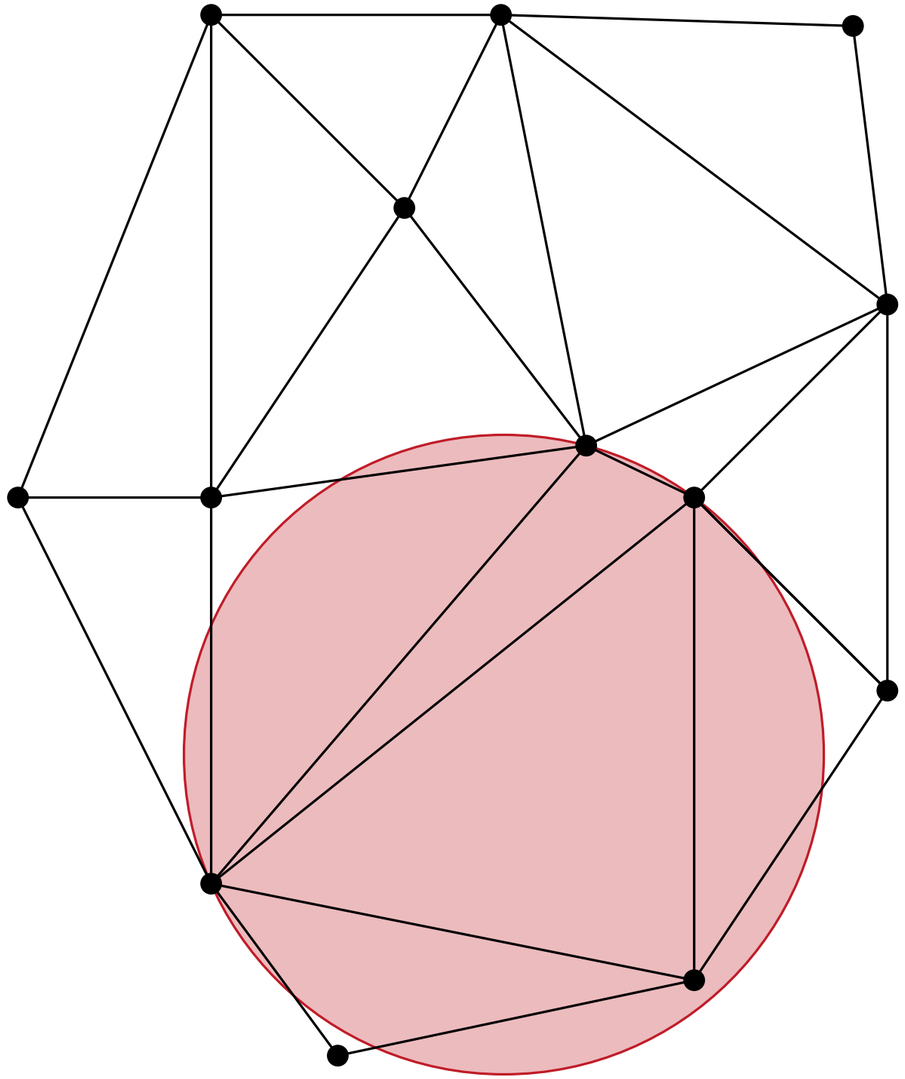

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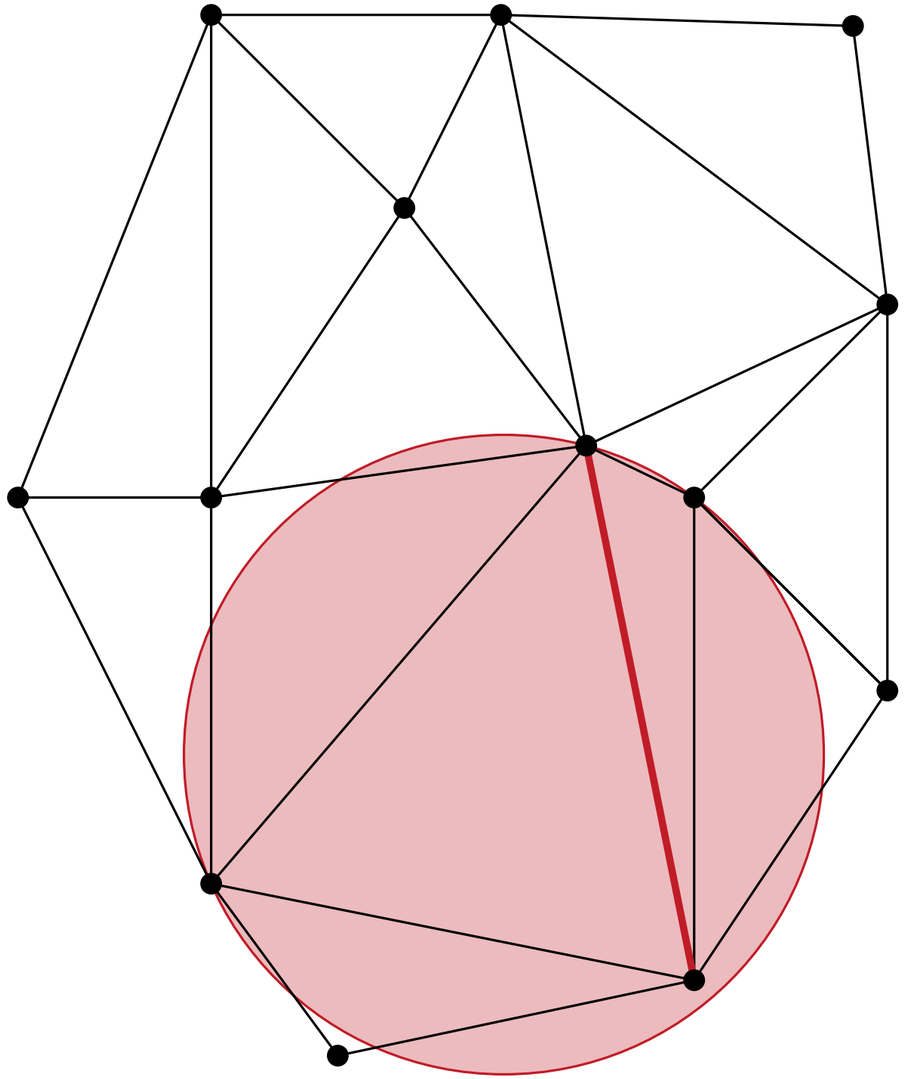

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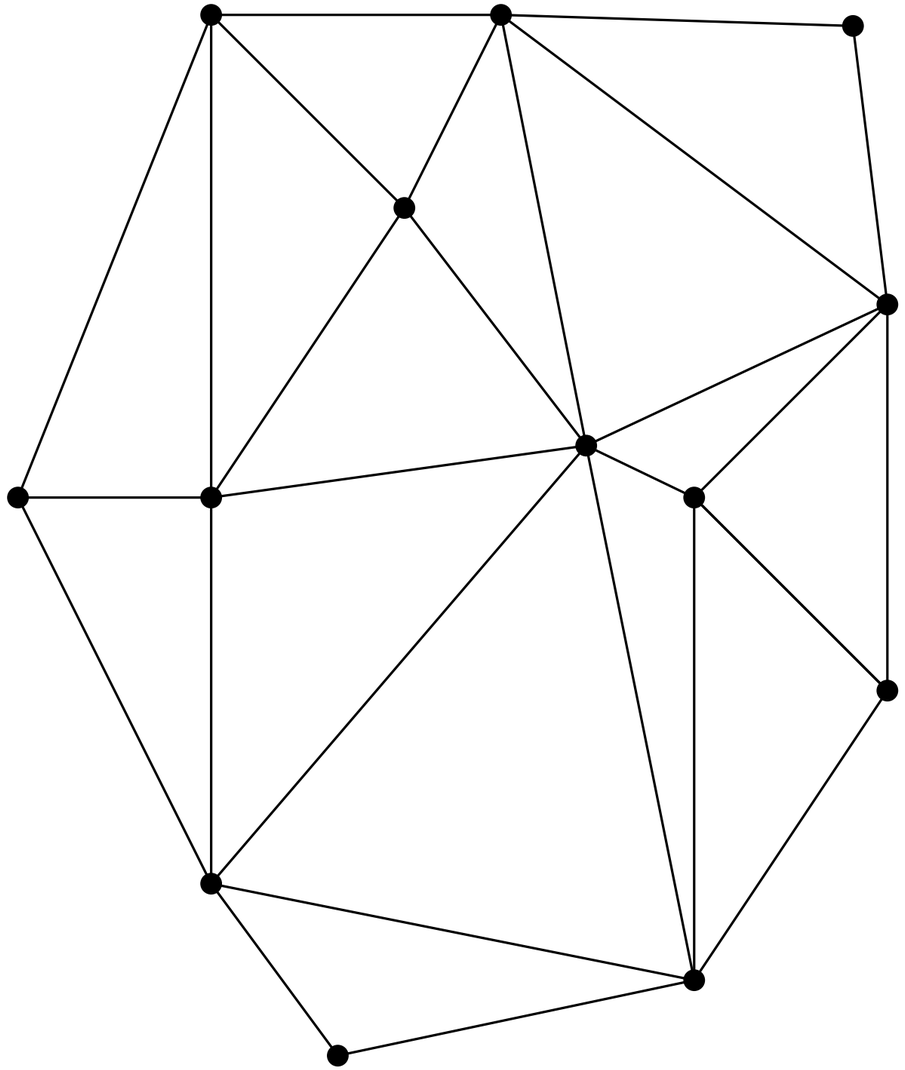

1) Start from a triangulation T

2) While T is not Delaunay :

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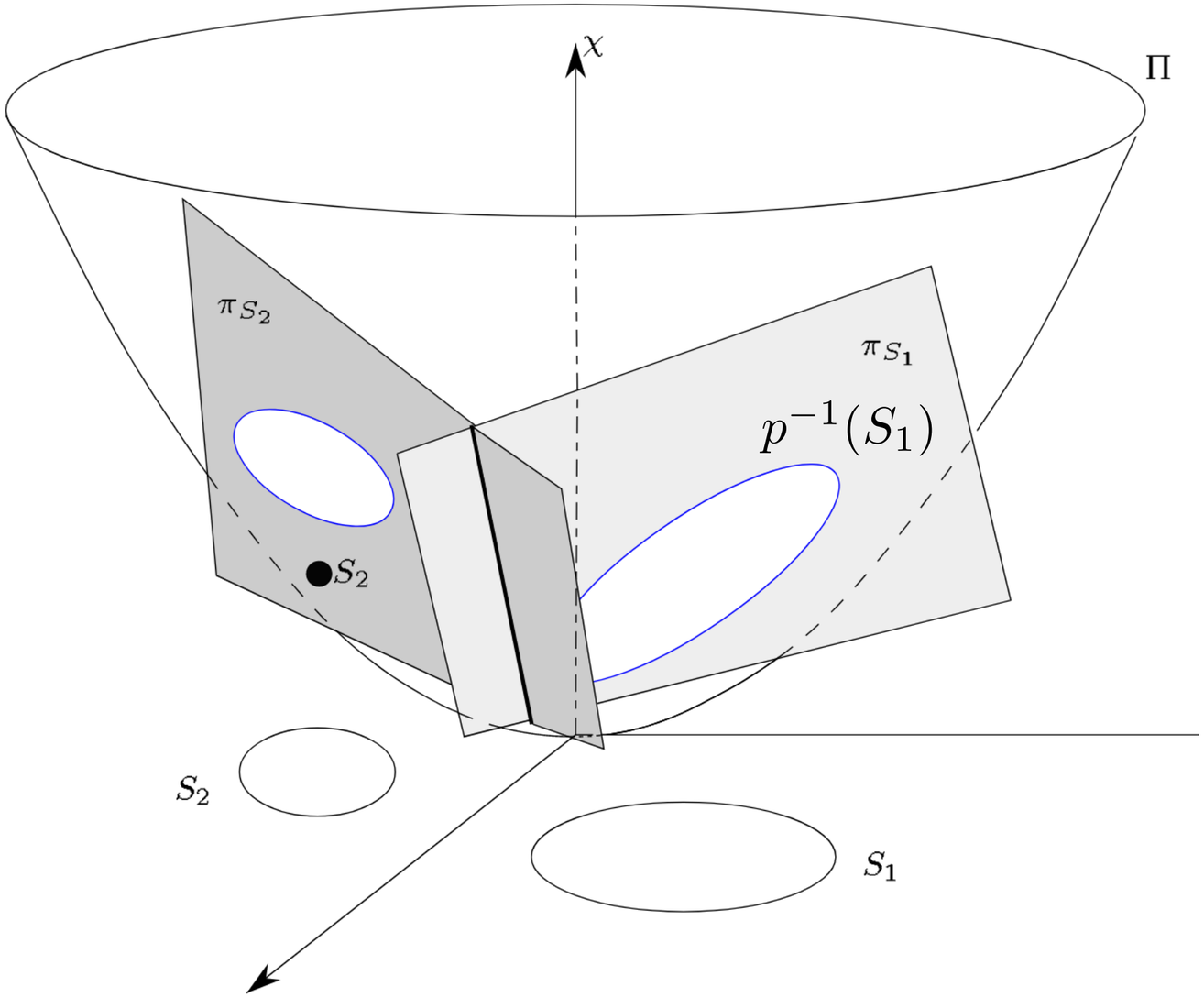
3) Return T

Delaunay triangulation and hyperboloid

Delaunay triangulation = lower convex hull of points on a paraboloid [Bro79]



Paraboloid Π:

$$x^2 + y^2 - z = 0$$

Projection p:

$$p : (x, y, x^2 + y^2) \in \Pi \mapsto (x, y, 0)$$

Prop: $p^{-1}(\text{circle}) = \Pi \cap$ a hyperplane

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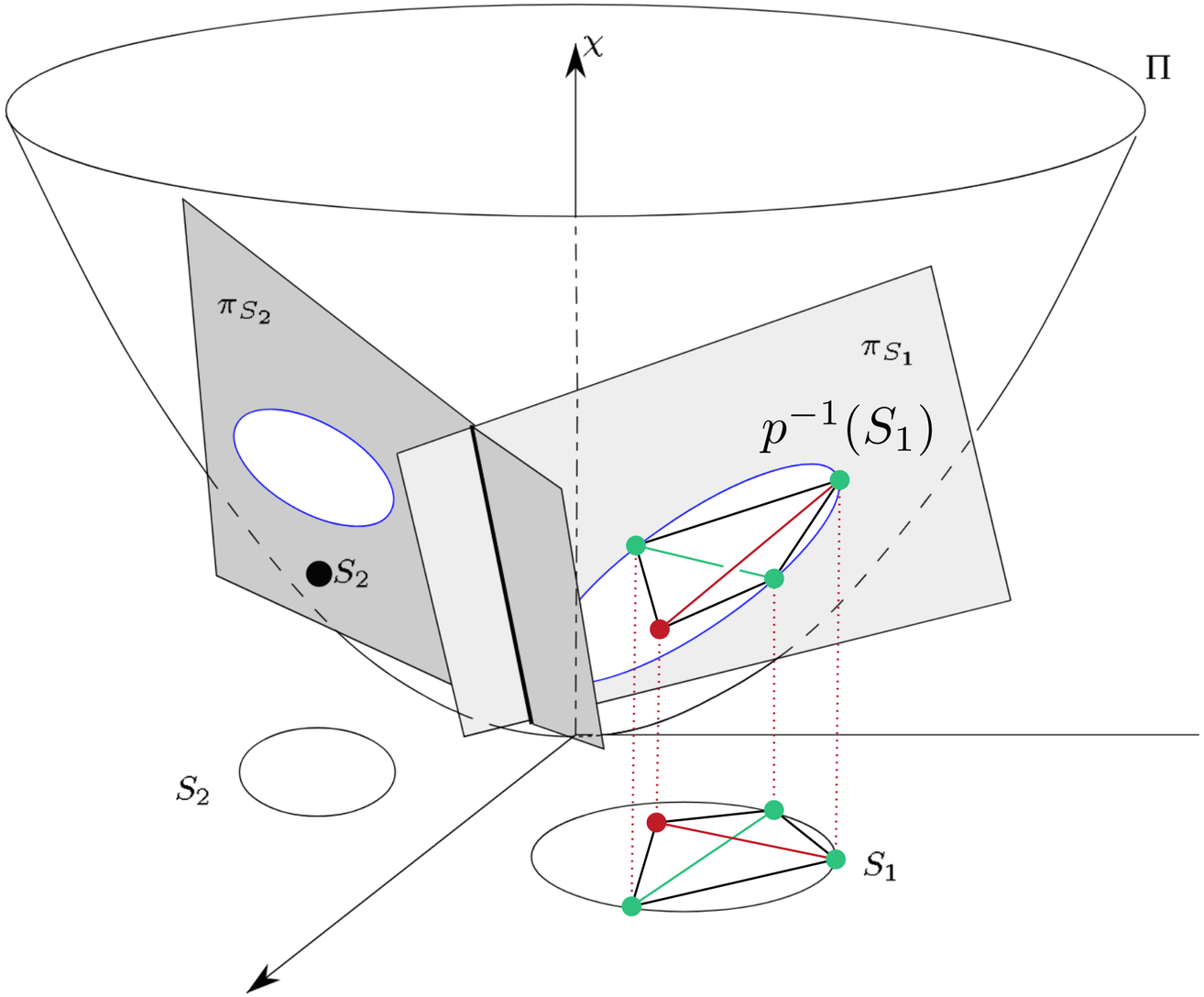

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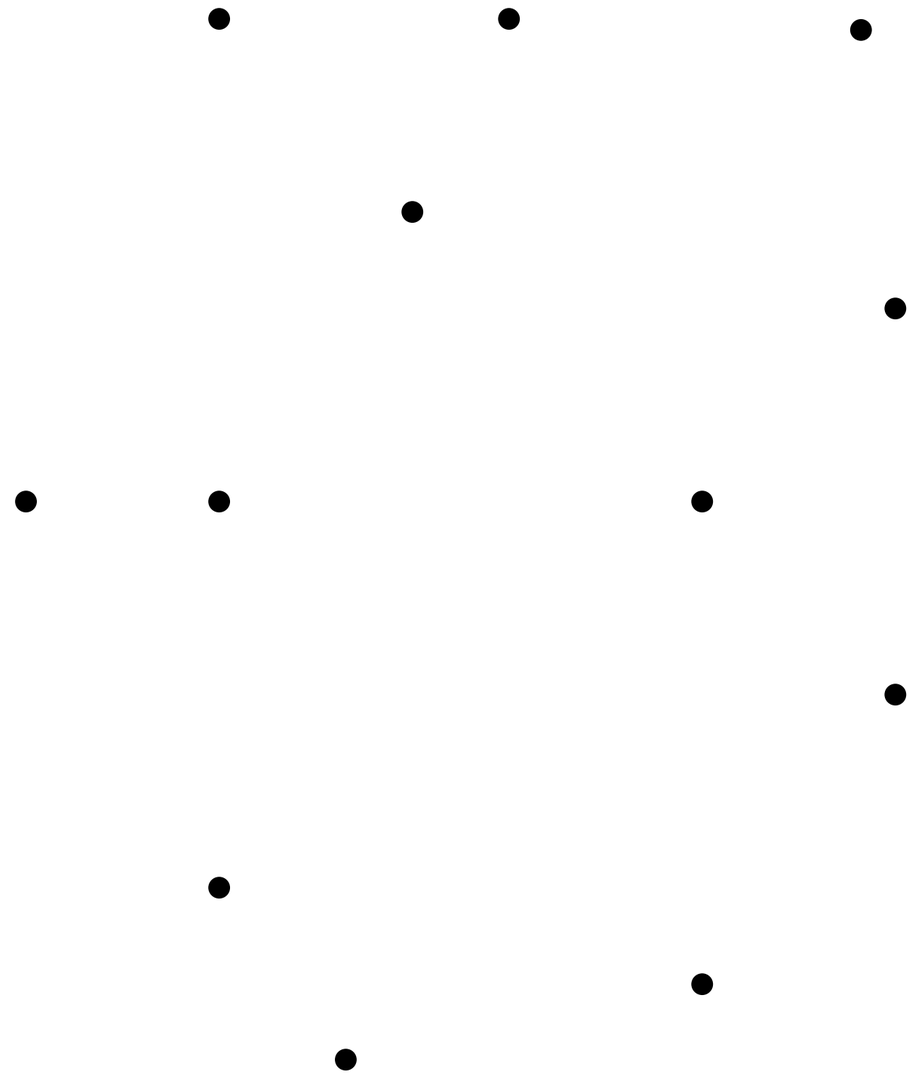
Prop: Let V be a set of points in the plane, let C be the lower convex hull of $p^{-1}(V)$. Then $\text{DL}(V) = p(C)$.

→ Delaunay triangulation exists

→ Flip algorithm is correct

Divide and conquer algorithm

→ Divide and conquer algorithm that computes a Delaunay triangulation



Recursion principle :

Function DL(V) :

If $|V| \leq 3$:

Return convhull(V)

Else :

$V = V' \cup V''$

Return Merge(DL(V'), DL(V''))

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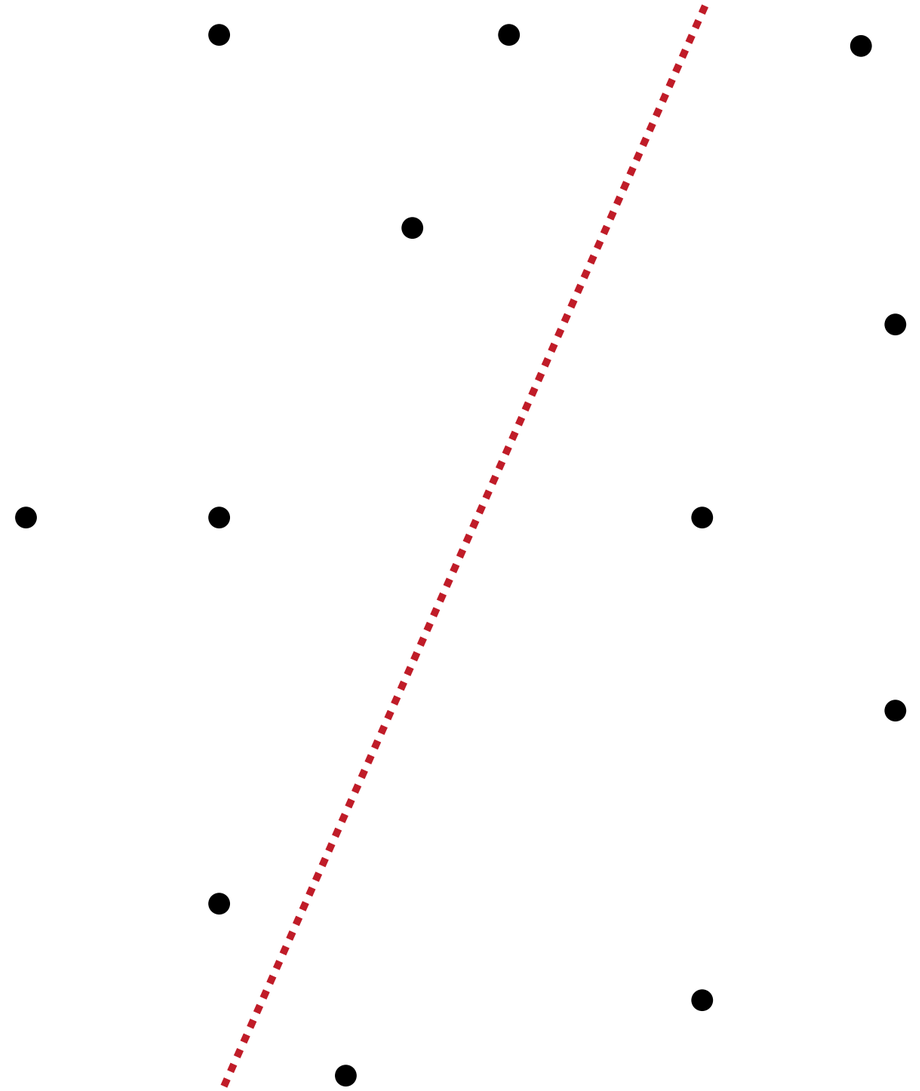

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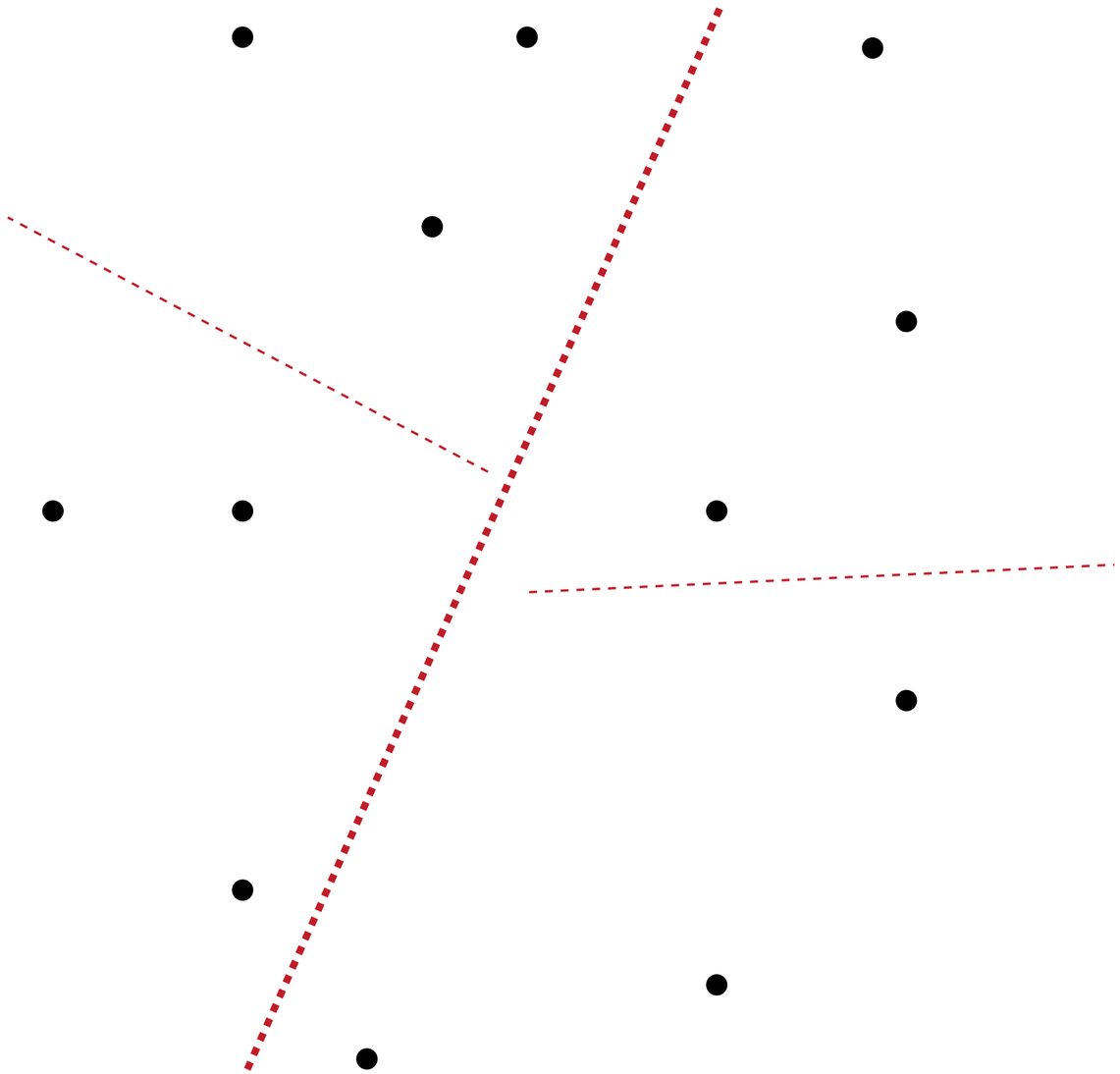

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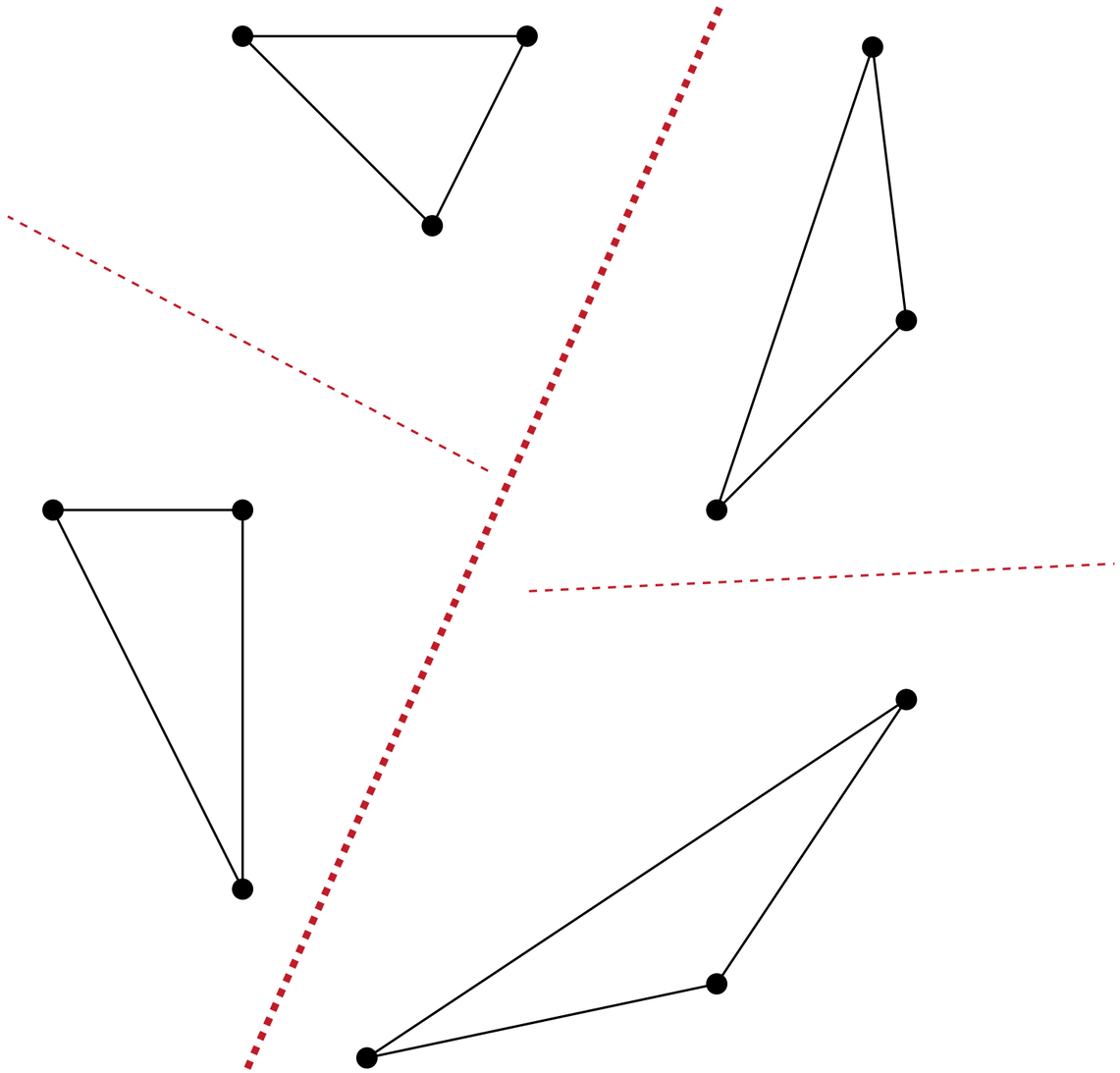

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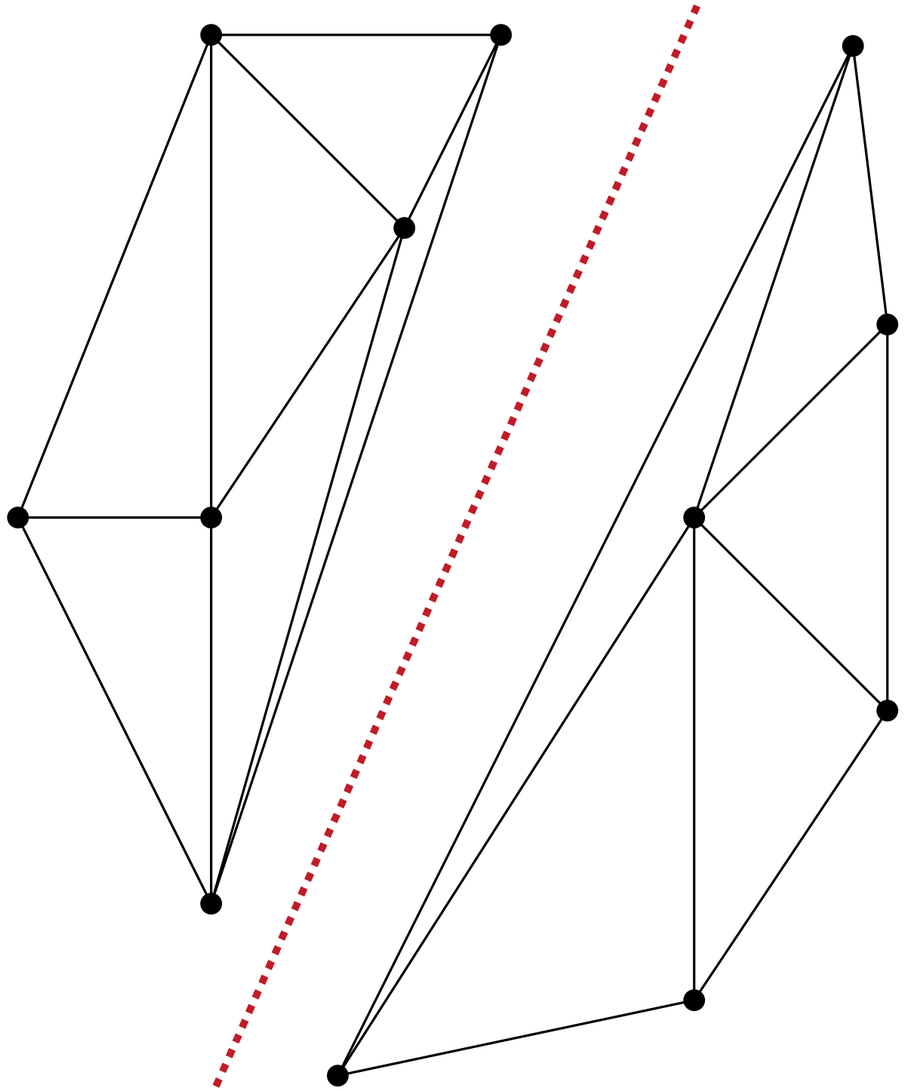

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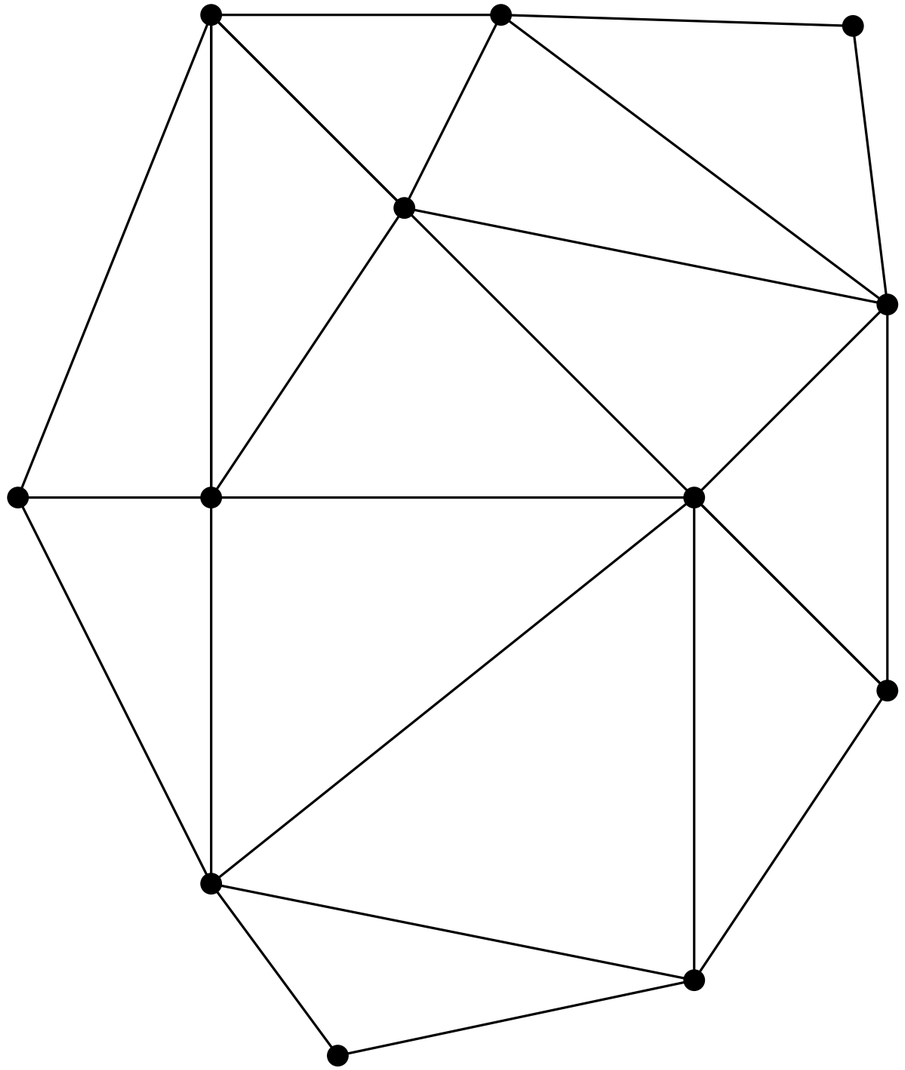

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Bowyer-Watson (BW) algorithm

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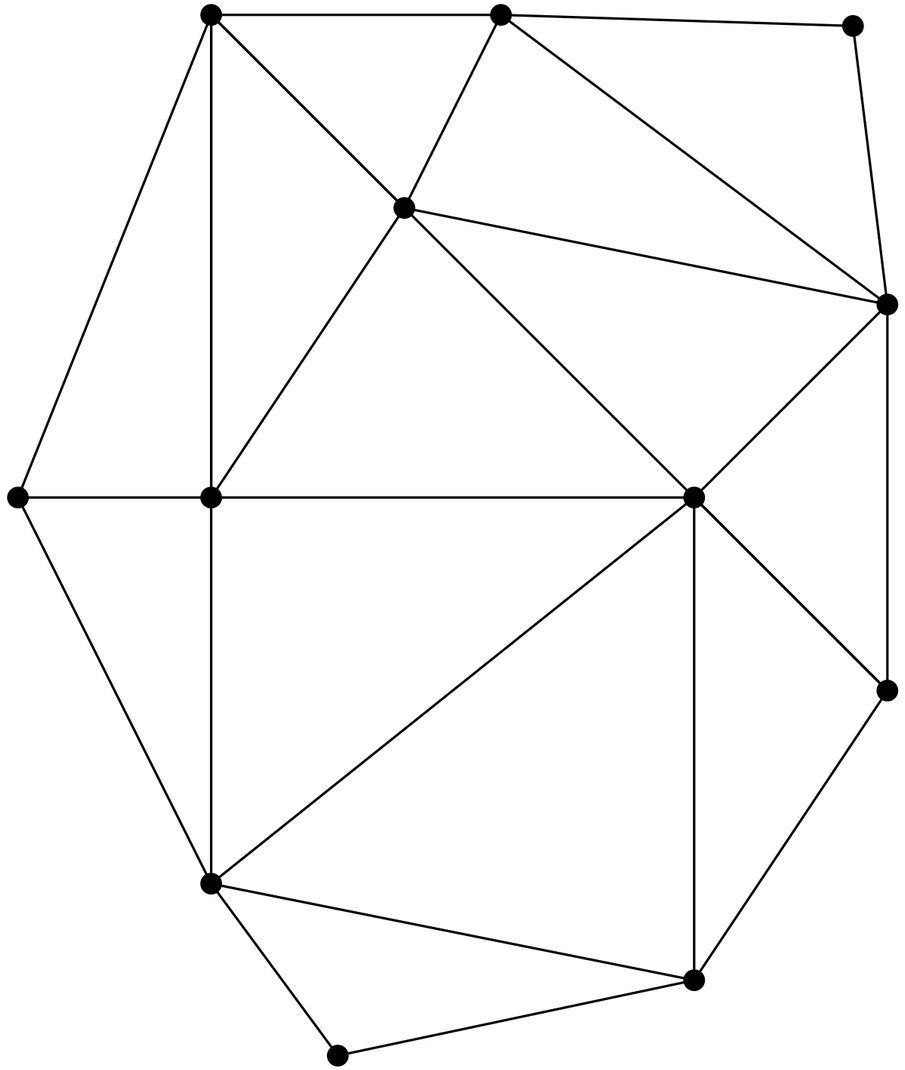

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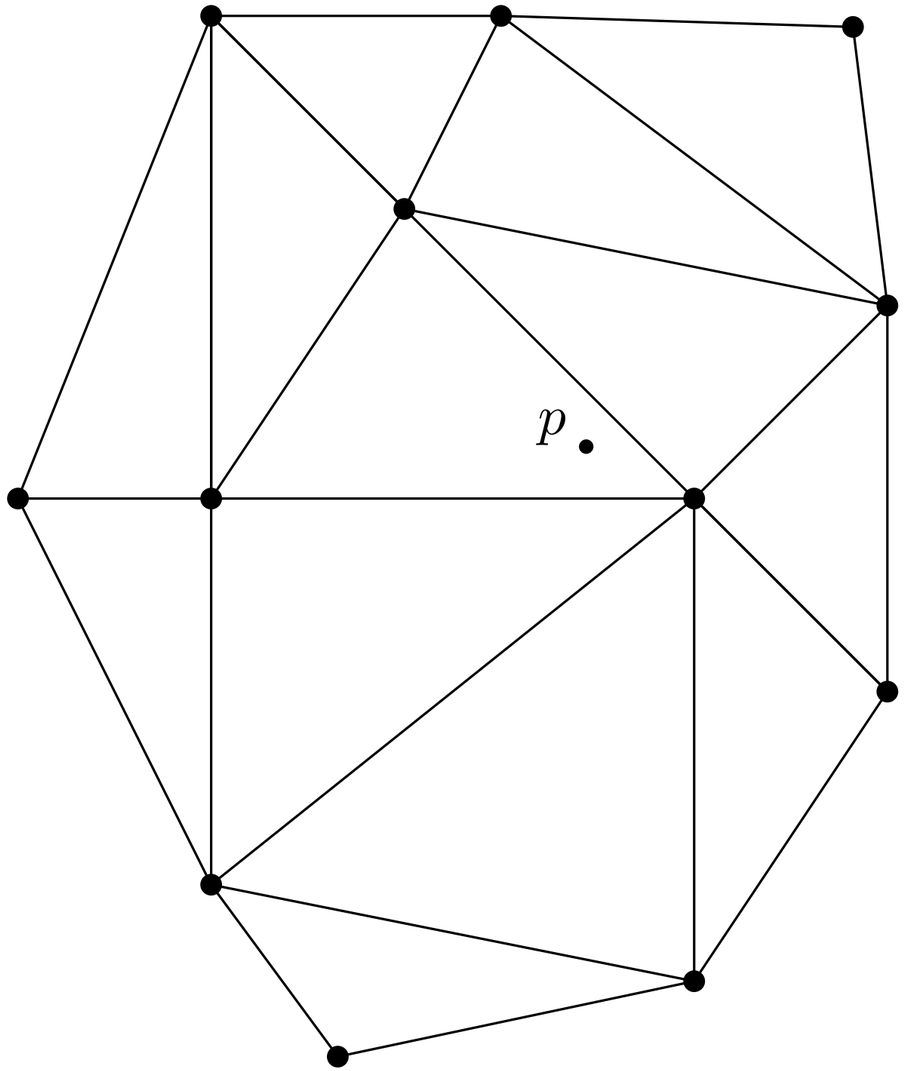

Insertion step :

1) Insert a point p

Bowyer-Watson (BW) algorithm

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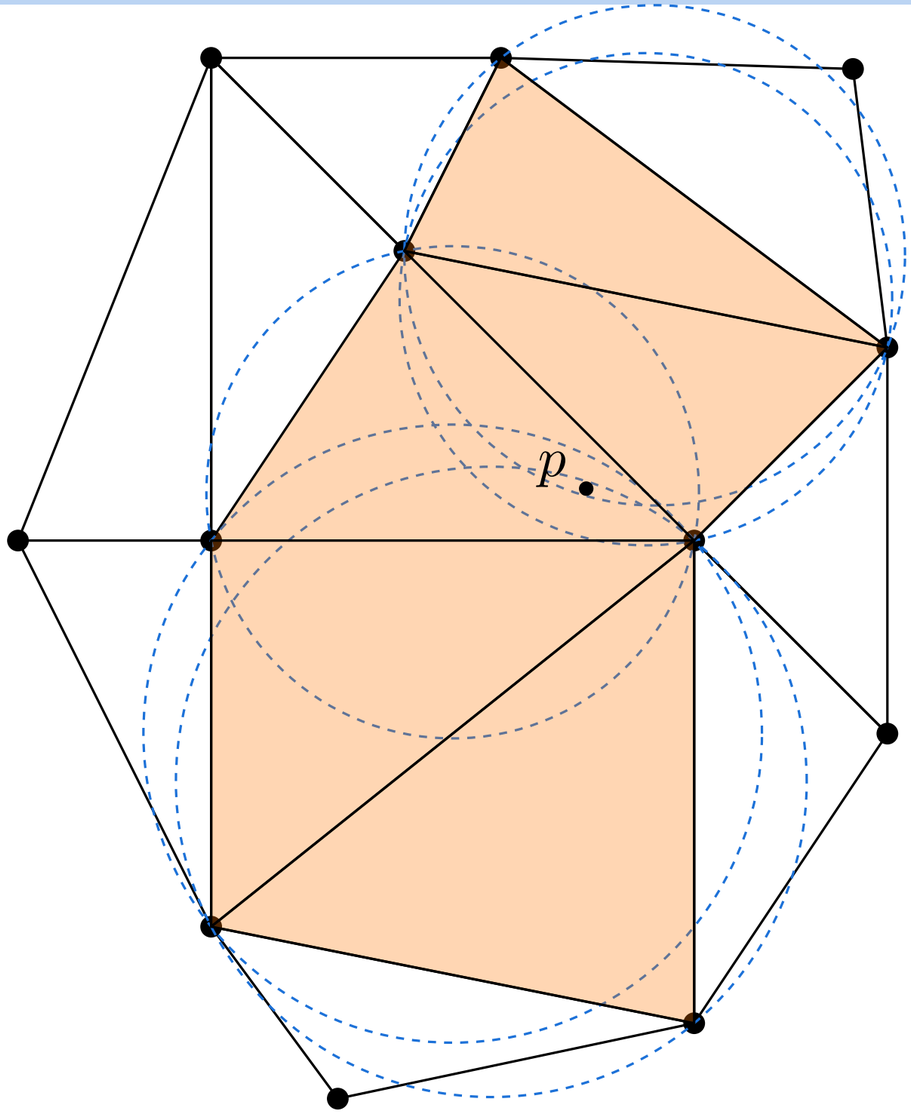

Insertion step :

1) Insert a point p

2) Compute triangles whose circumscribed disk contains p : the conflict zone.

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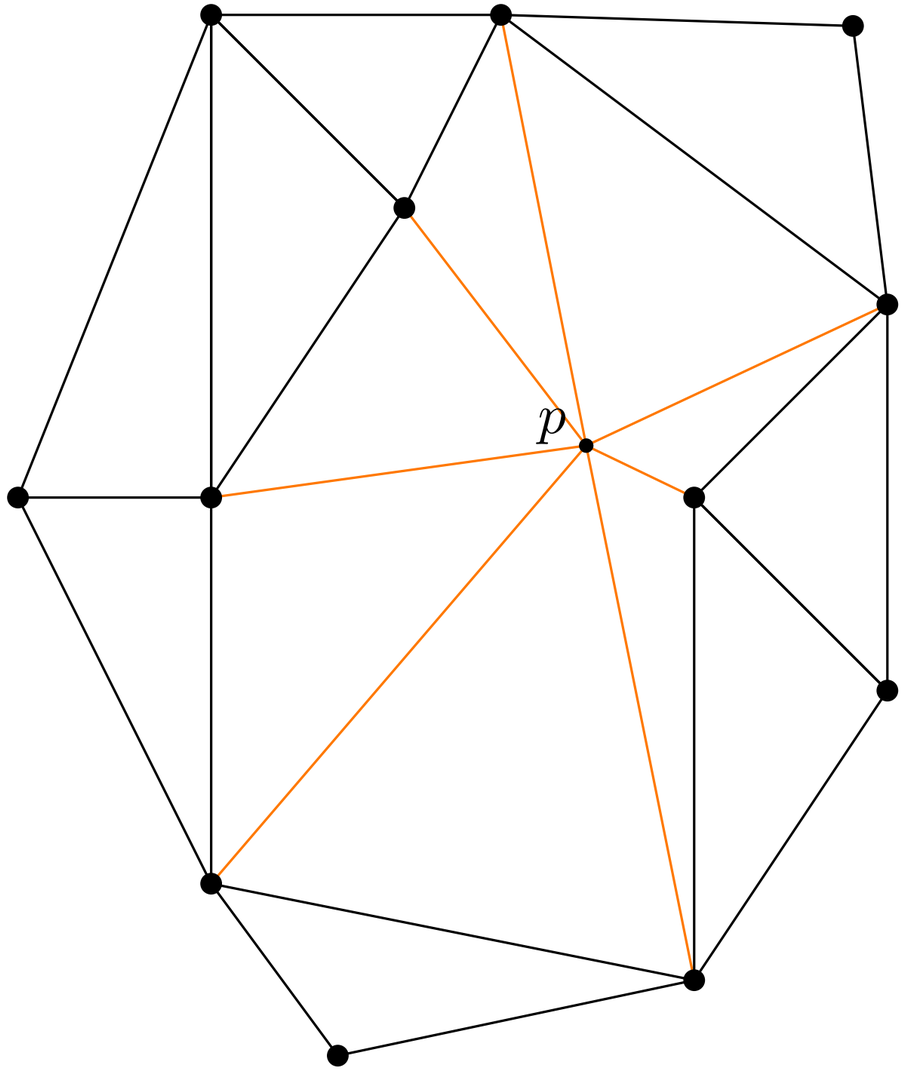


Insertion step :

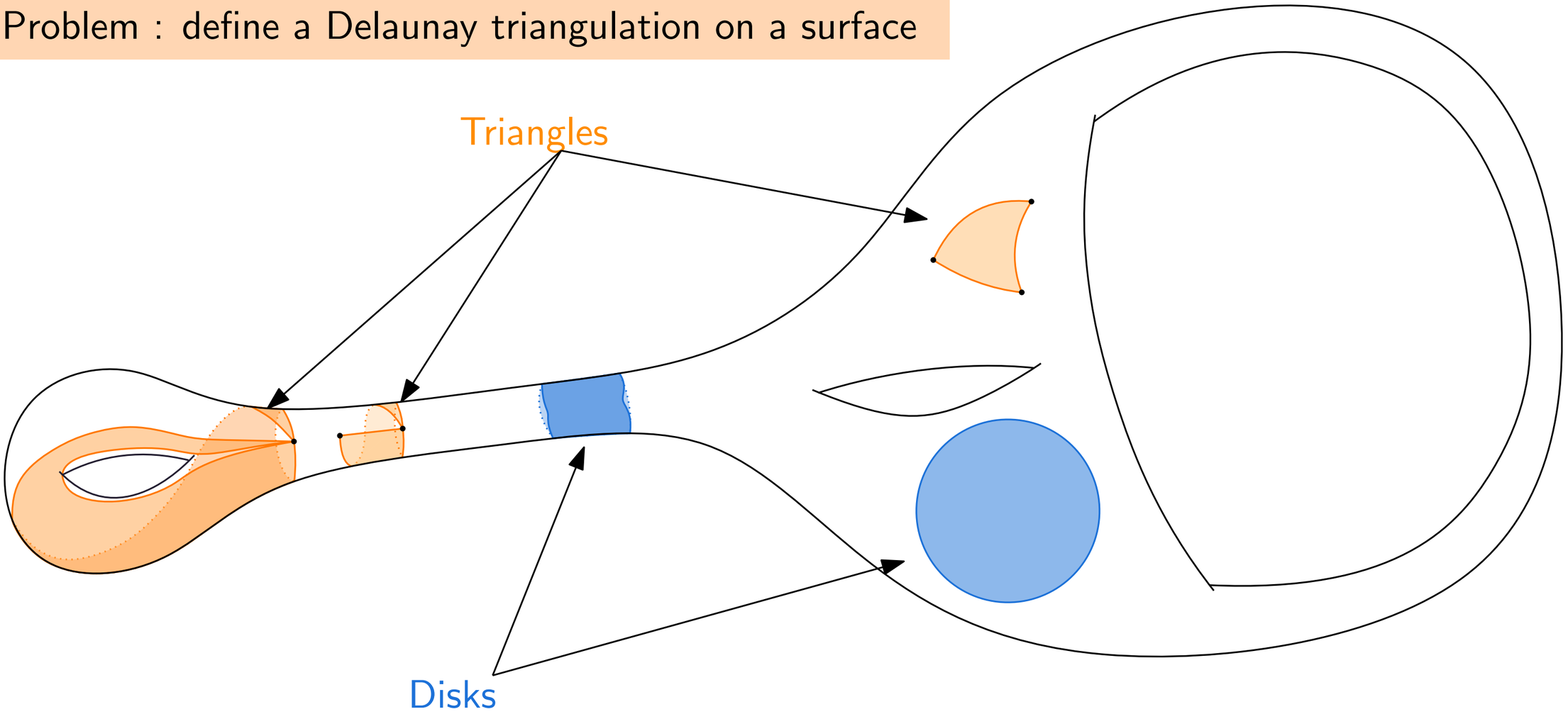
1) Insert a point p

2) Compute triangles whose circumscribed disk contains p : the conflict zone.

3) Connect p to vertices of these triangles.

Topological problem : weird triangles and disks

Problem : define a Delaunay triangulation on a surface



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- ▶ BW-algorithm on a torus ($g = 1$)
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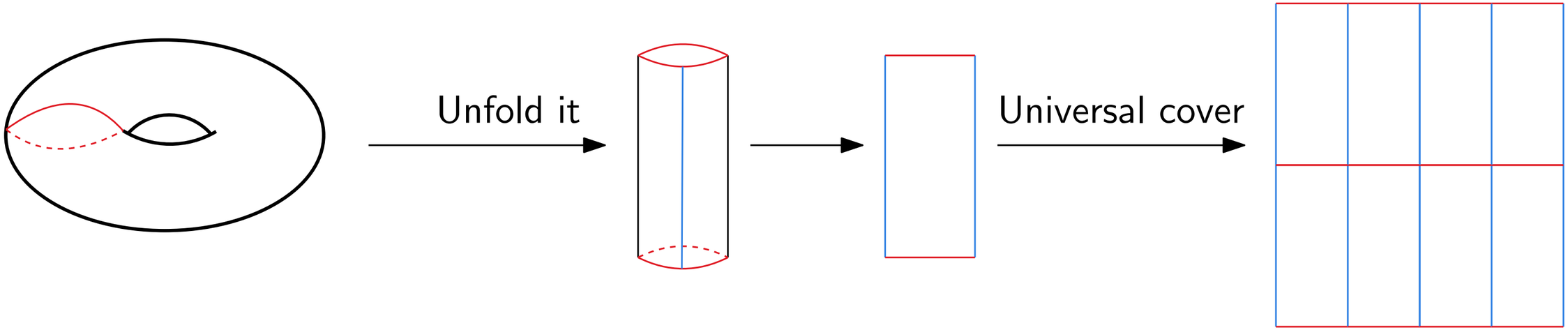
BW-algorithm on a flat Torus



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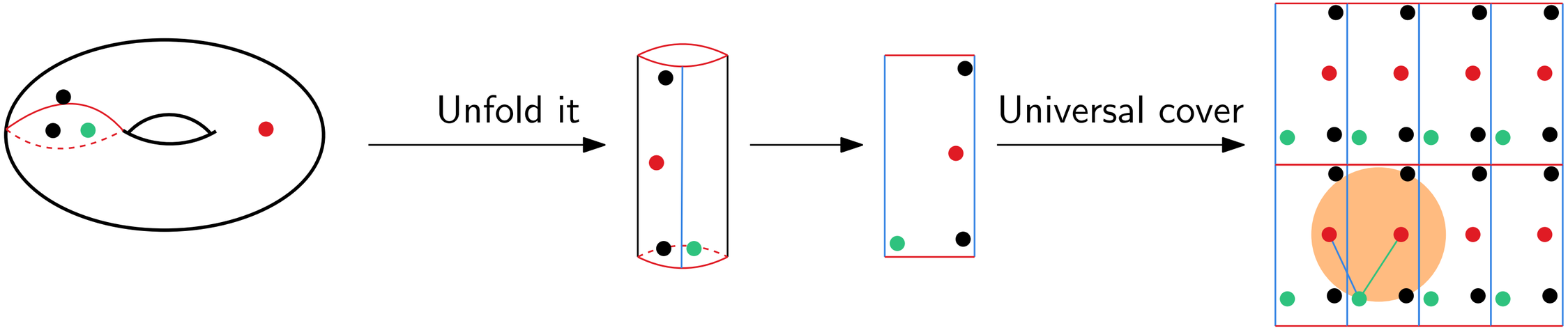
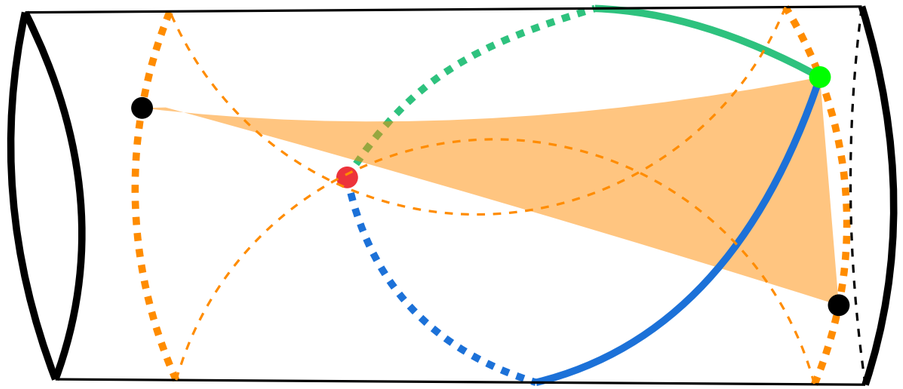


Avoid non-embedded disk!

↳ Which path ?



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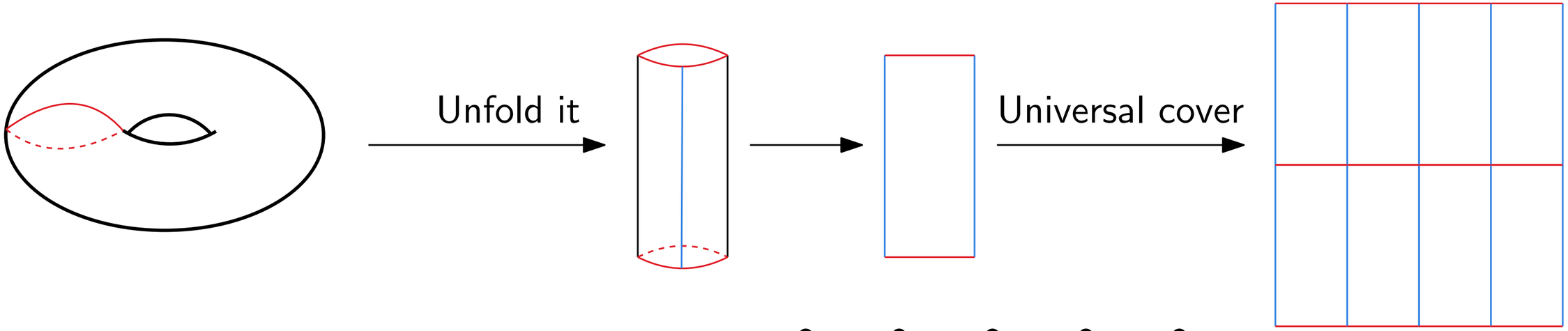
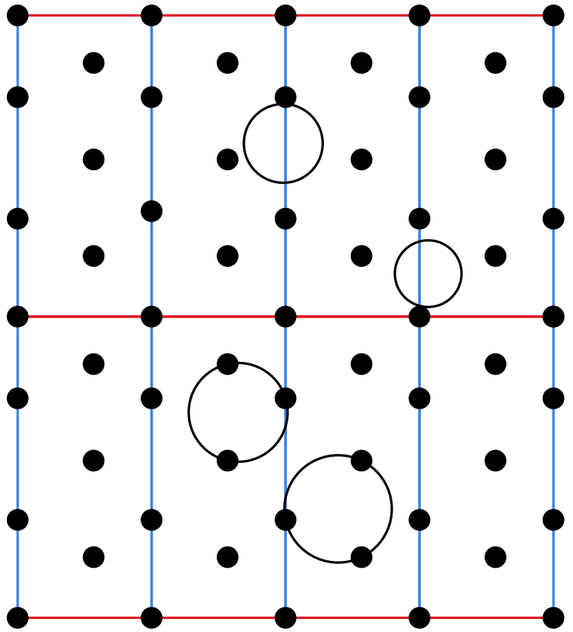


1) Add some "dummy" points forming a "ε-net" to avoid non-embedded disk.

2) Continue the insertion with the BW-algorithm.



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Disks models of hyperbolic plane

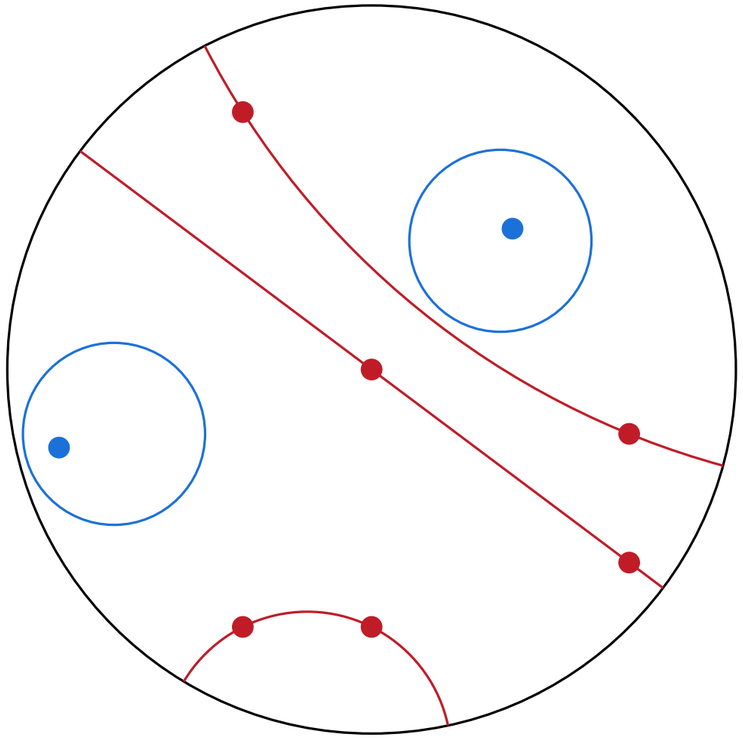
Poincaré disk model

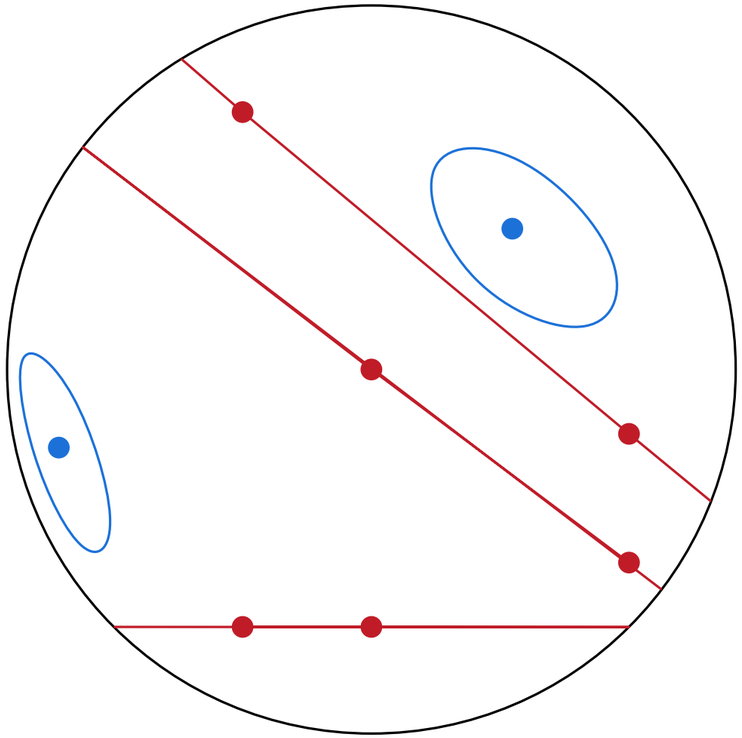
- circle = Euclidean circle
- line = circle wich is orthogonal to the unit circle
- preserves angles

Do your Choice!

Klein-Beltrami disk model

- circle = Euclidean ellipse
- line = Euclidean line
- (Useful for the hyperbolic extended plane model)





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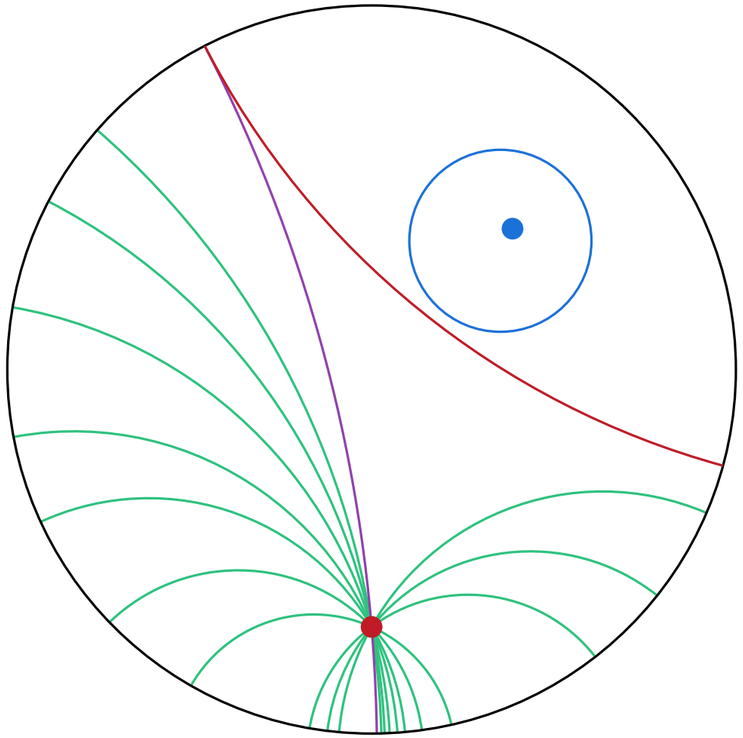
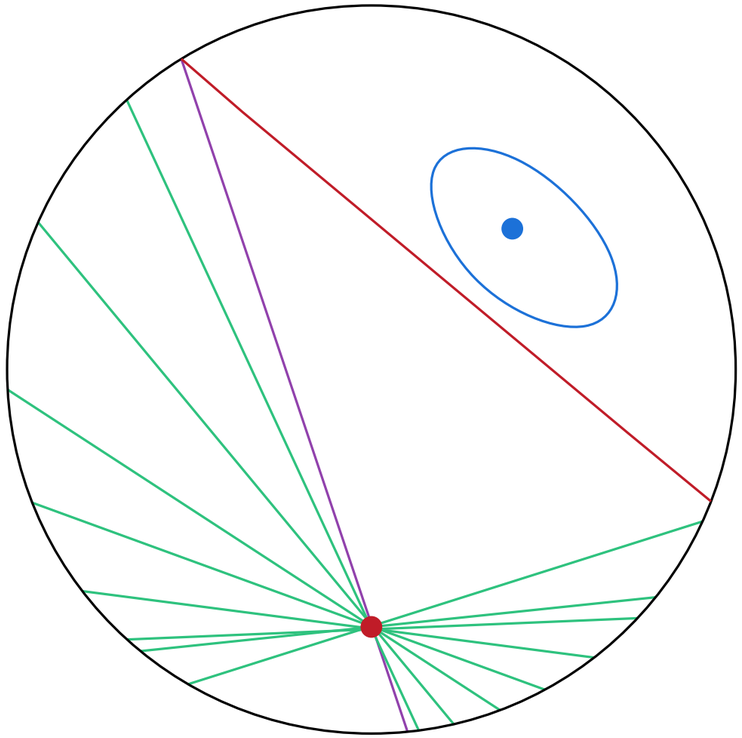


Infinite number of parallel lines to the red one passing through a point

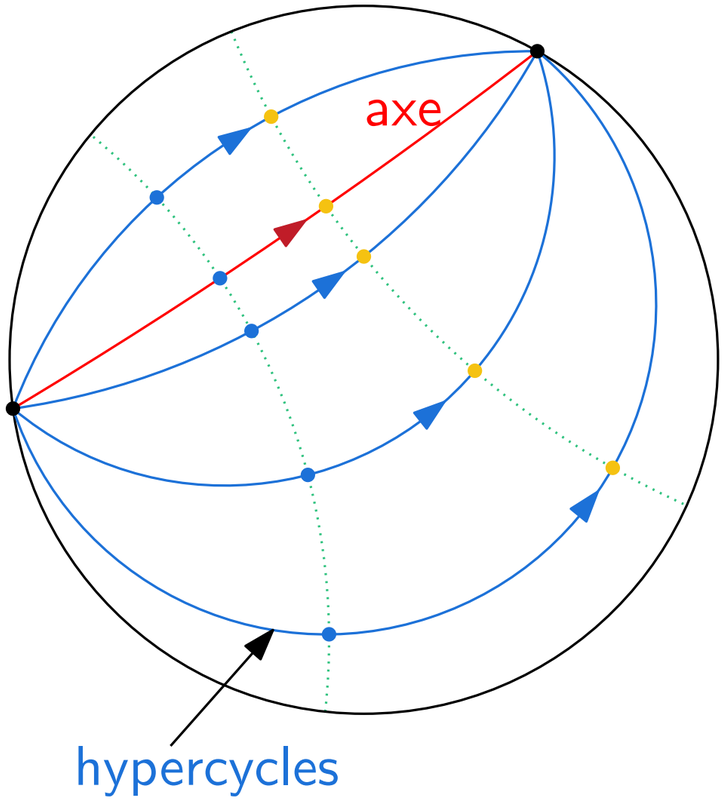


Triangles in hyperbolic plane

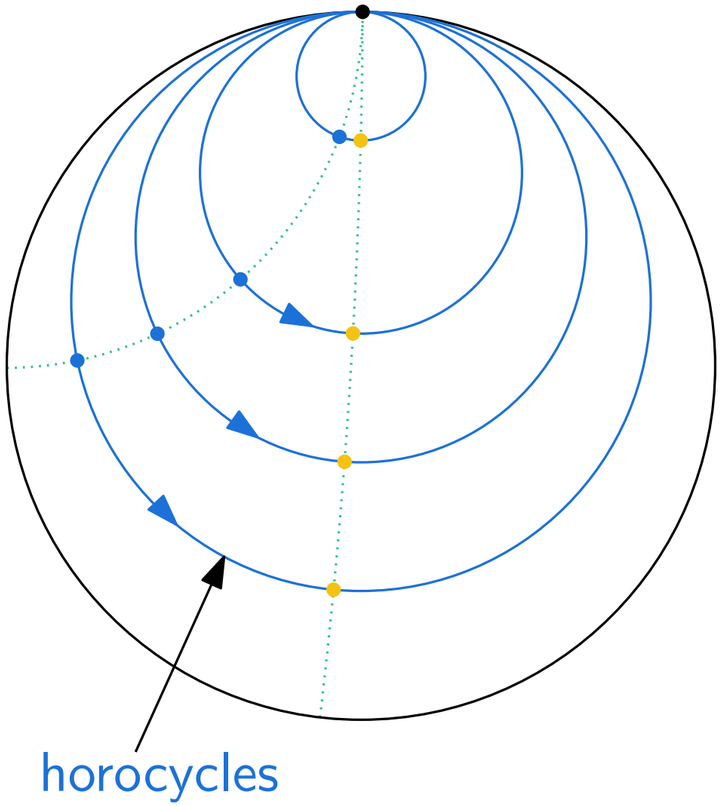
Prop : $\mathrm{Biholom}(D) = \{z \mapsto \lambda \frac{\rho - z}{1 - \overline{\rho} z} \mid \lambda \in U, \rho \in D\}$

3 types of orientation-preserving isometries.

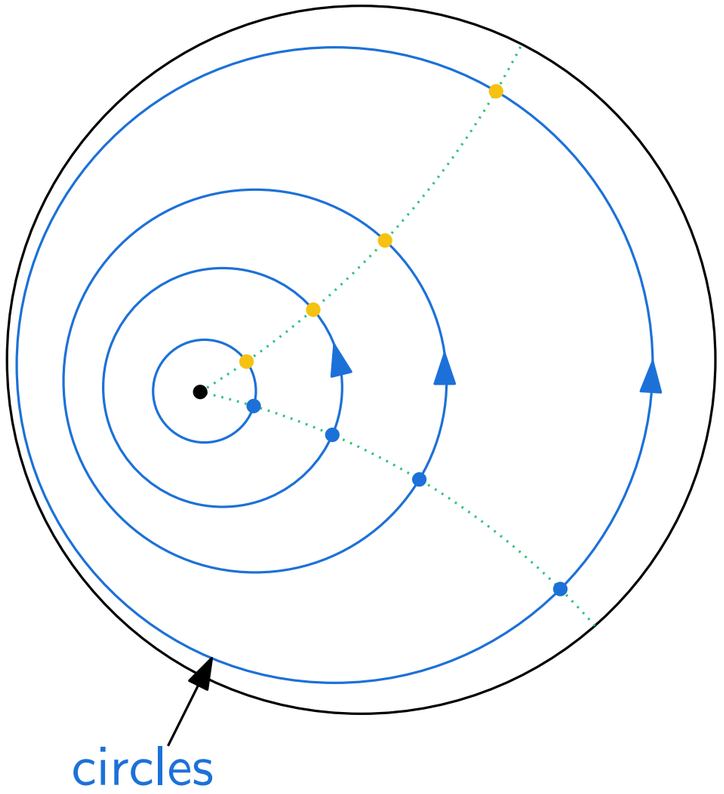
Hyperbolique



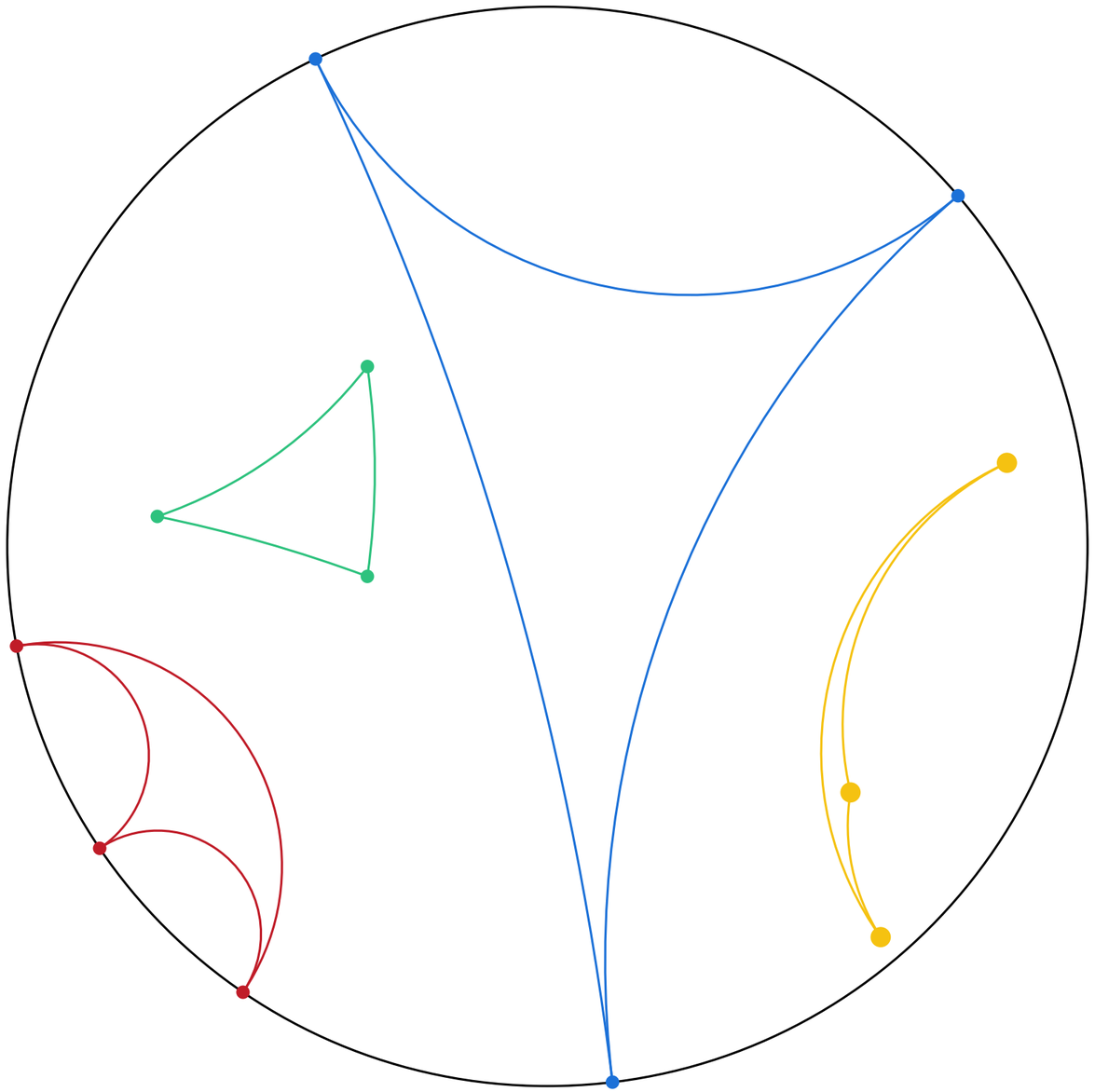
Parabolique



Elliptique

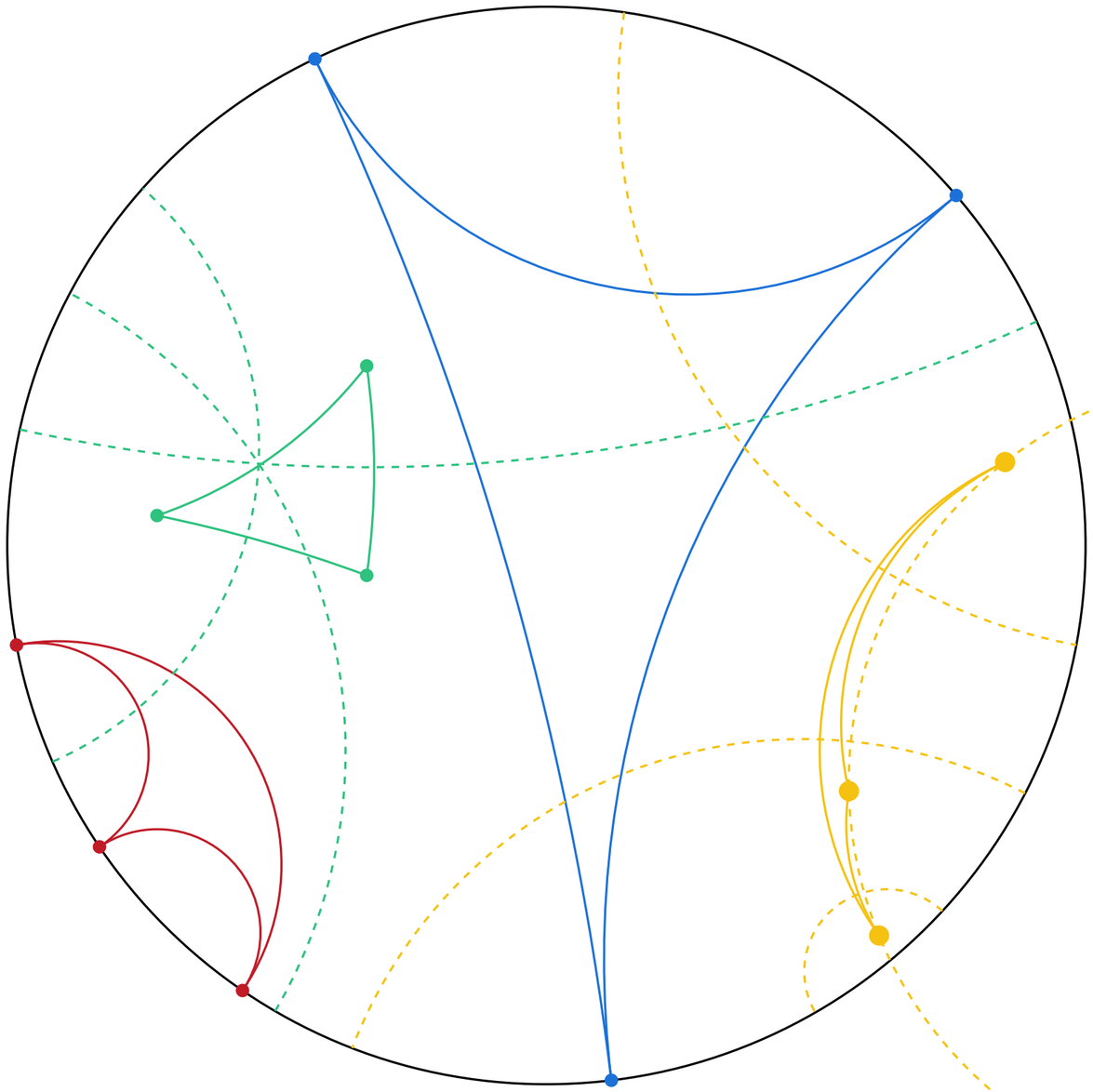


Triangles in hyperbolic plane



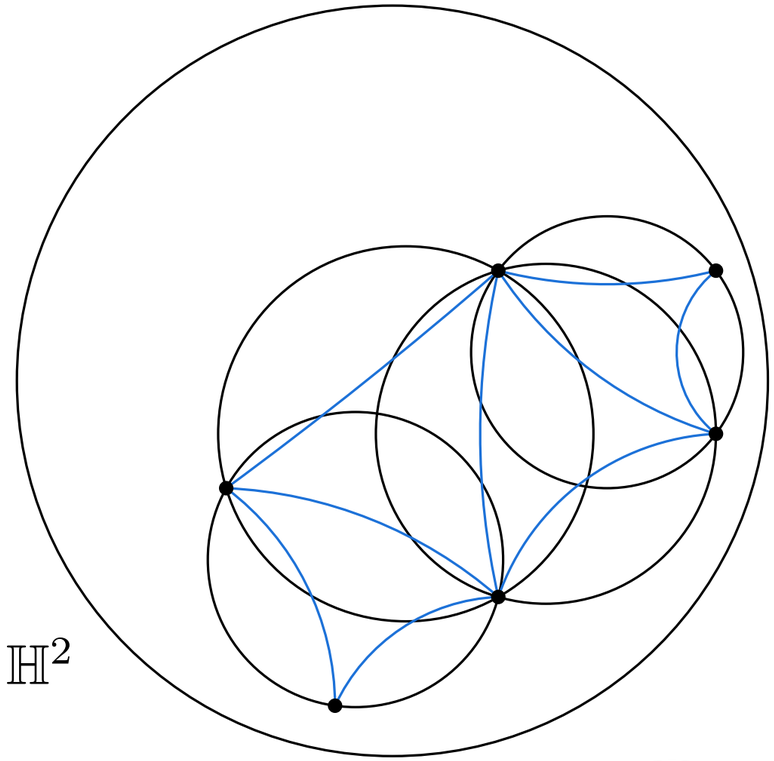
- Area(triangle) $= \pi - \sum$ angles

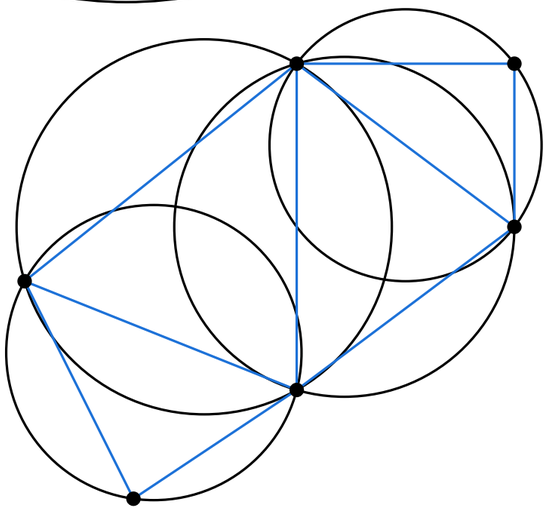
Triangles in hyperbolic plane



- Area(triangle) $= \pi - \sum$ angles
- Circumcircle may not exist!

Triangles in hyperbolic plane





Plane

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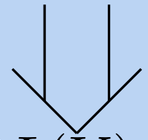
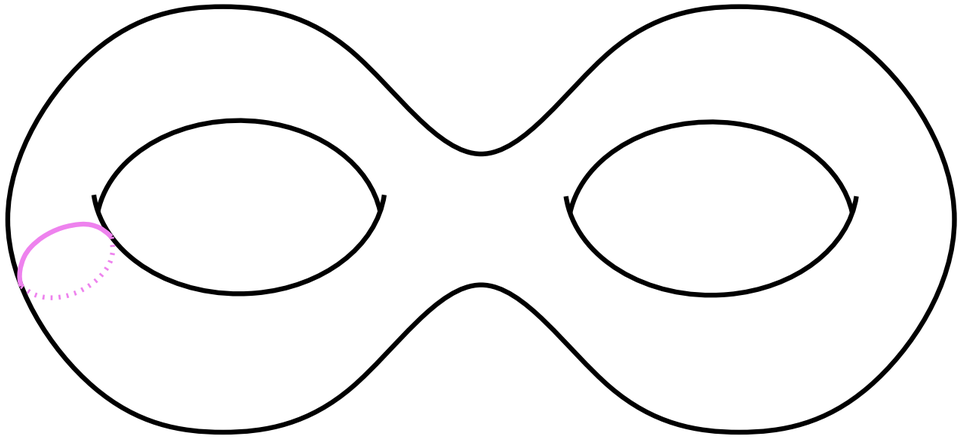
Existence of $DL(V)$ in Euclidean plane



Existence of $DL(V)$ in hyperbolic plane

- ▷ Delaunay triangulation in Euclidean plane
- ▷ BW-algorithm on a torus ($g = 1$)
- ▷ Hyperbolic plane
- ▶ BW-algorithm on the Bolza surface ($g = 2$)
- ▷ BW-algorithm on hyperbolic surface ($g \geq 2$)

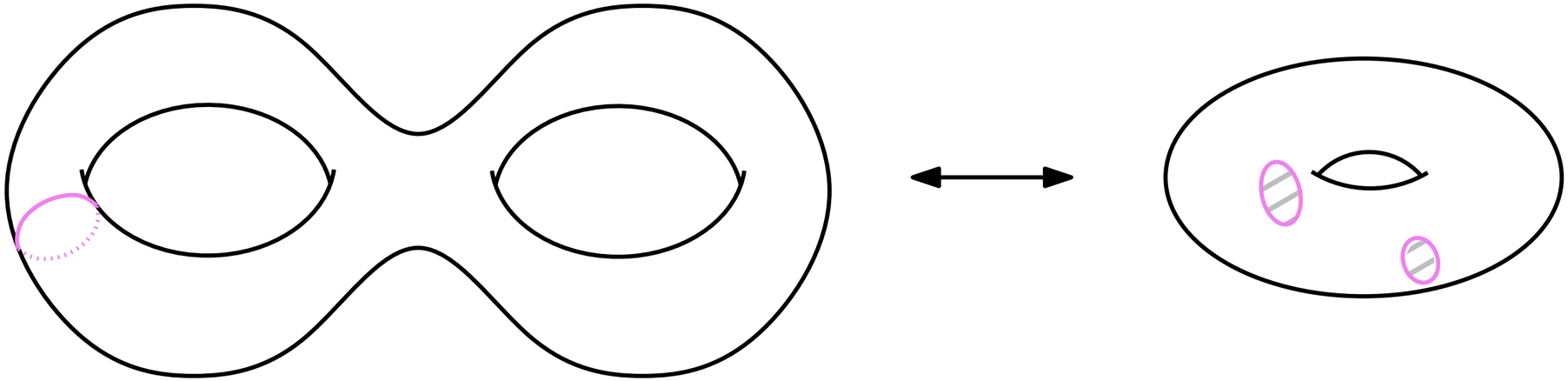
Unfolding the Bolza surface



hyperbolic surface: a surface endowed with a Riemannian metric of constant negative curvature.

Bolza surface: a hyperbolic surface with the simplest possible topology (genus 2).

Unfolding the Bolza surface



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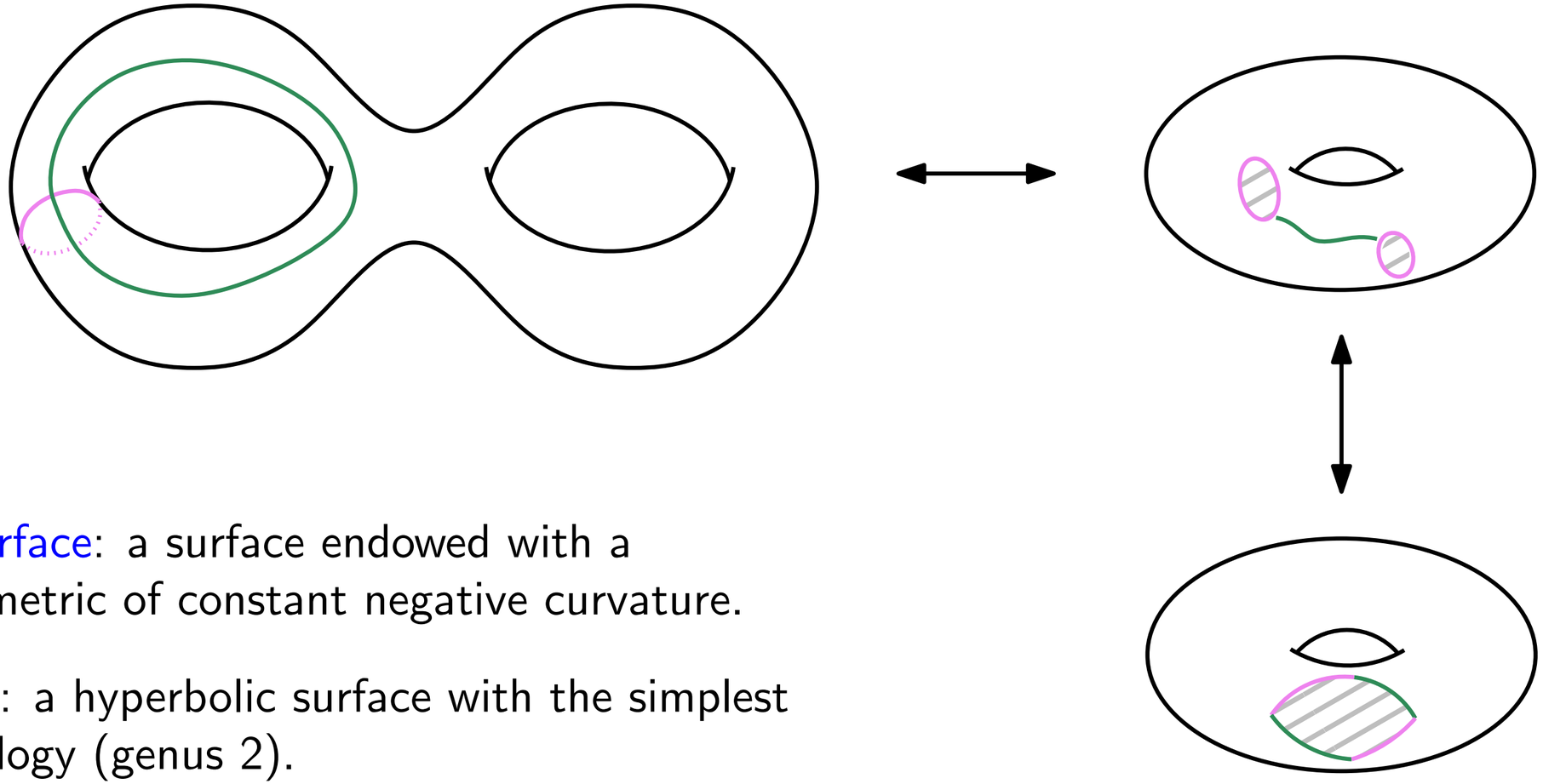

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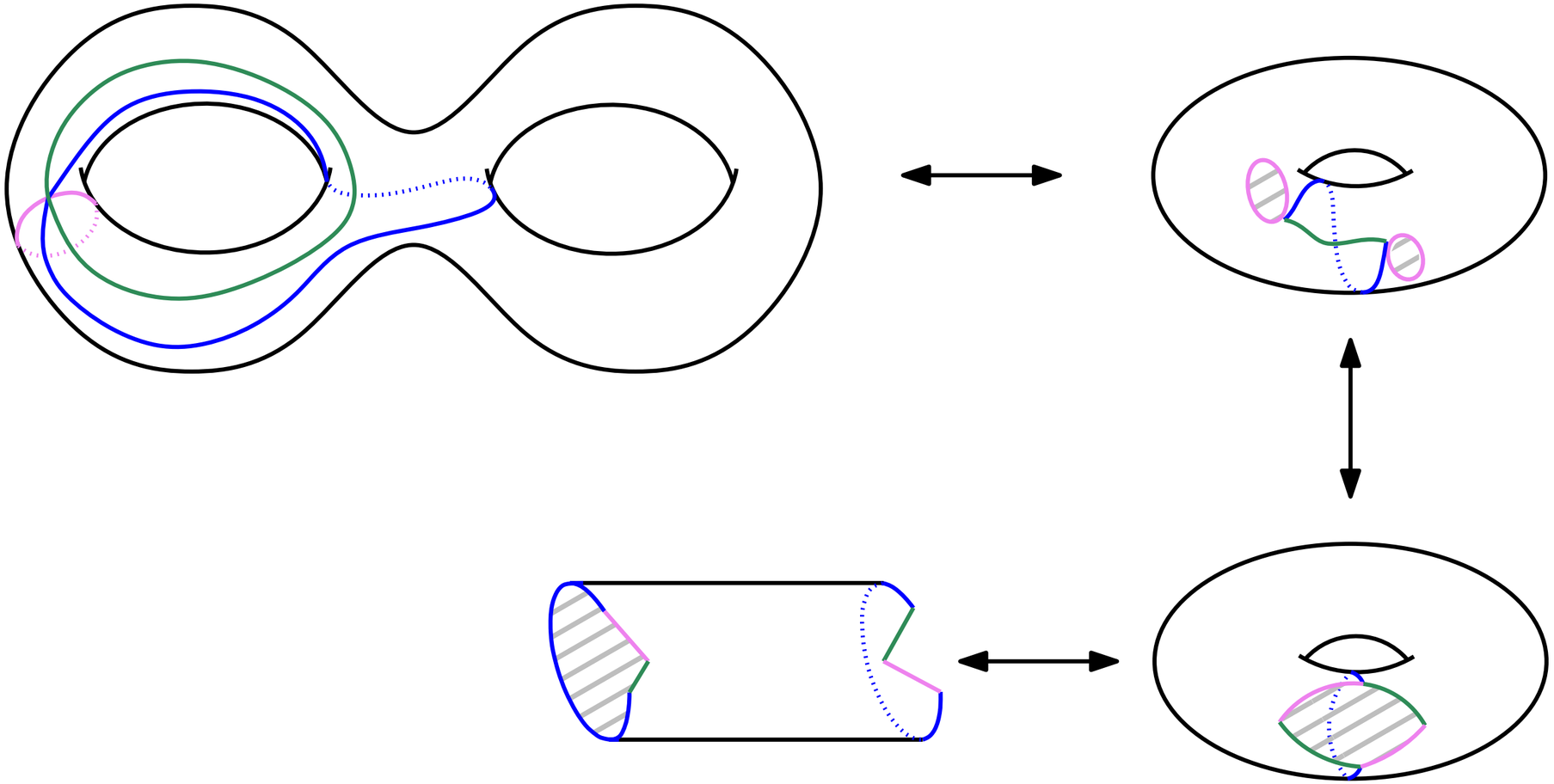

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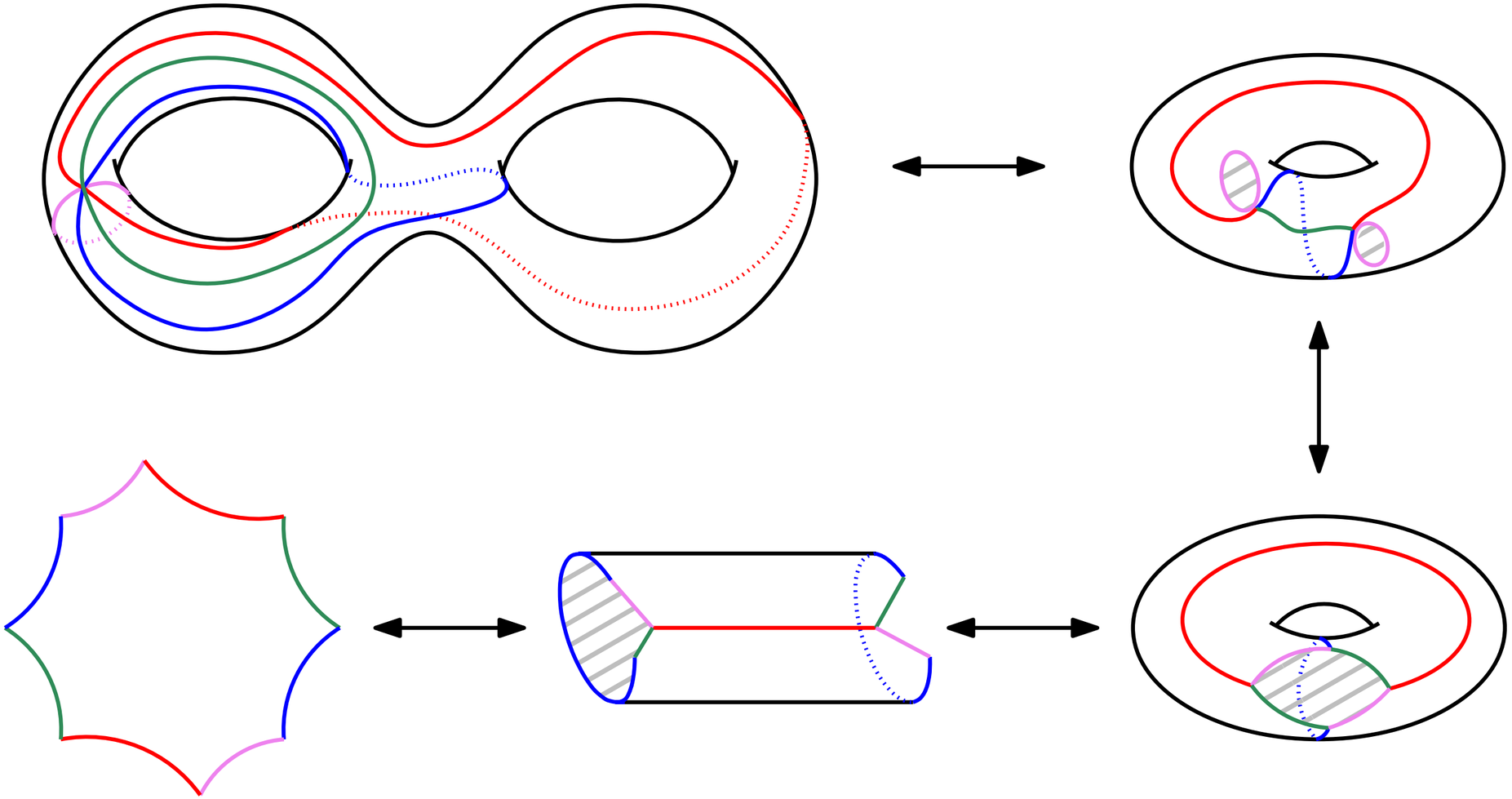
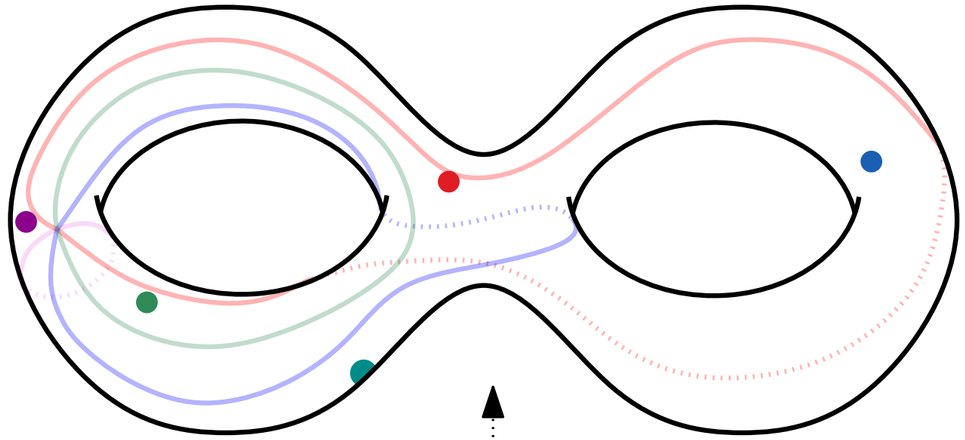
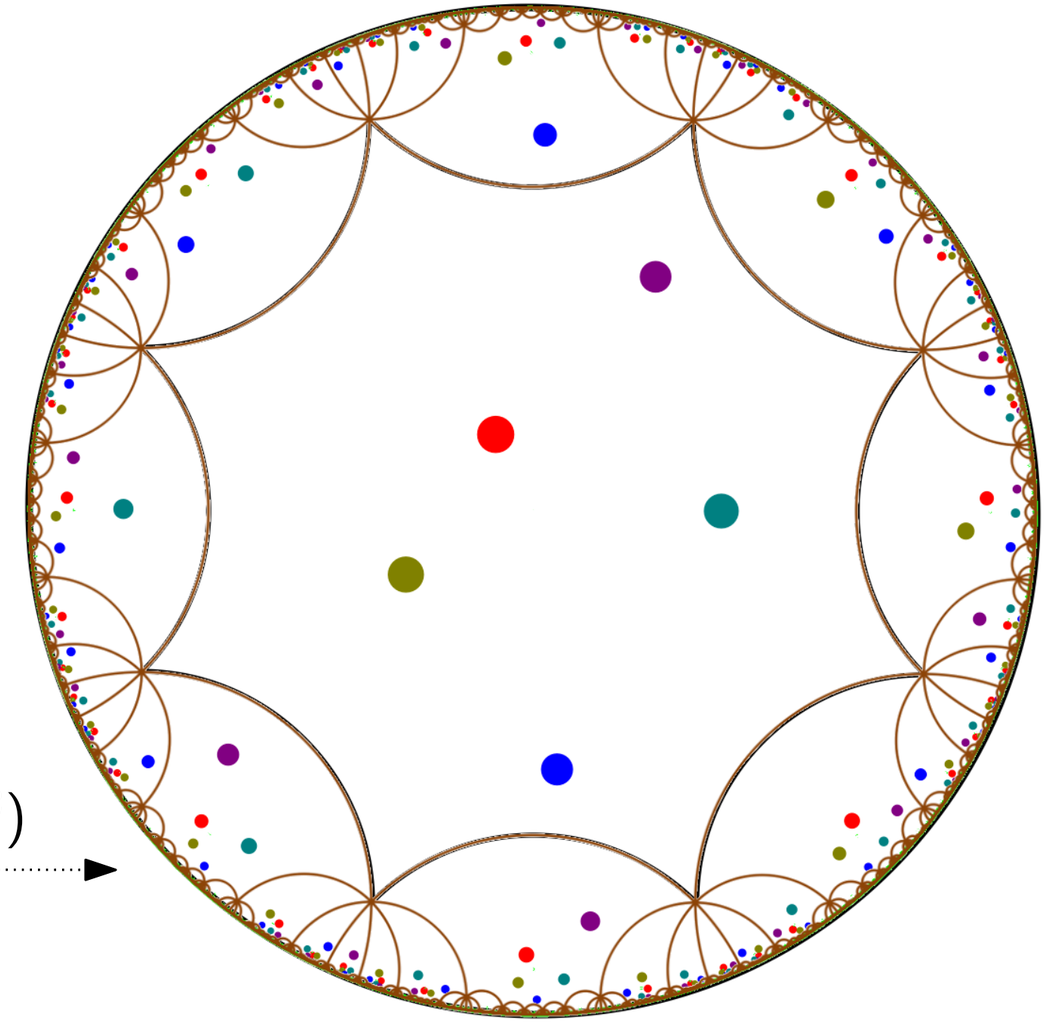


Fundamental polygon

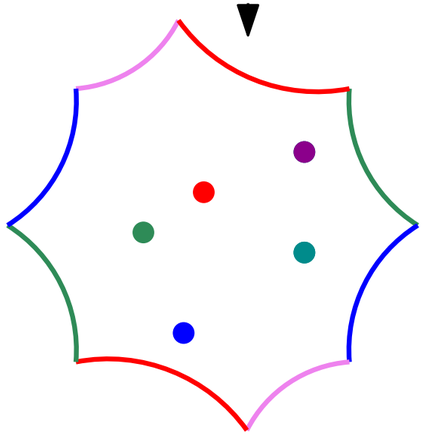
Unfolding the Bolza surface



Universal cover



"Unfolding" the surface
(Fundamental polygon)



Hyperbolic plane ($\mathbb{H}^2$)

tesselation

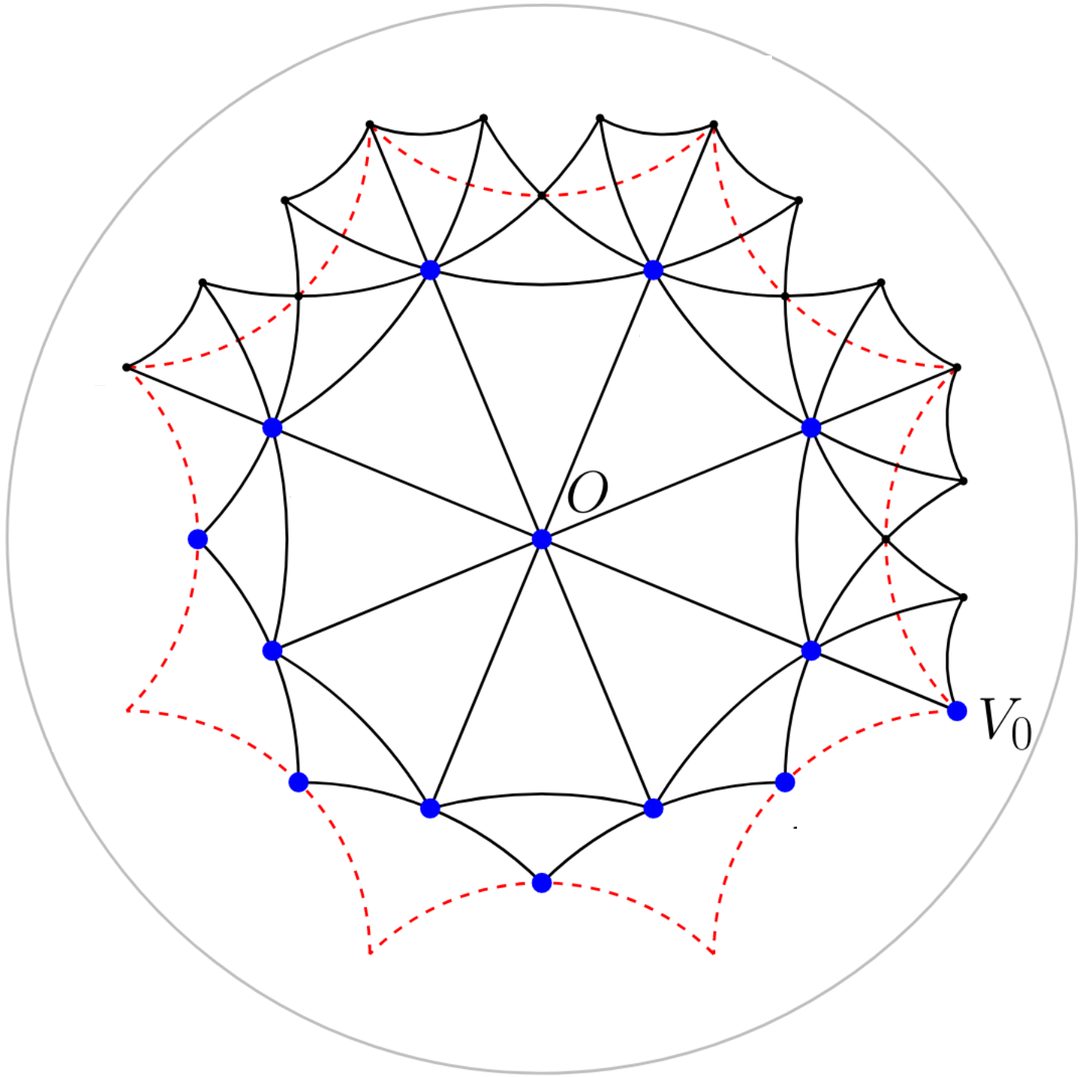
BW-algorithm on the Bolza surface

Results in [IT17]:

Add 14 dummy points such that $sys(M) > 2\delta$ where δ is the radius of the largest disks that do not contain any dummy point.

Use

- Knowledge of the systole
- A good fondamental domain
- Symetries of the surfaces



Summary of BW-algorithm on surface

Keys ideas:

- Initialization by adding points
- Use the universal cover -plane or $\mathbb{H}^2$-

For Torus and Bolza we used:

- Knowledge of the systole
- A good fondamental domain
- Symetries of the surfaces

For any surface S of genus ≥ 2:

- Initialization by adding points
- Use the universal cover $\mathbb{H}^2$

Adapted settings:

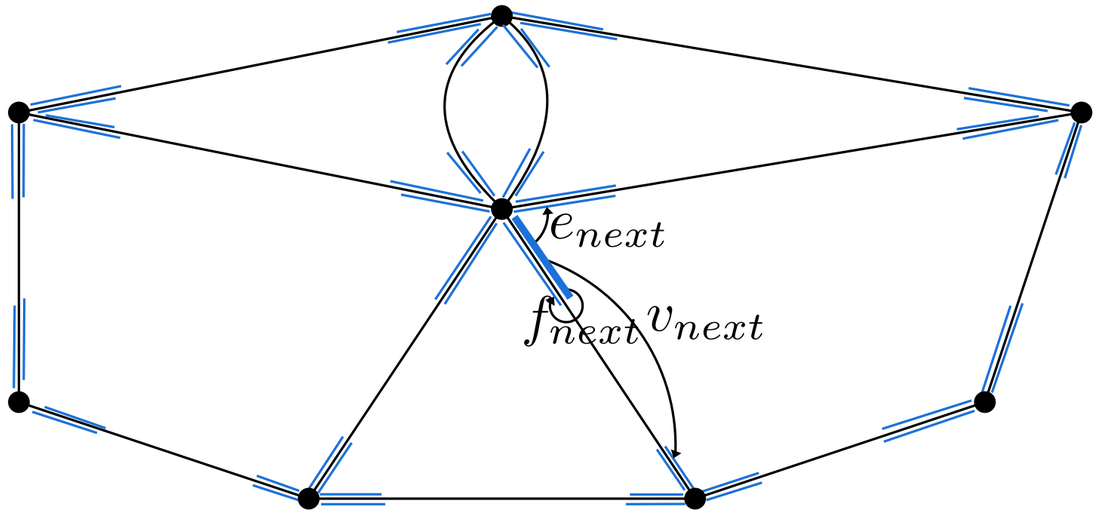
- Decompose S in "thin" and "thick" parts
- Compute a Dirichlet domain
- Use points satisfying a "density" property.

- ▷ Delaunay triangulation in Euclidean plane
- ▷ BW-algorithm on a torus ($g = 1$)
- ▷ Hyperbolic plane
- ▷ BW-algorithm on the Bolza surface ($g = 2$)
- ▶ BW-algorithm on the Bolza surface ($g = 2$)

Data structure

A Flag is a triple (v, e, f) (a vertex, an edge, a face)

Combinatorial map = set of flags + 3 involutions $(v_{next}, e_{next}, f_{next})$



- v_{next} change only the vertex incidence.
- e_{next} change only the edge incidence.
- f_{next} change only the face incidence.

Data structure

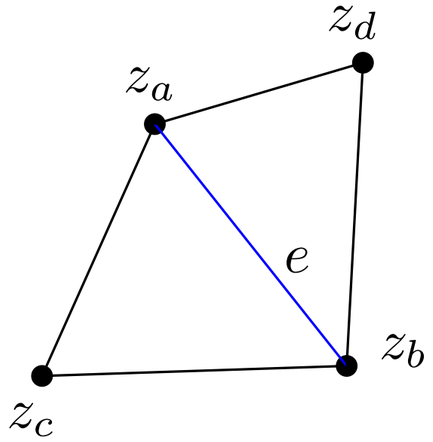
A Flag is a triple (v, e, f) (a vertex, an edge, a face)

Combinatorial map = set of flags + 3 involutions $(v_{next}, e_{next}, f_{next})$

To encode the geometry, we attach at each edge e a complex number ζ_e.to reconstruct the triangles.

1) Fix a triangle $z_a z_c z_b$.

2) Glue the triangle $z_a z_b z_d$ such that $\frac{(z_c - z_a)(z_d - z_b)}{(z_c - z_b)(z_d - z_a)} = \zeta_{[z_a, z_b]}$.



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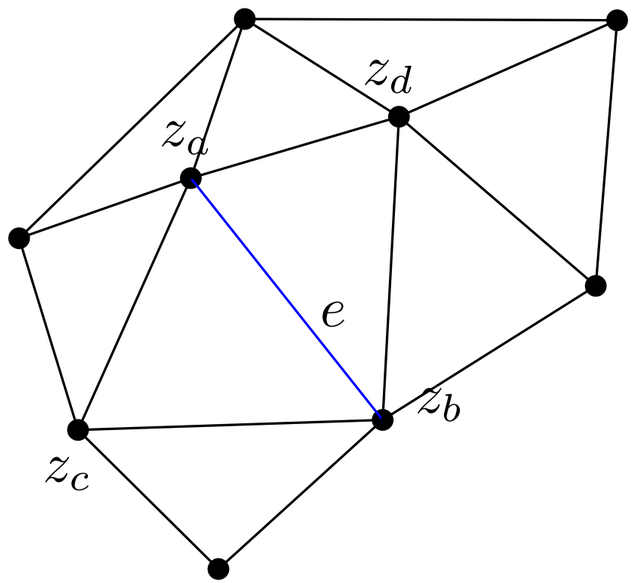
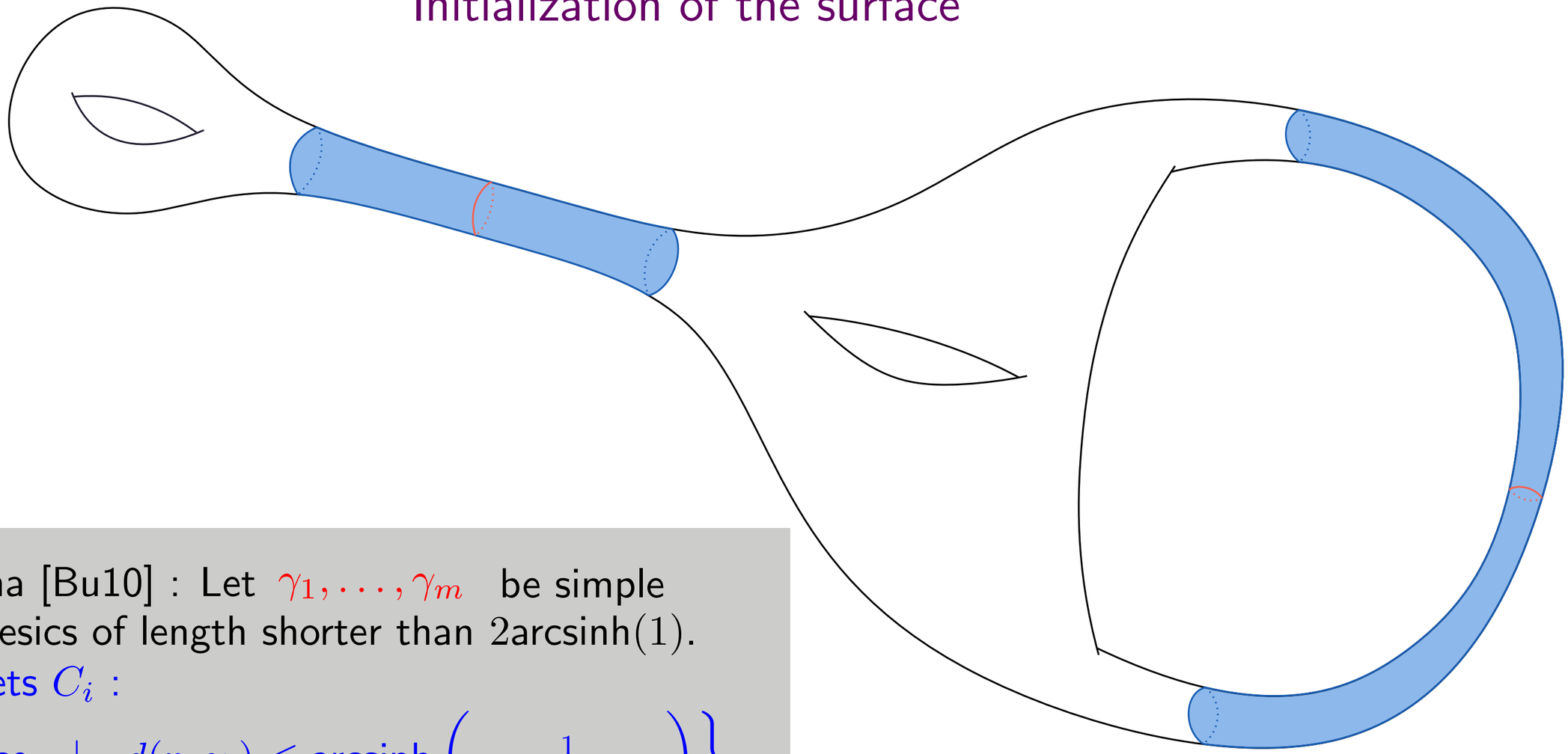
3) And so on...



Initialization of the surface

Initialization of the surface

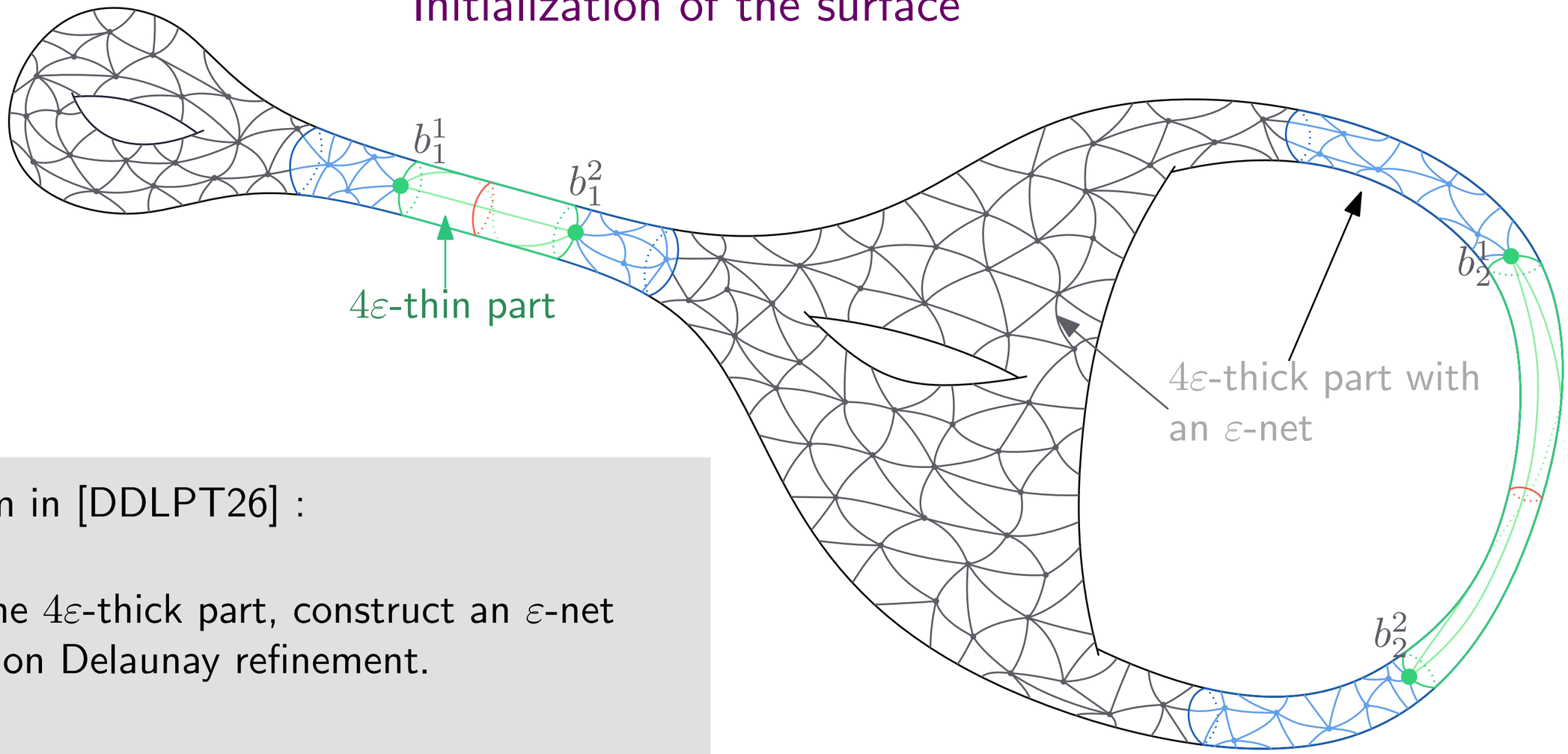


Collar lemma [Bu10] : Let $\gamma_1, \dots, \gamma_m$ be simple closed geodesics of length shorter than $2\text{arcsinh}(1)$. Then the sets C_i :

$$\left\{ p \in \text{Surface} \quad | \quad d(p, \gamma_i) \leq \text{arcsinh}\left(\frac{1}{\sinh\left(\frac{1}{2} l(\gamma_i)\right)} \right) \right\}$$

are pairwise disjoint and homeomorphic to a cylinder.

Initialization of the surface

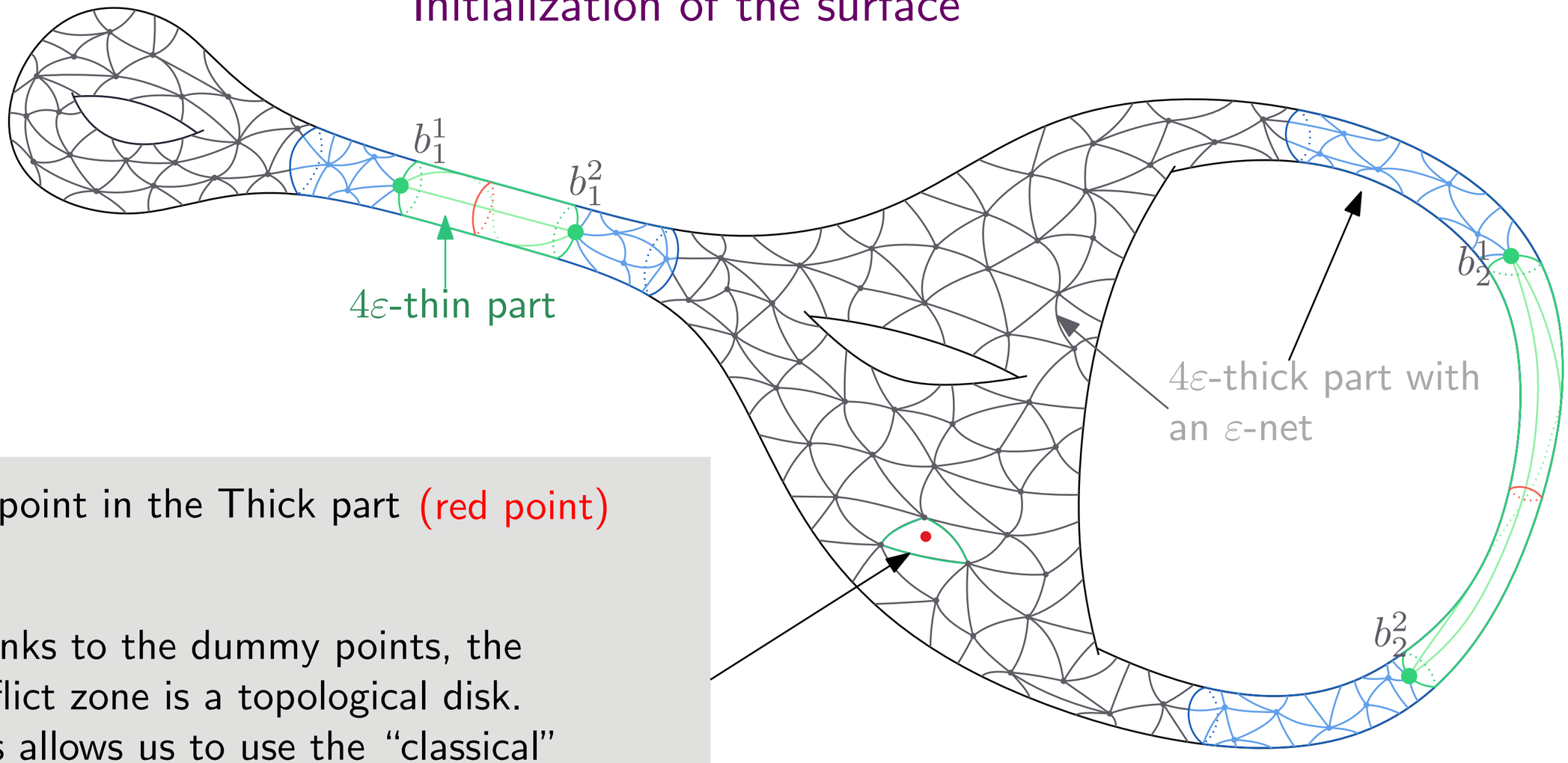


Algorithm in [DDLPT26] :

- In the 4ε-thick part, construct an ε-net based on Delaunay refinement.

- Detect the collars (4ε-thin parts) and place points $b_{1,2}^i$ on their borders.

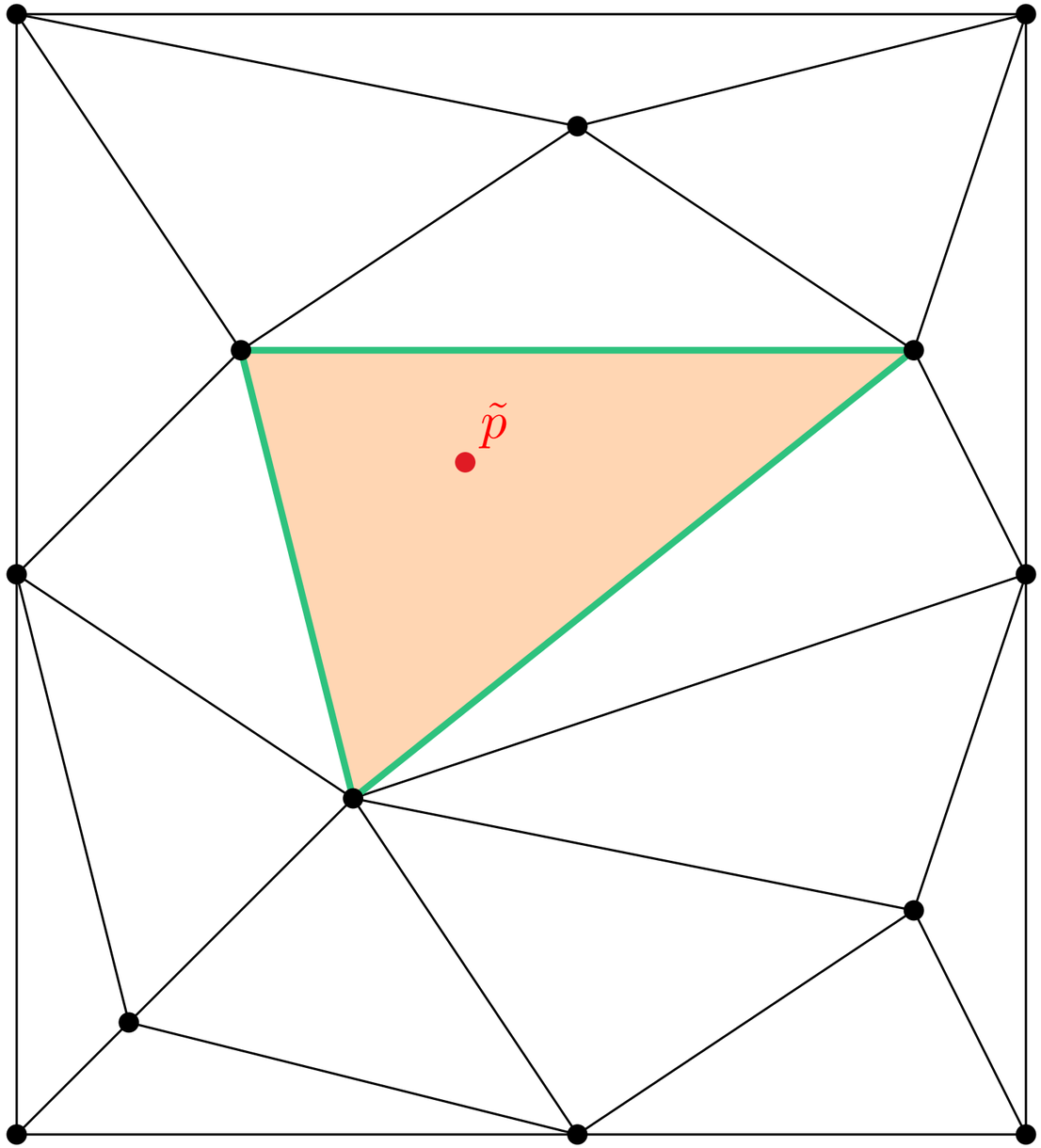
Initialization of the surface



Insert a point in the Thick part (red point)

Thanks to the dummy points, the conflict zone is a topological disk. This allows us to use the "classical" BW-algorithm.

Insertion in 4ε-thick part

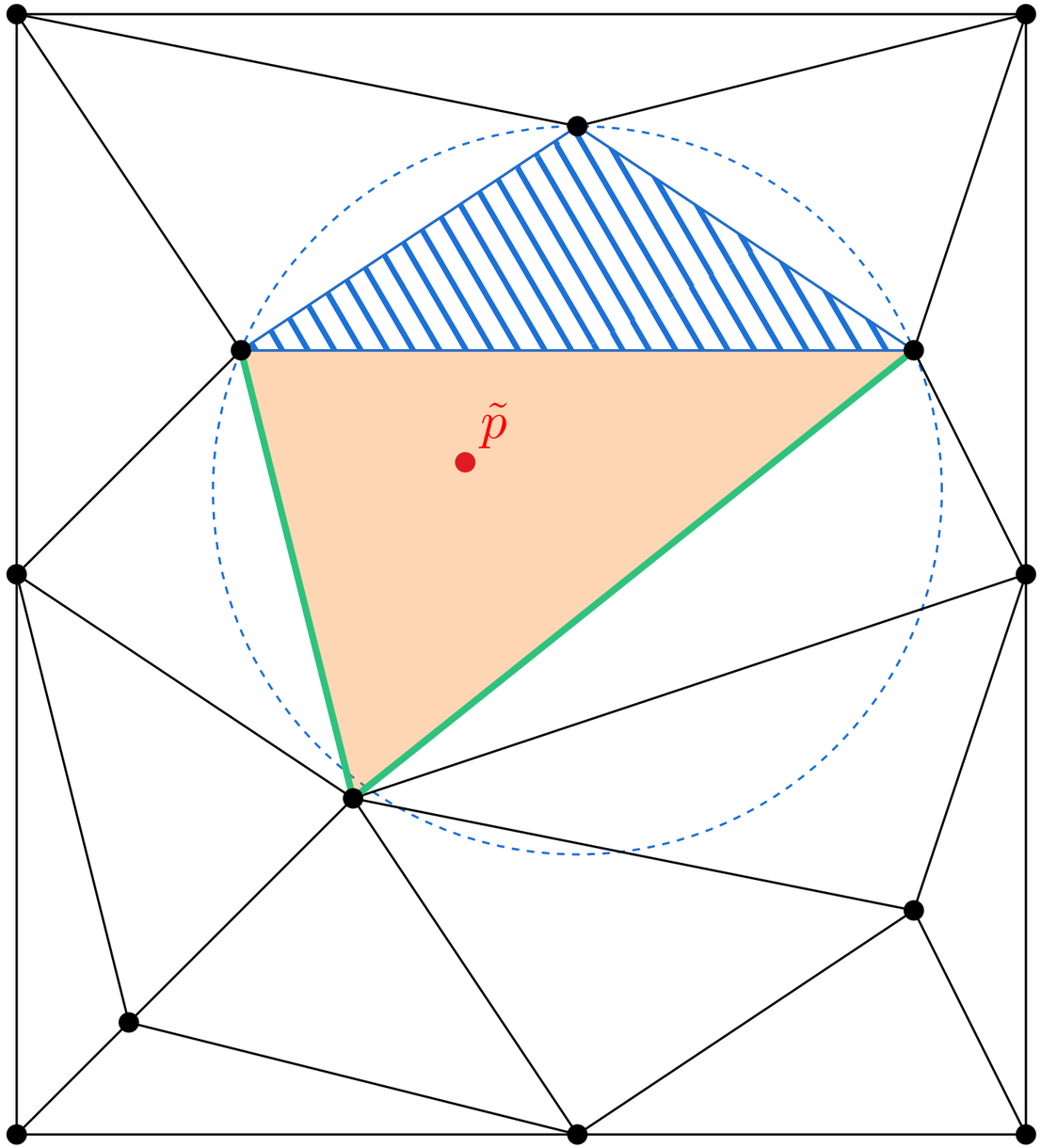


Processing in the universal cover

→ Find a triangle which contains $\widetilde{p}$.

→ Start a DFS to find conflict zone.

Insertion in 4ε-thick part

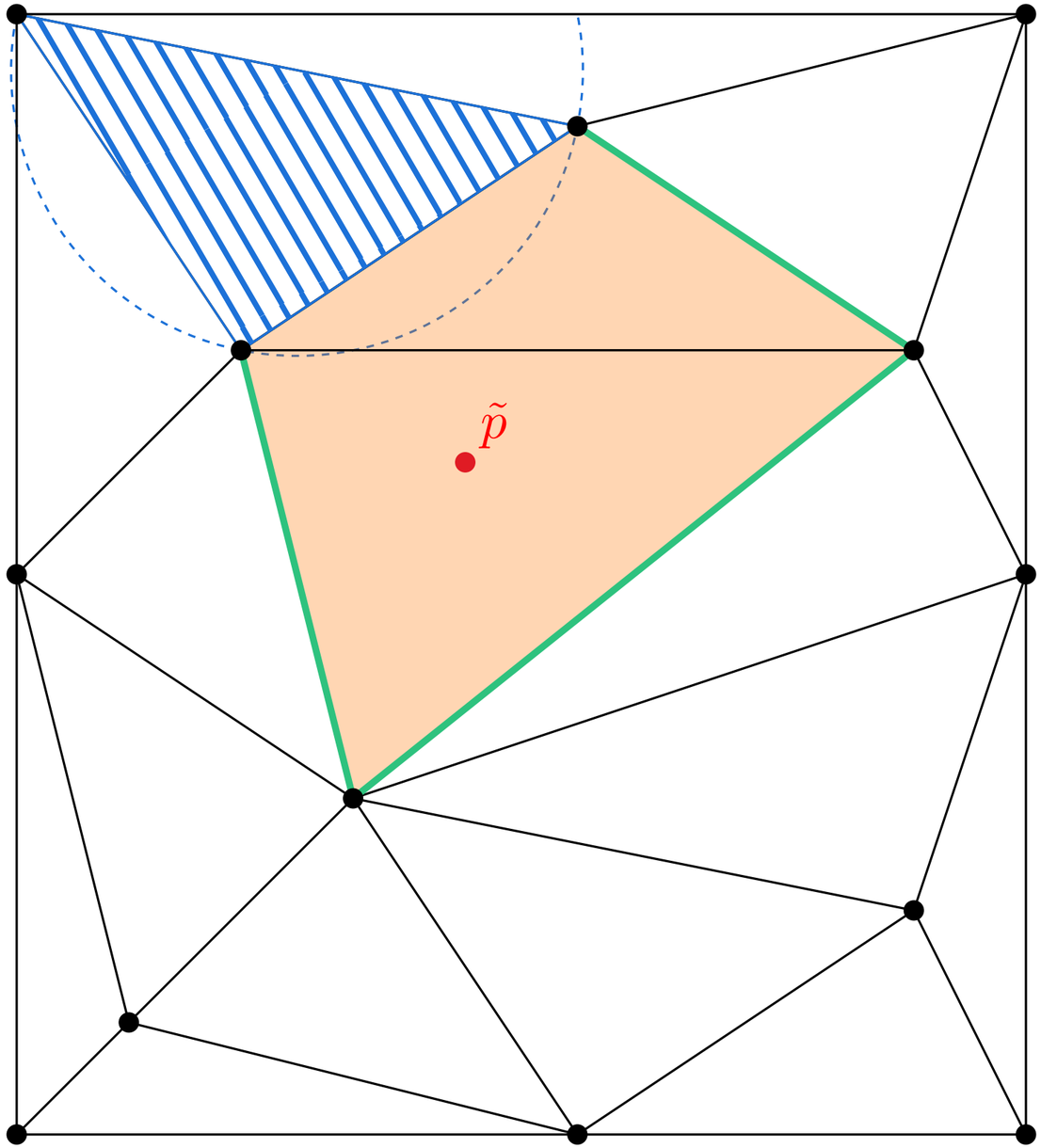


Processing in the universal cover

$\longrightarrow$ Conflict with $\widetilde{p}$.

$\longrightarrow$ Push the two other sides in the stack.

Insertion in 4ε-thick part

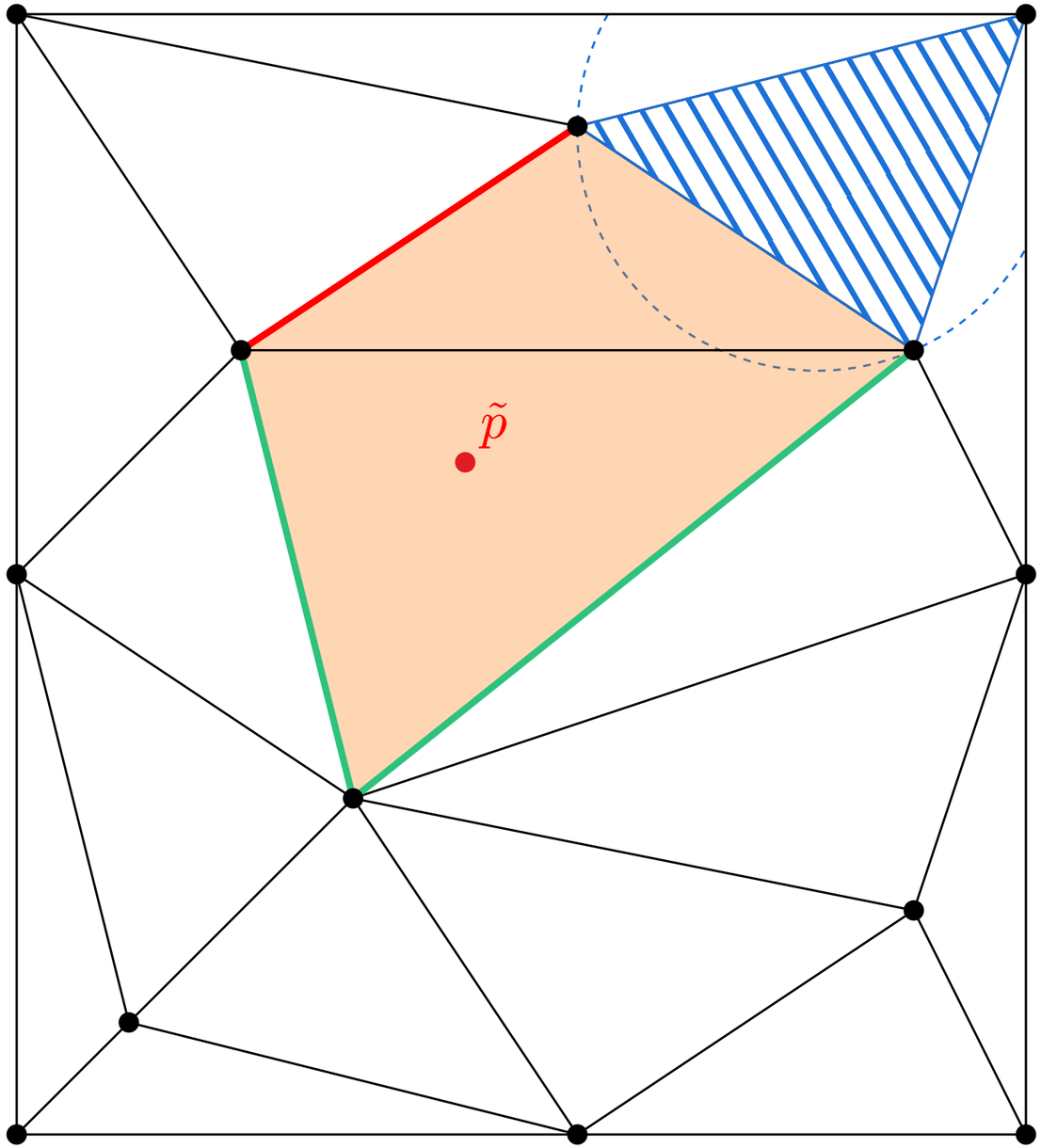


Processing in the universal cover

→ Not in conflict with $\widetilde{p}$.

→ The side in commun with the conflict zone belongs to the border.

Insertion in 4ε-thick part



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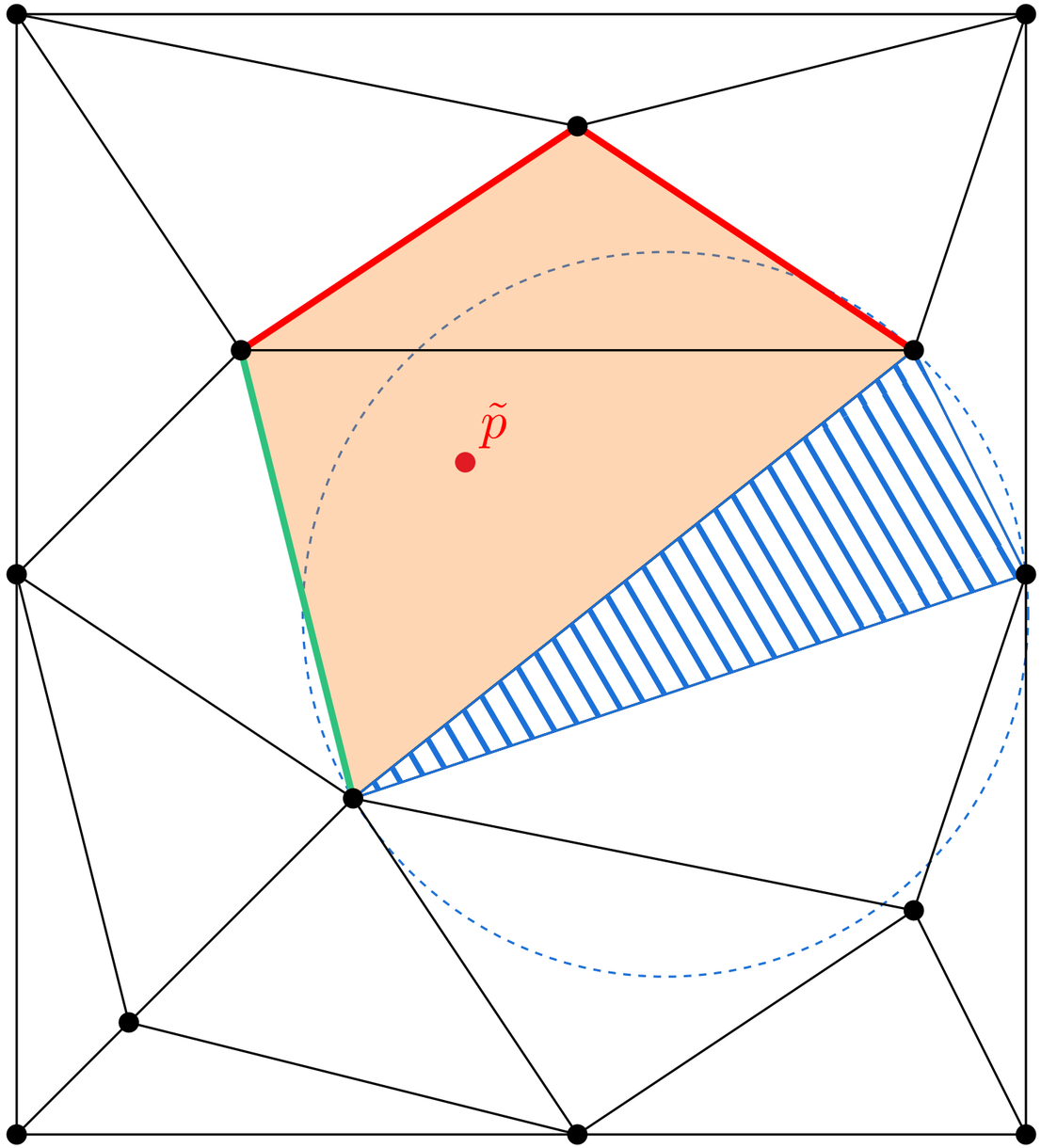

Processing in the universal cover

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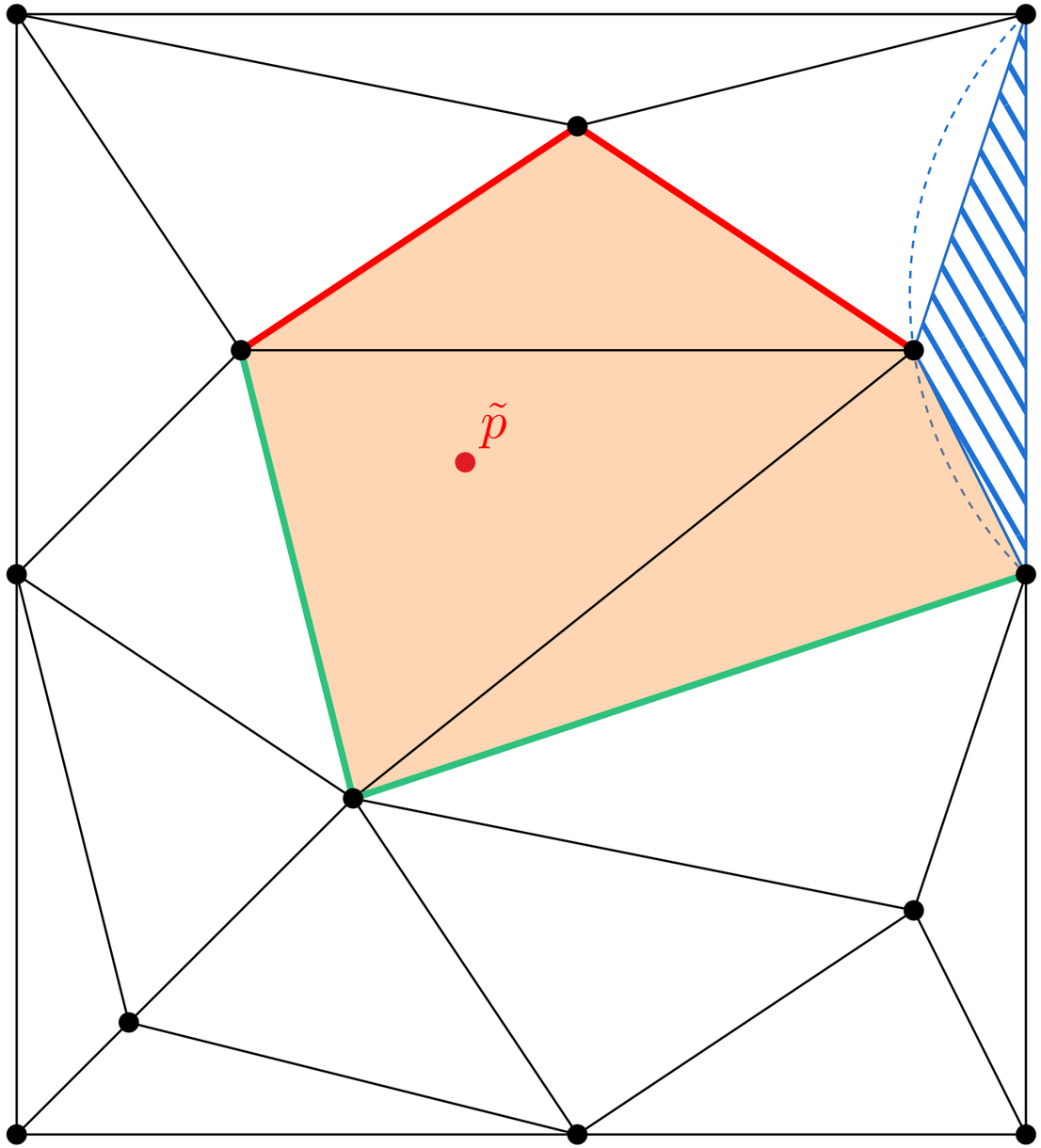

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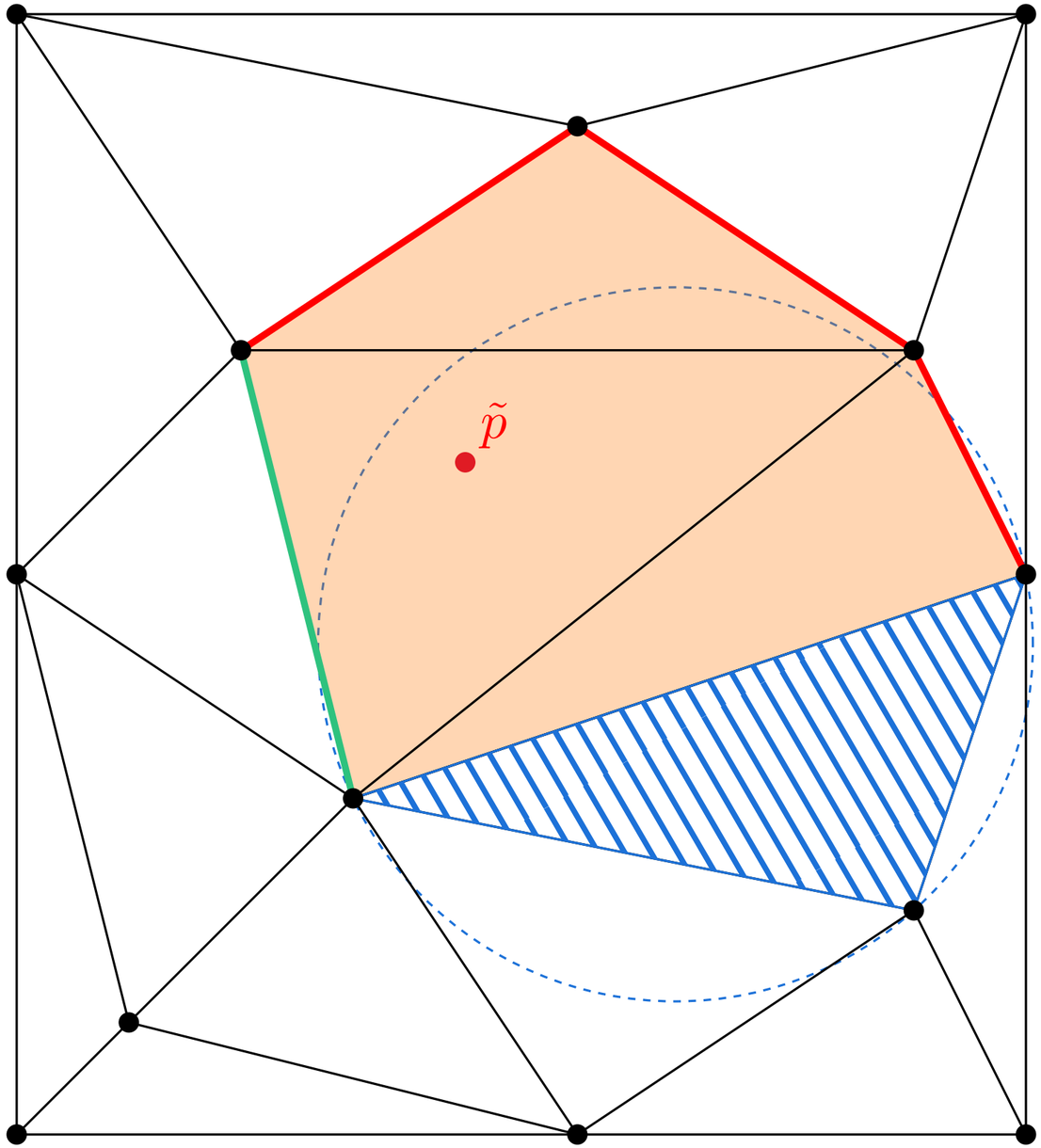

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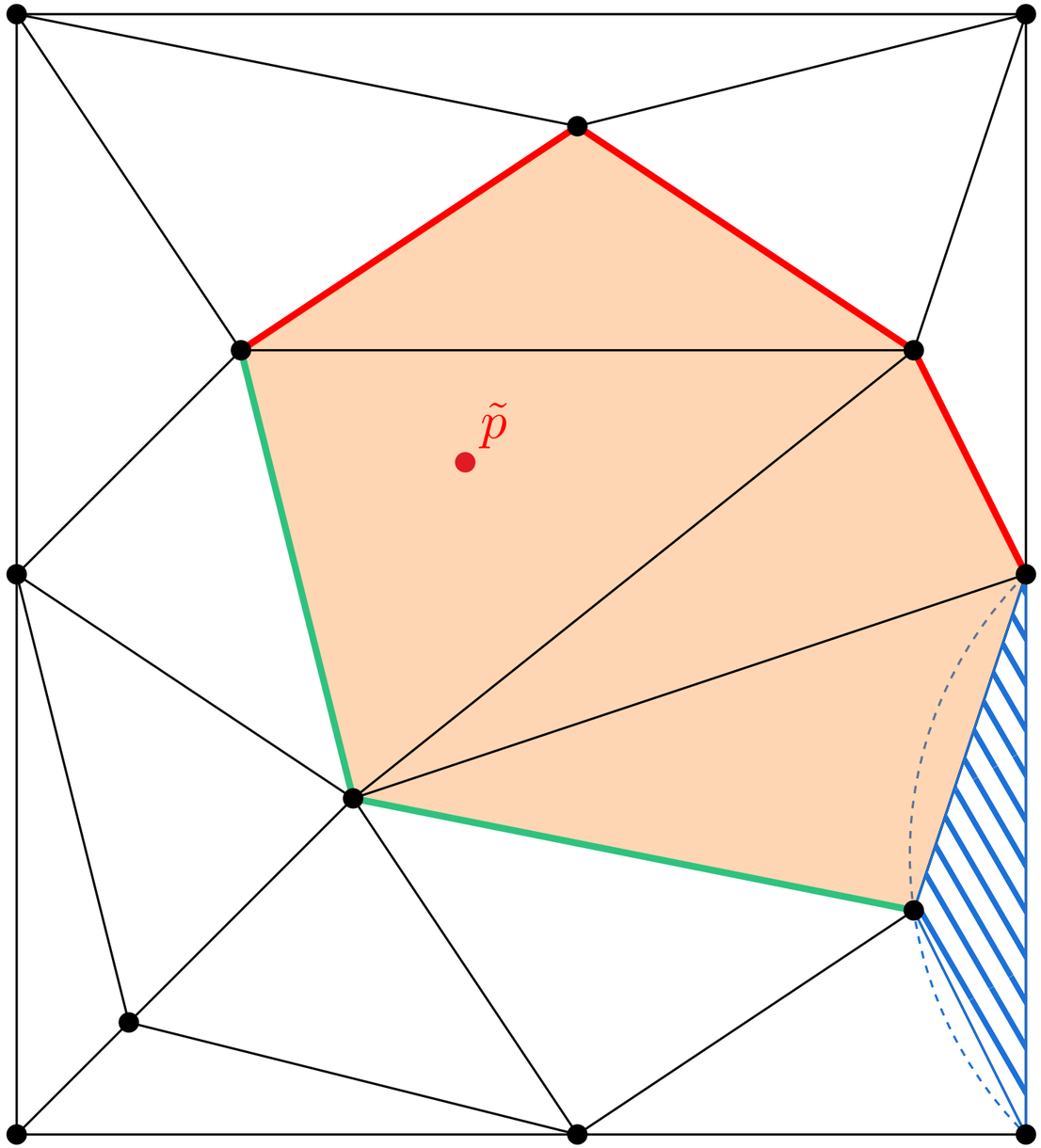

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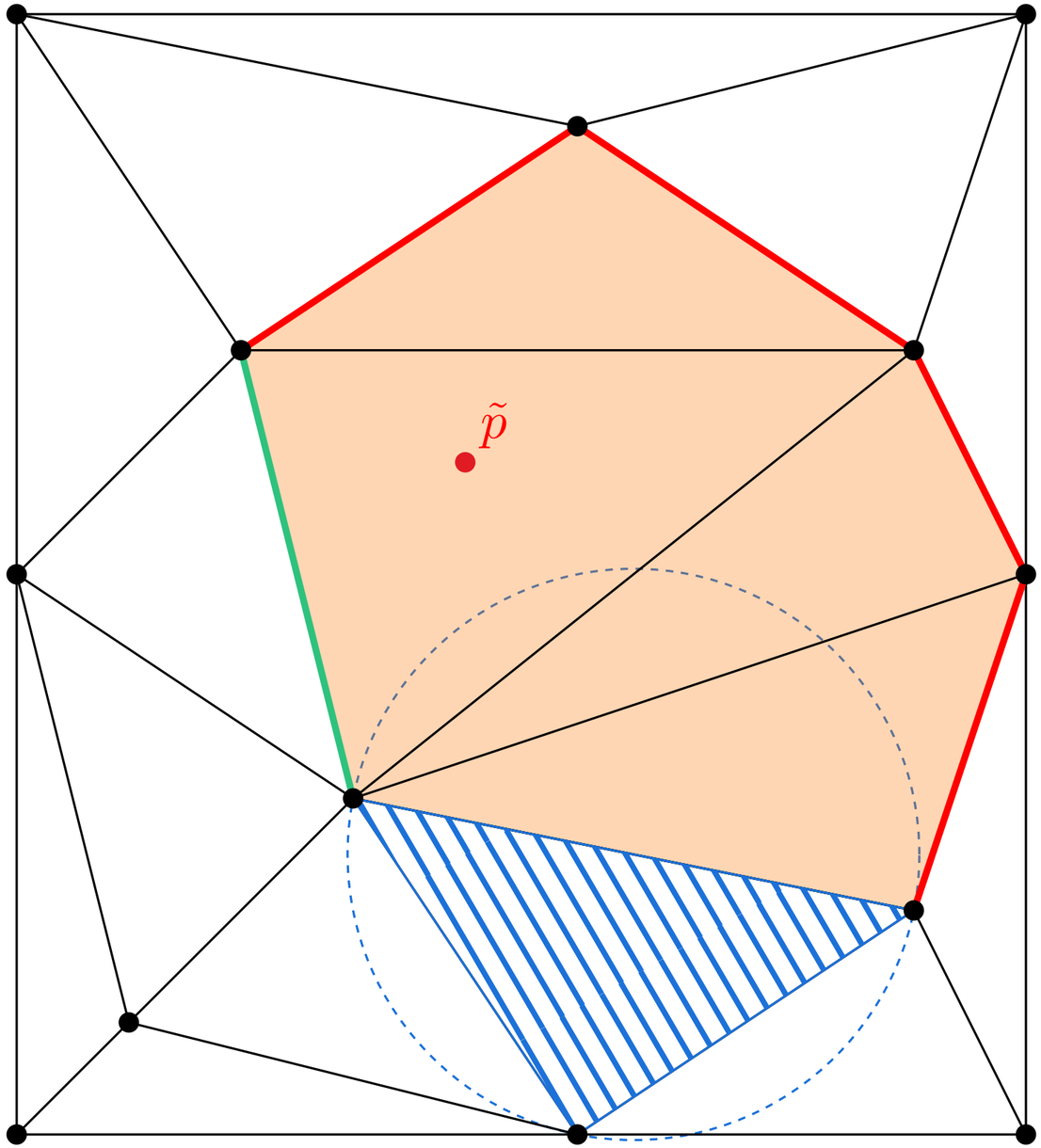

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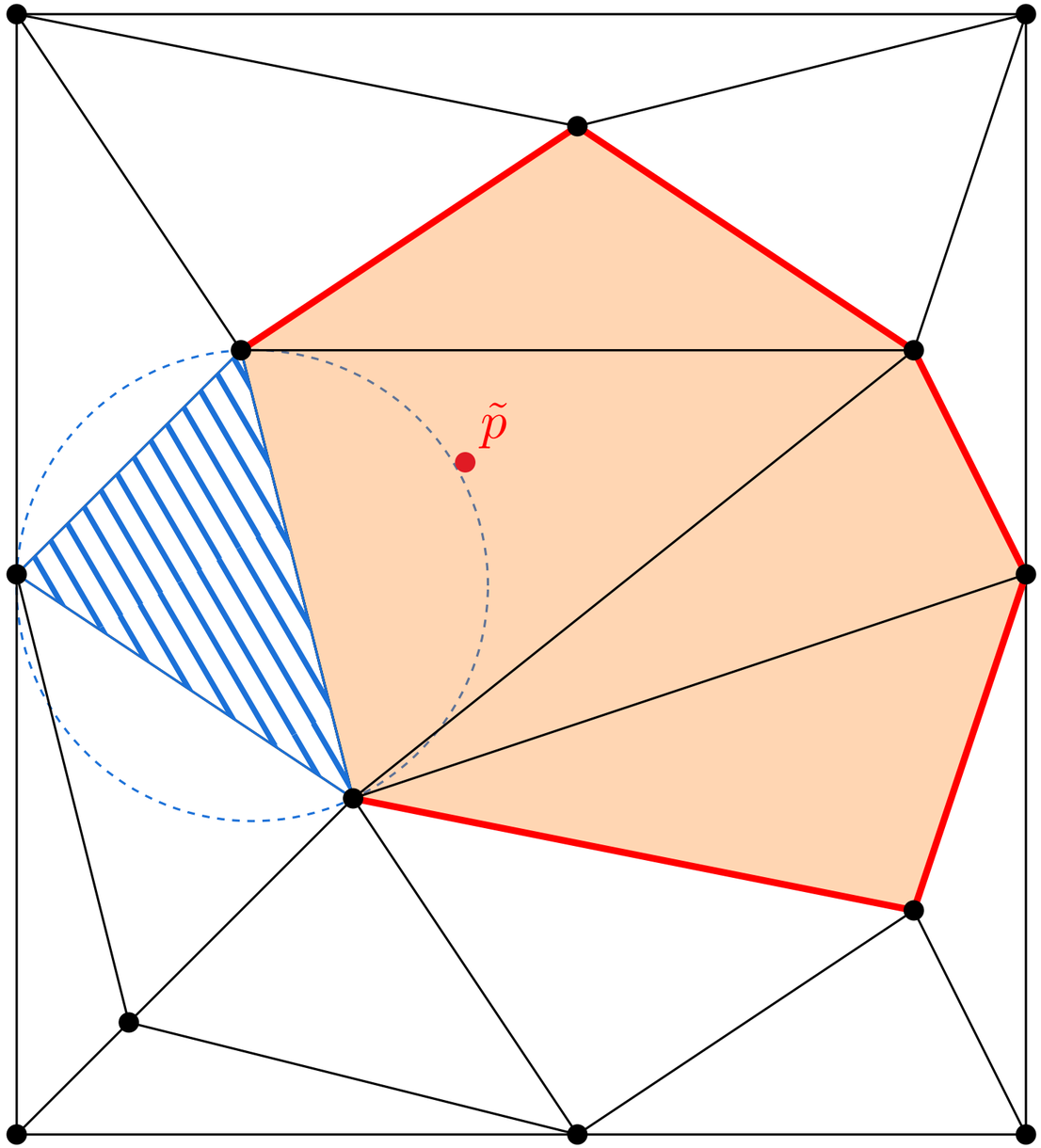

Processing in the universal cover

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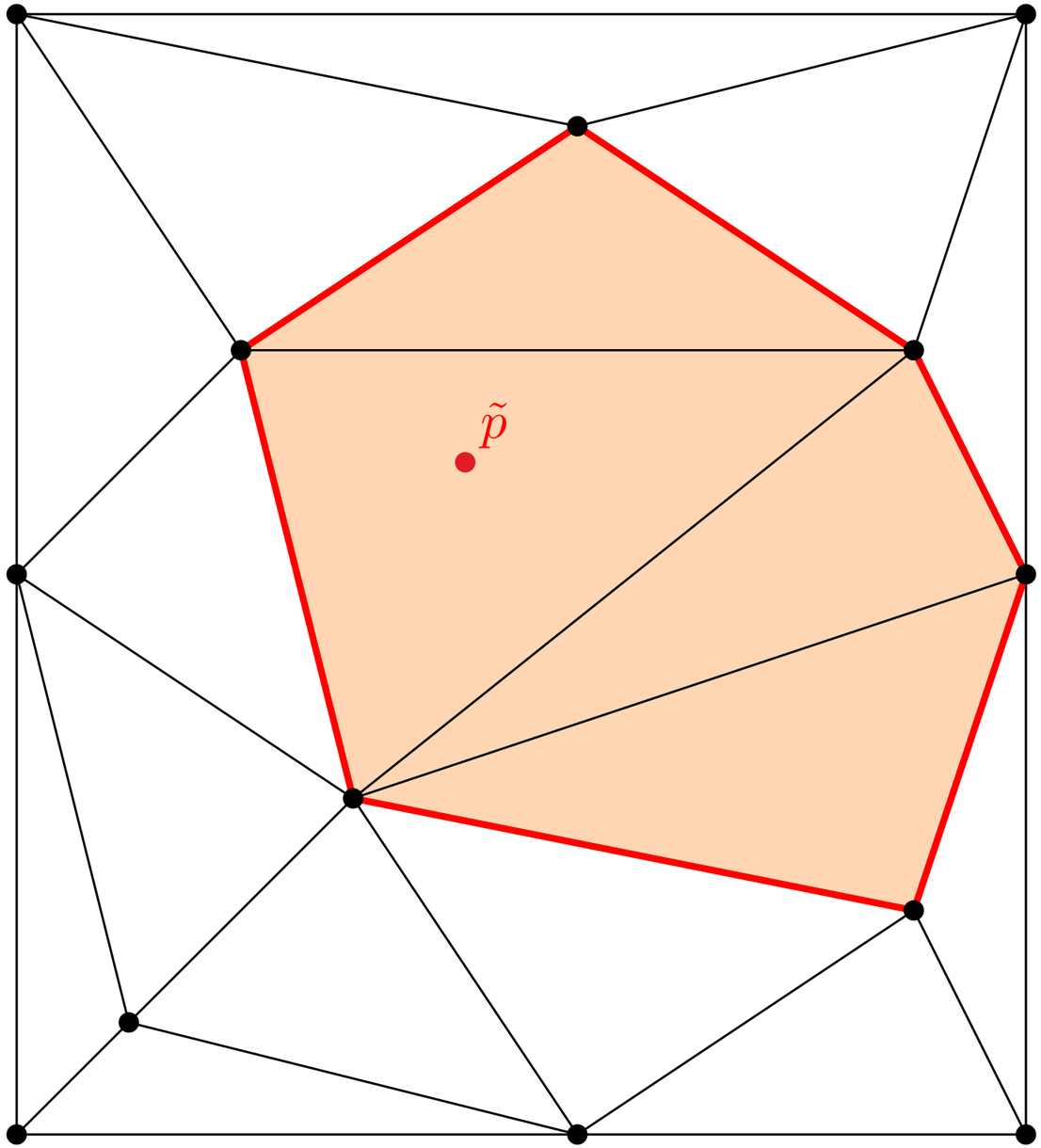
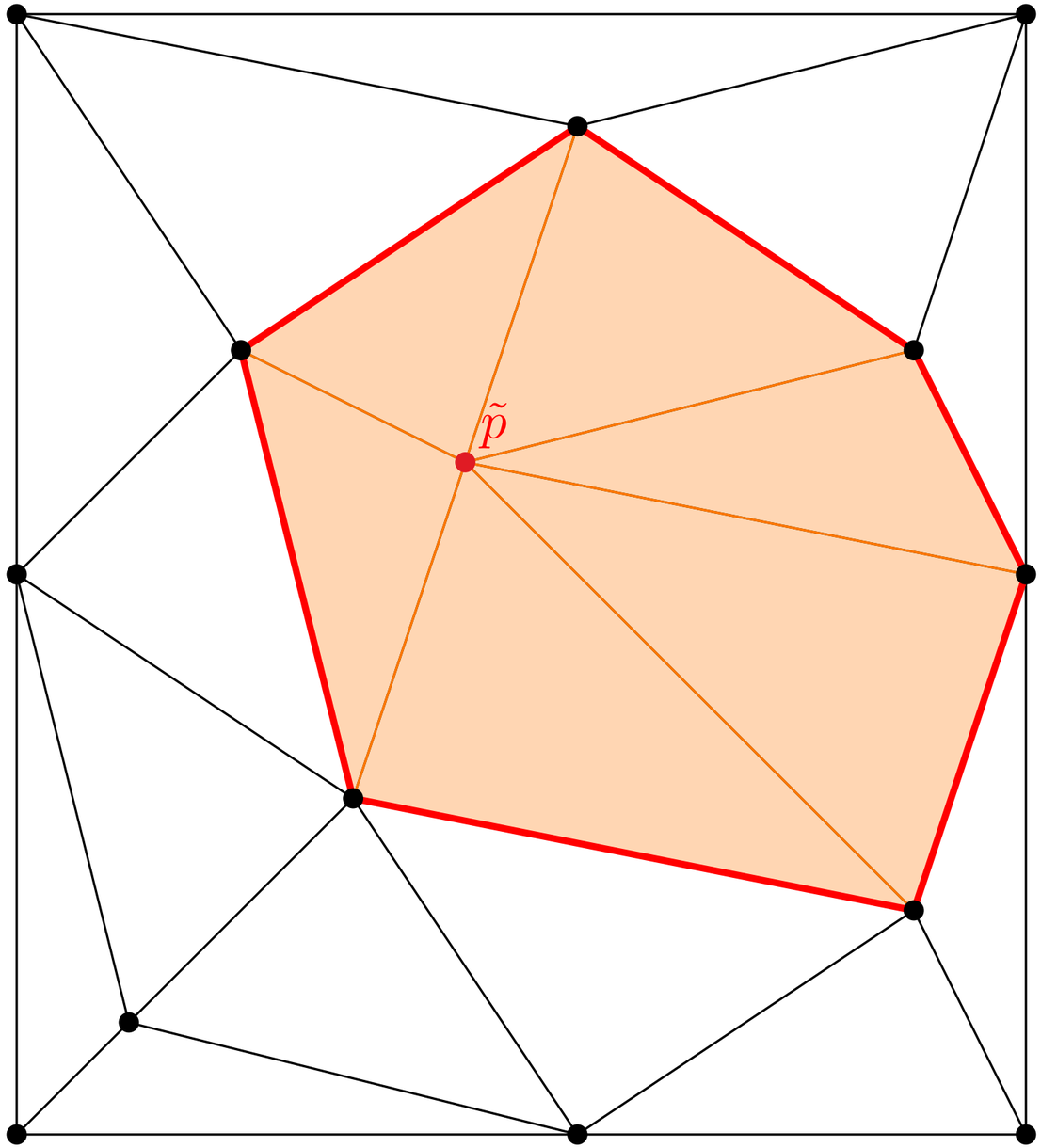
Insertion in 4ε-thick part



Processing in the universal cover

$\longrightarrow$ End of the BFS, we find the conflict zone.

Insertion in 4ε-thick part

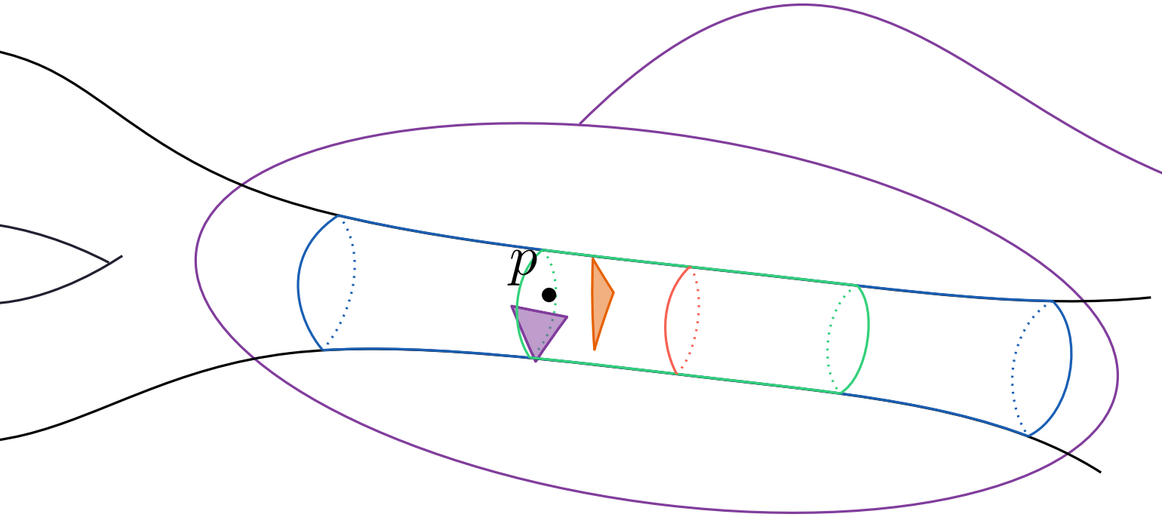


Processing in the universal cover

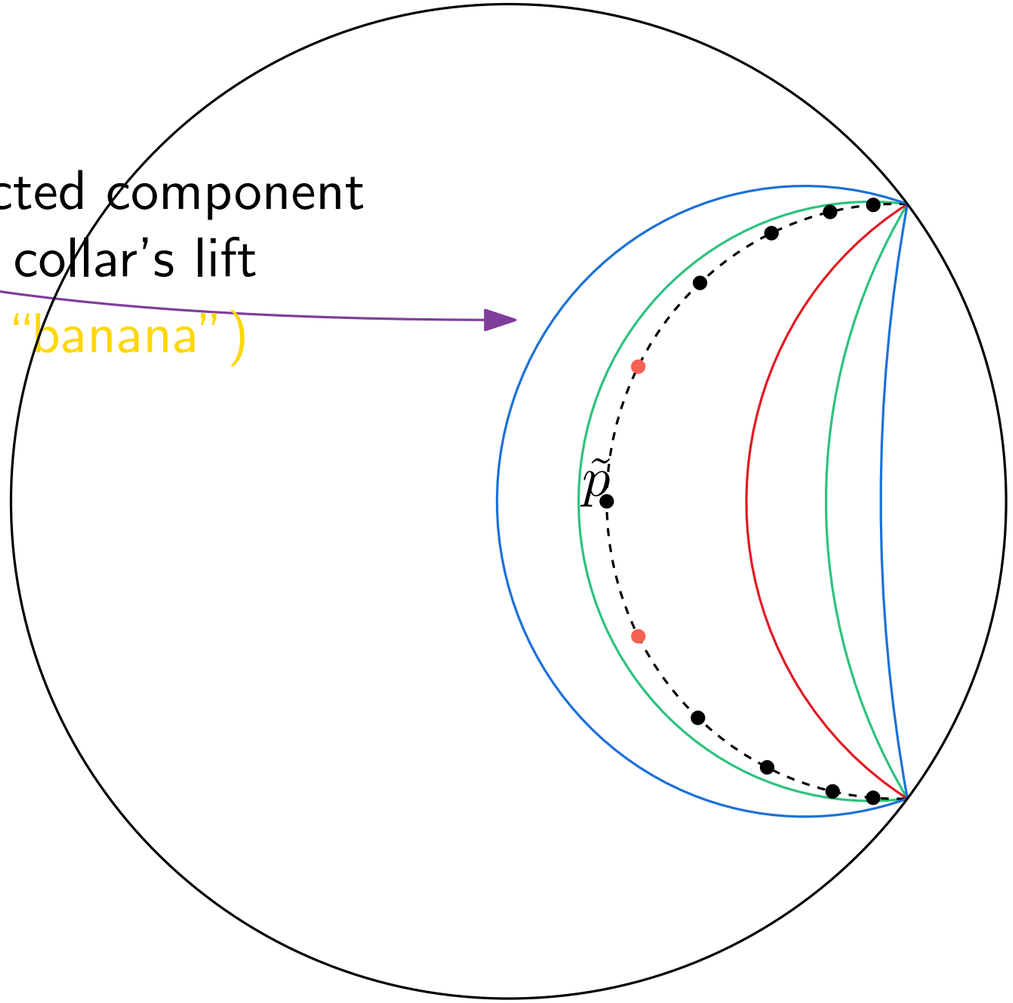
→ Remove all triangles in the conflict zone.

→ Connect $\widetilde{p}$ to the border's vertices.

4ε-thin part : lift of a collar

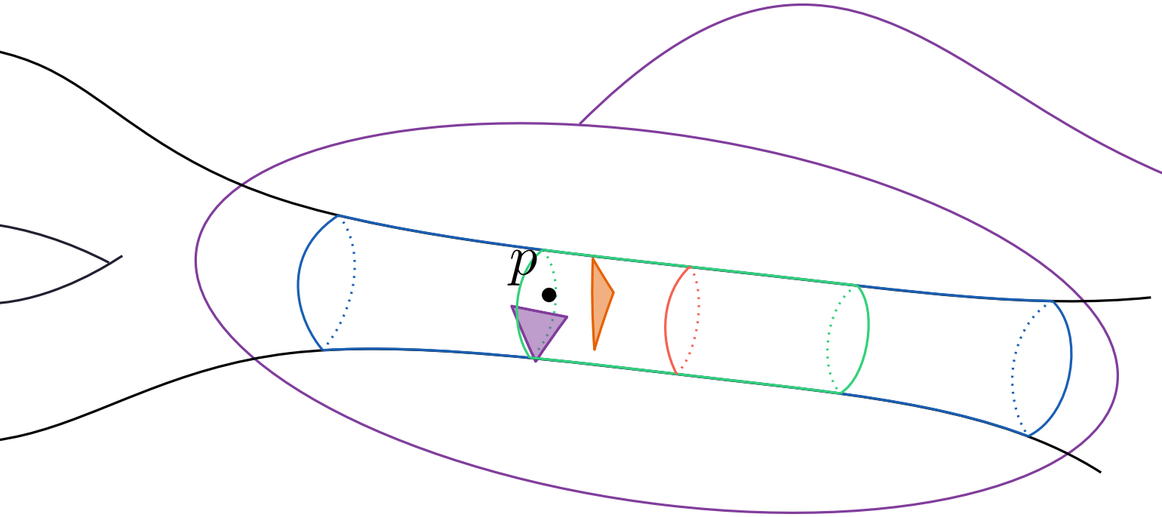


A connected component
of a collar's lift
(a "banana")



$\mathbb{H}^2$, universal cover

4ε-thin part : lift of a collar



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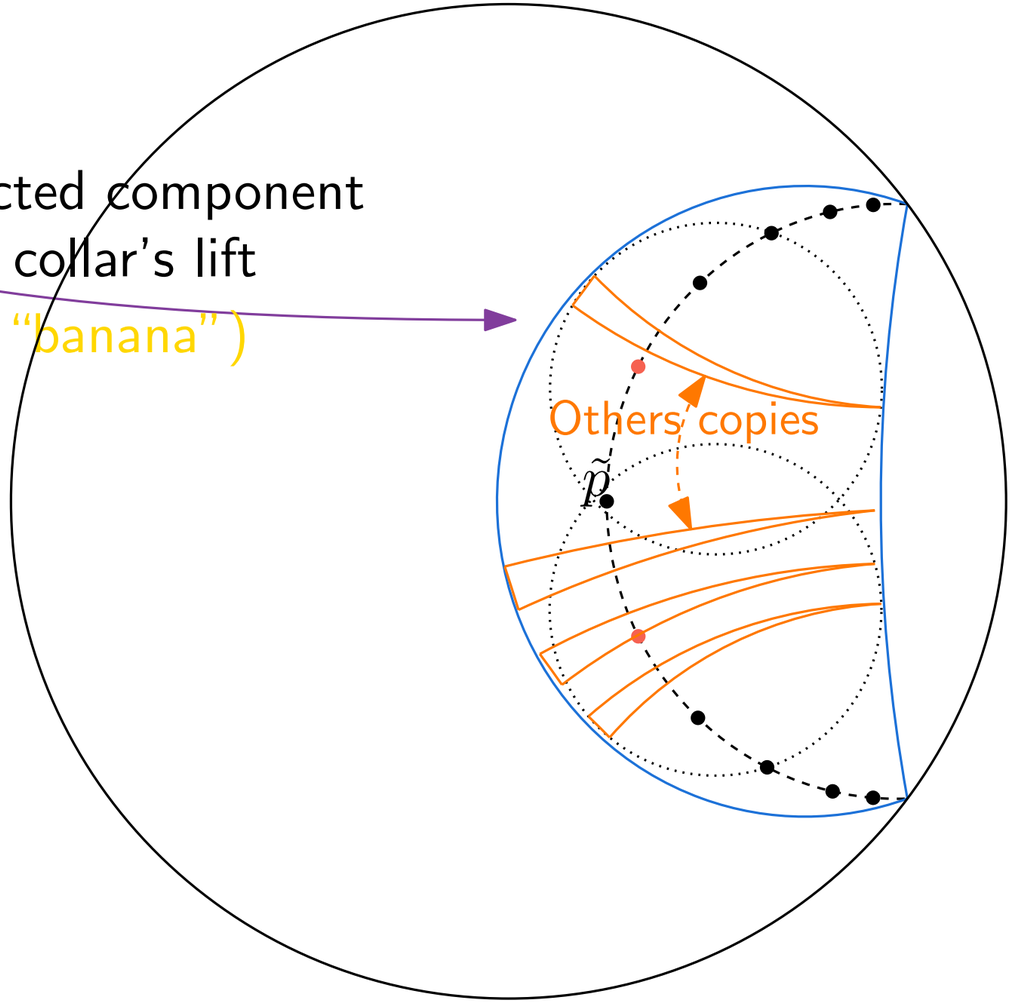
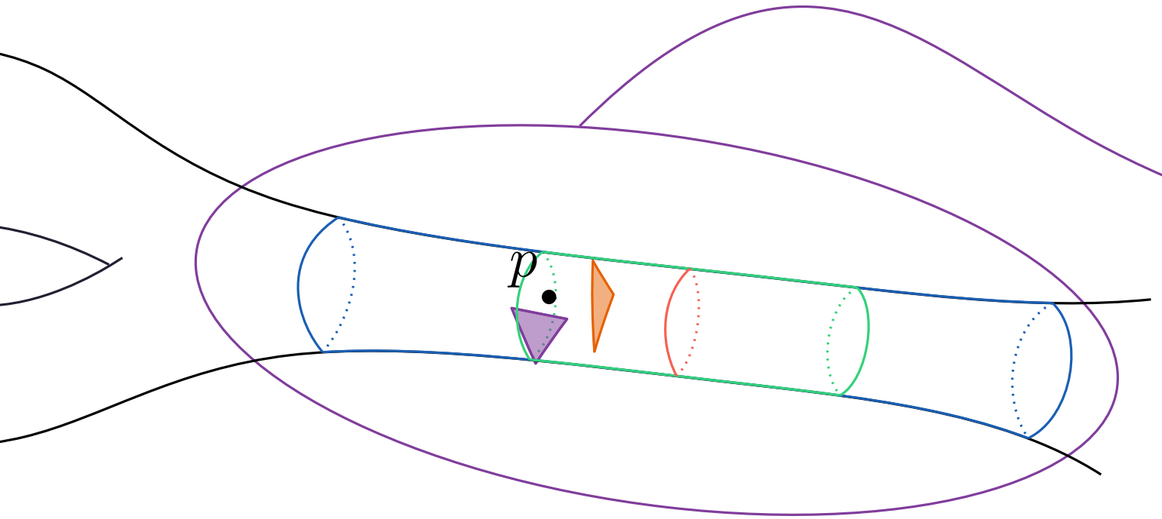


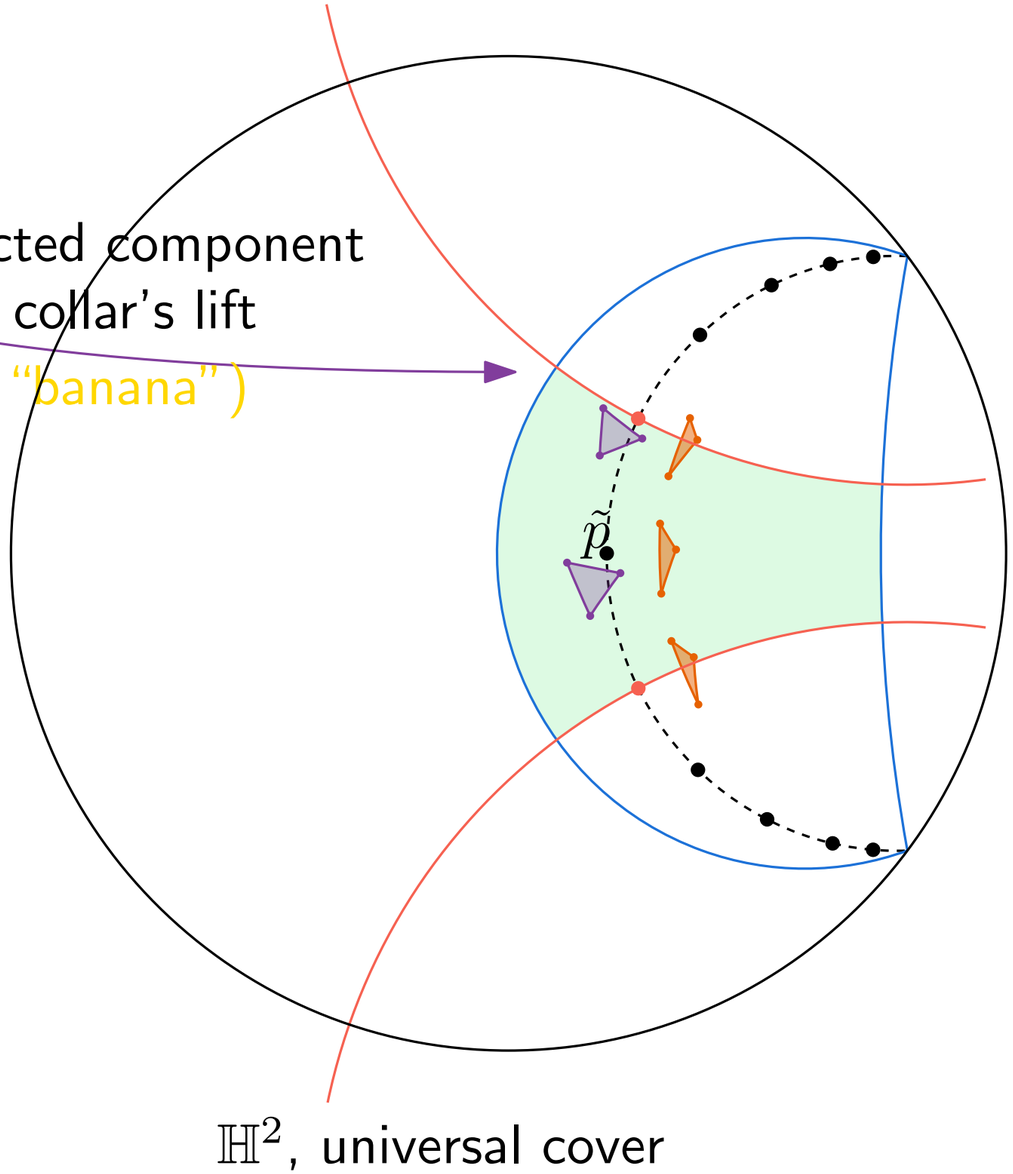
Prop : If p is in the 4ε-thin part,
then the conflict zone is included in the collar.

$\mathbb{H}^2$, universal cover

4ε-thin part : lift of a collar



A connected component
of a collar's lift
(a "banana")

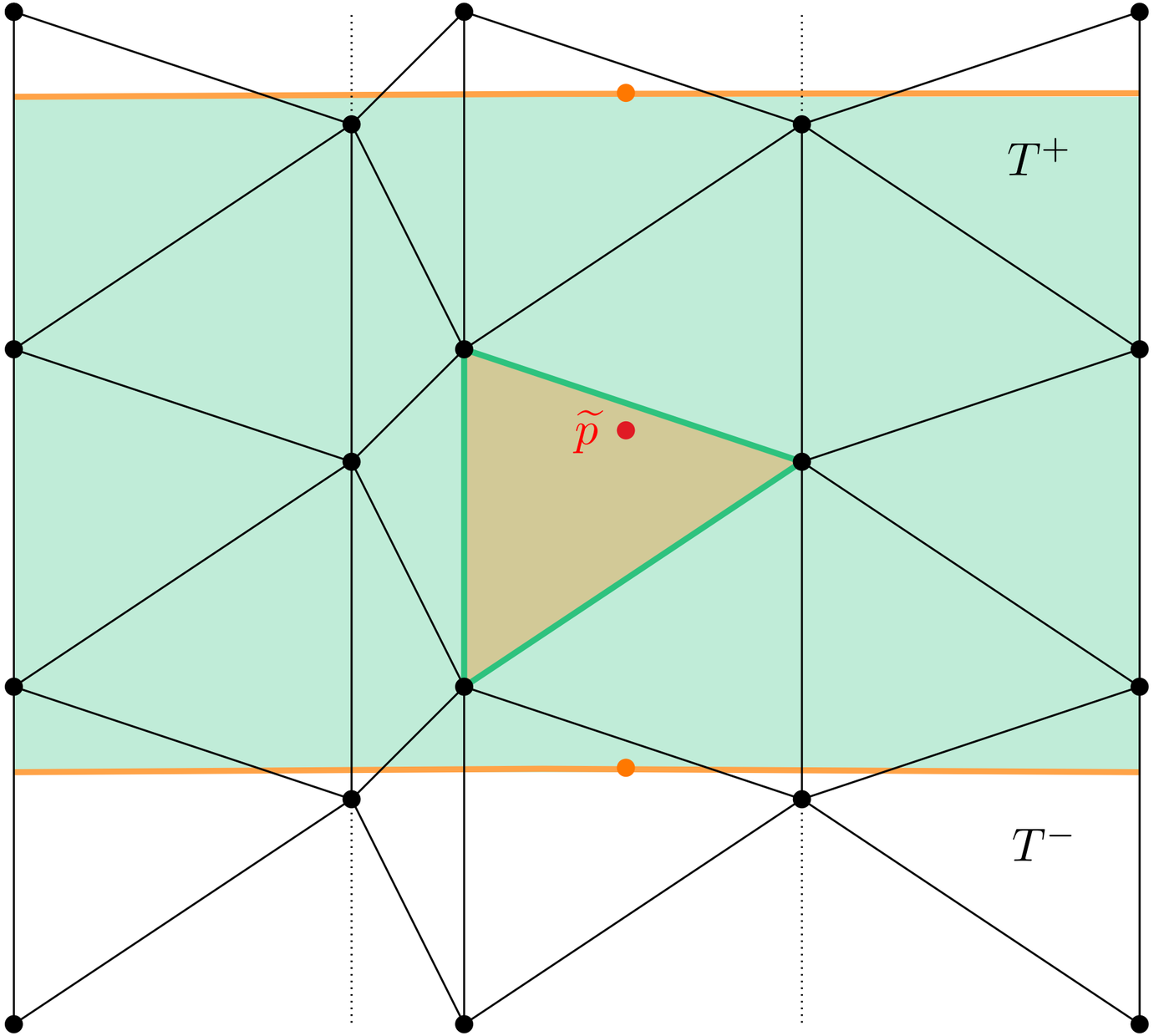


$\mathbb{H}^2$, universal cover

Prop : If p is in the 4ε-thin part,
then the conflict zone is included in the collar.

Restrict the conflict zone to triangles
having a vertex in the green region.

Insertion in the 4ε-thin part

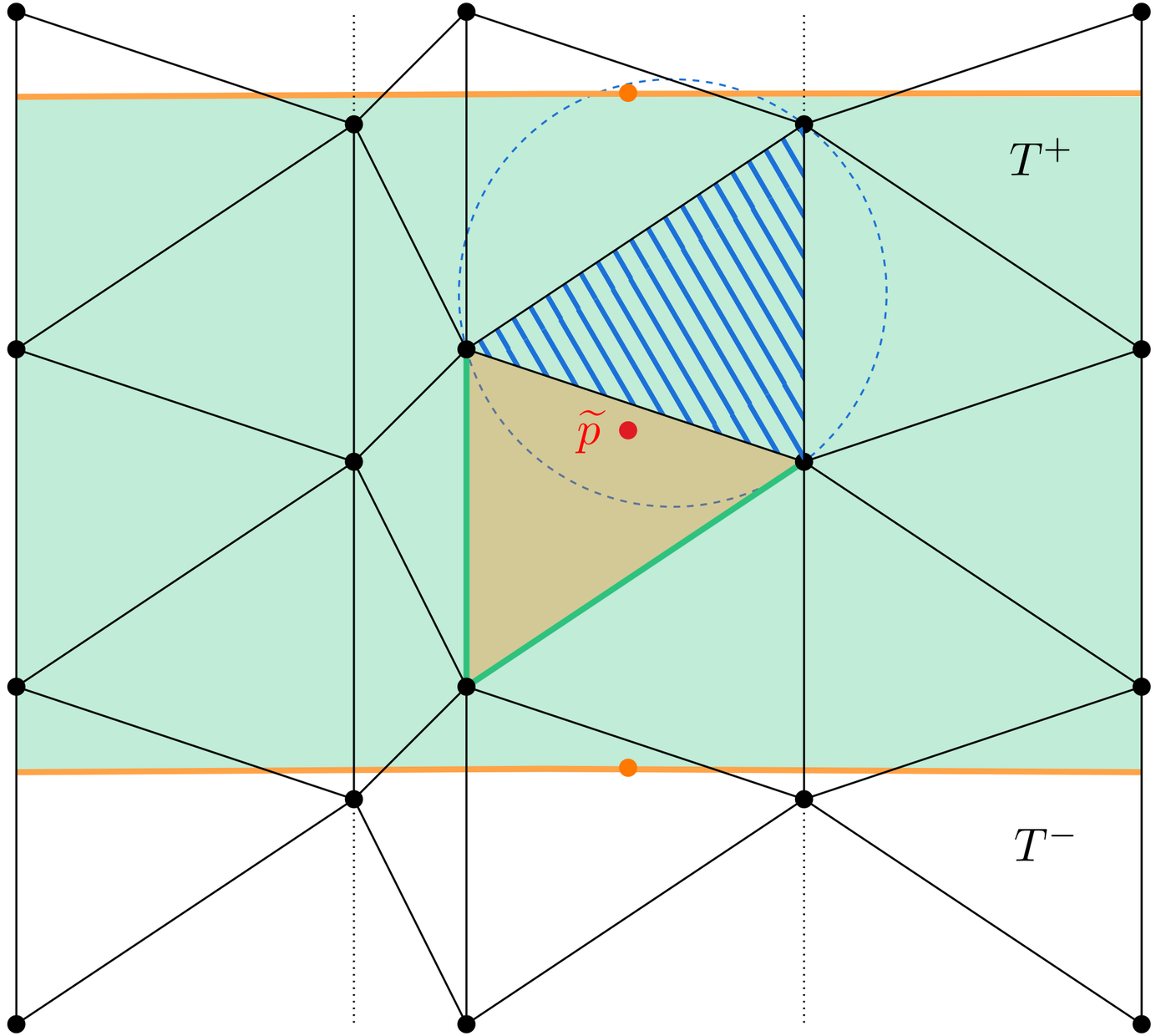


Processing in a banana

→ Find a triangle which contains $\widetilde{p}$.

→ Start a DFS to find conflict zone.

Insertion in the 4ε-thin part

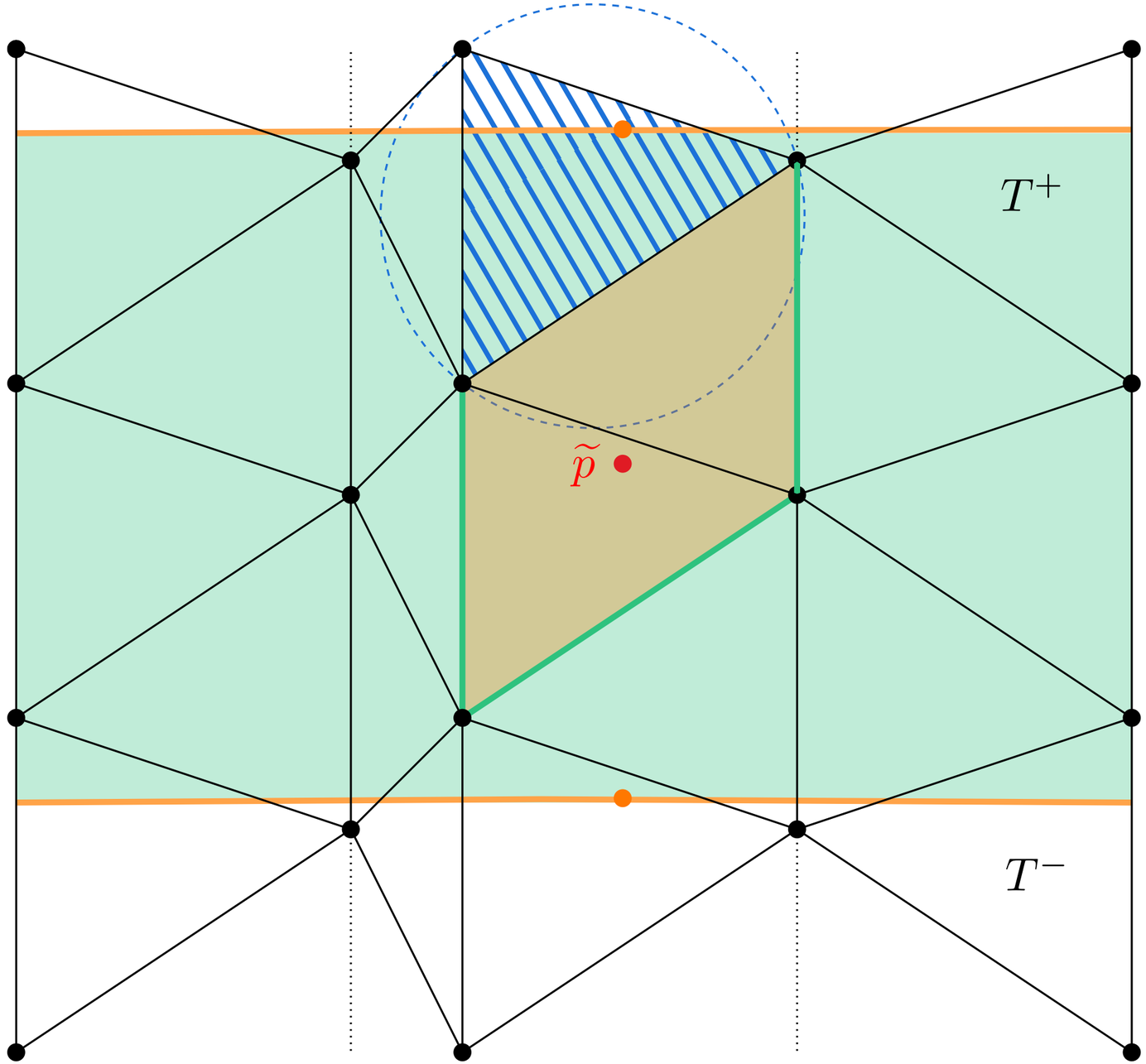


Processing in a banana

→ Conflict with $\widetilde{p}$ **and** the vertices are between the two lines.

→ Push the two other sides in the stack.

Insertion in the 4ε-thin part

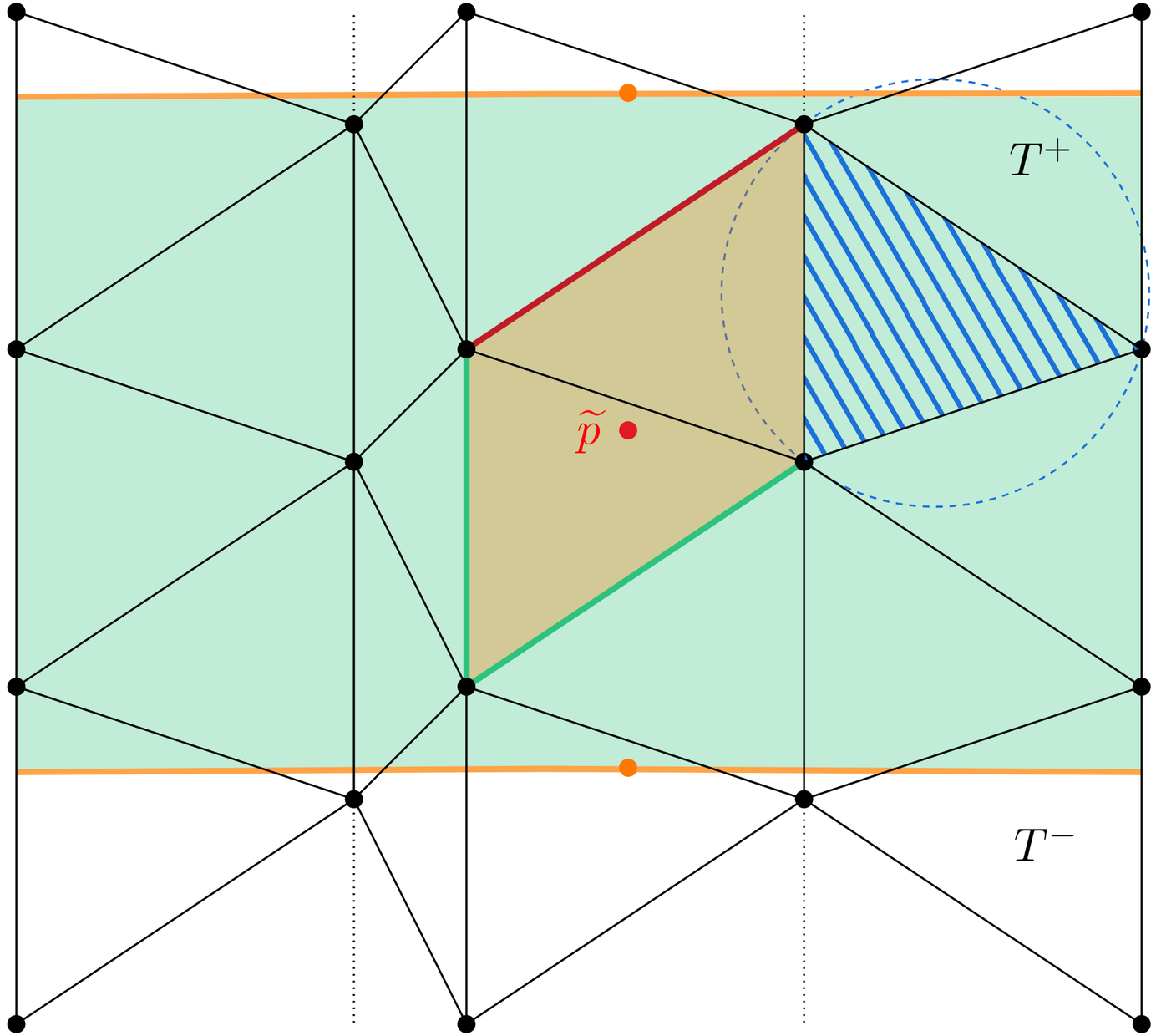


Processing in a banana

→ Not in conflict with $\widetilde{p}$.

→ The side in commun with the conflict zone belongs to the border.

Insertion in the 4ε-thin part



Processing in a banana

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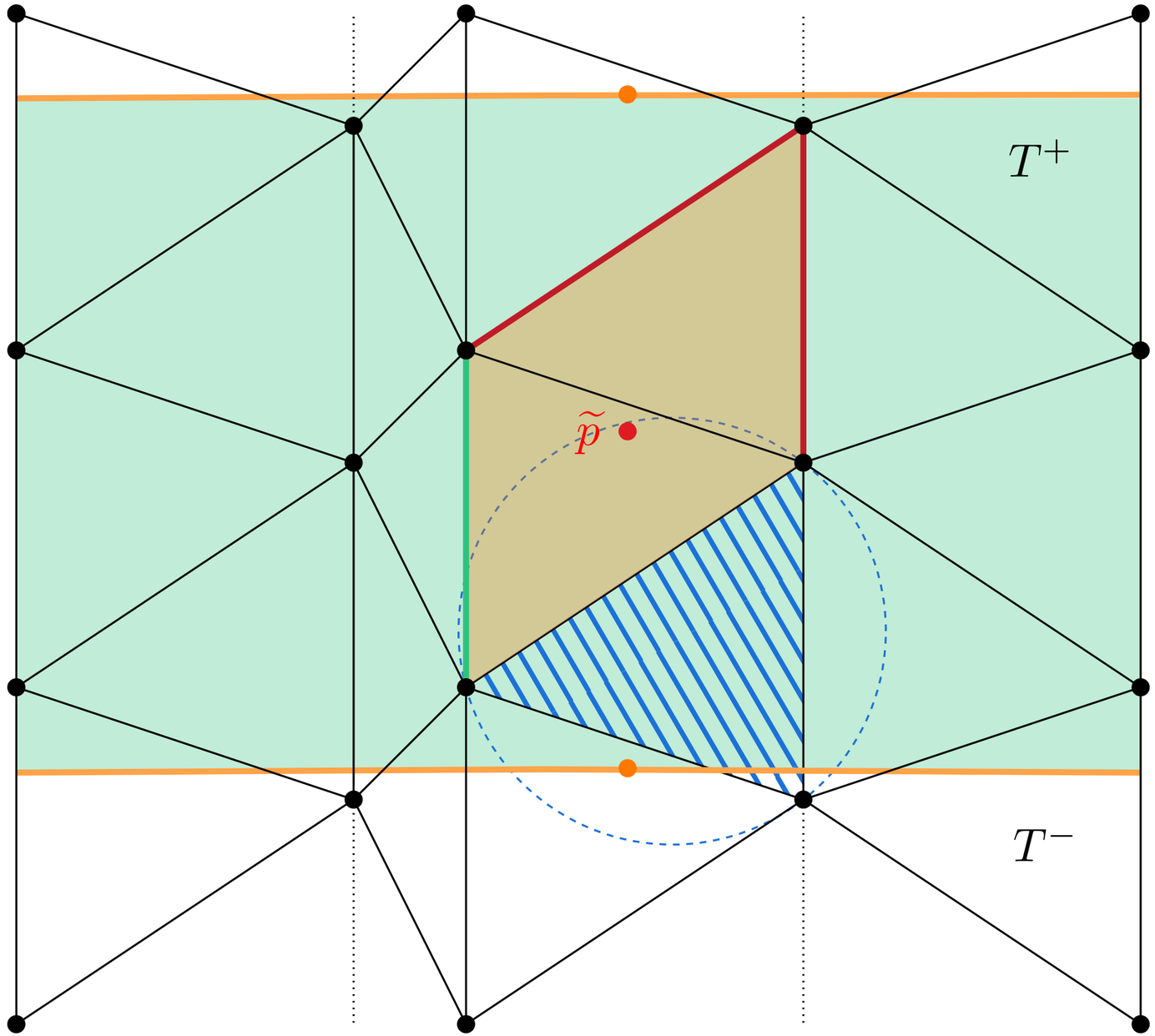
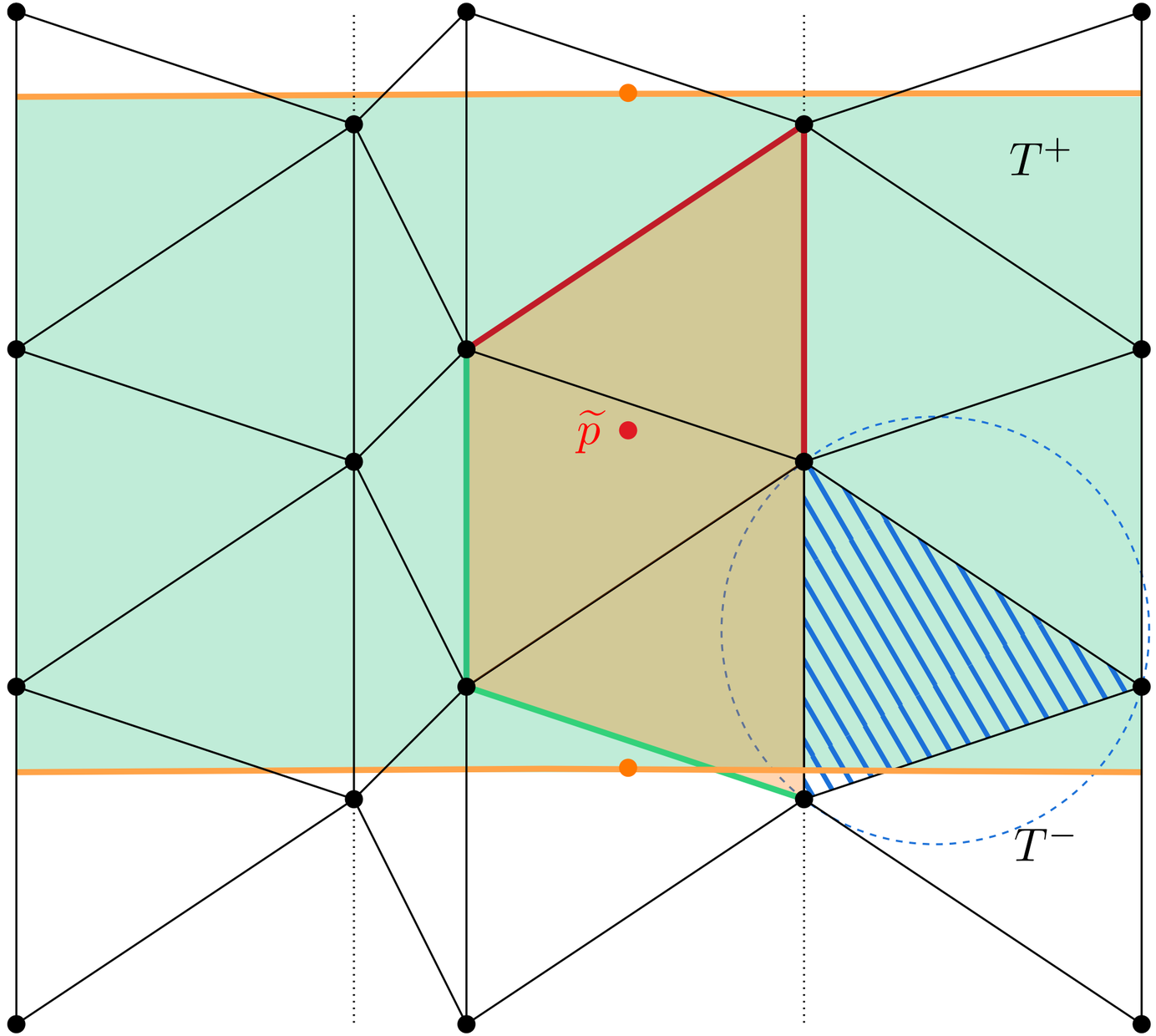


Processing in a banana

→ Conflict with $\widetilde{p}$ **and** has a vertex between the two lines.

→ Push the two other sides in the stack.

Insertion in the 4ε-thin part



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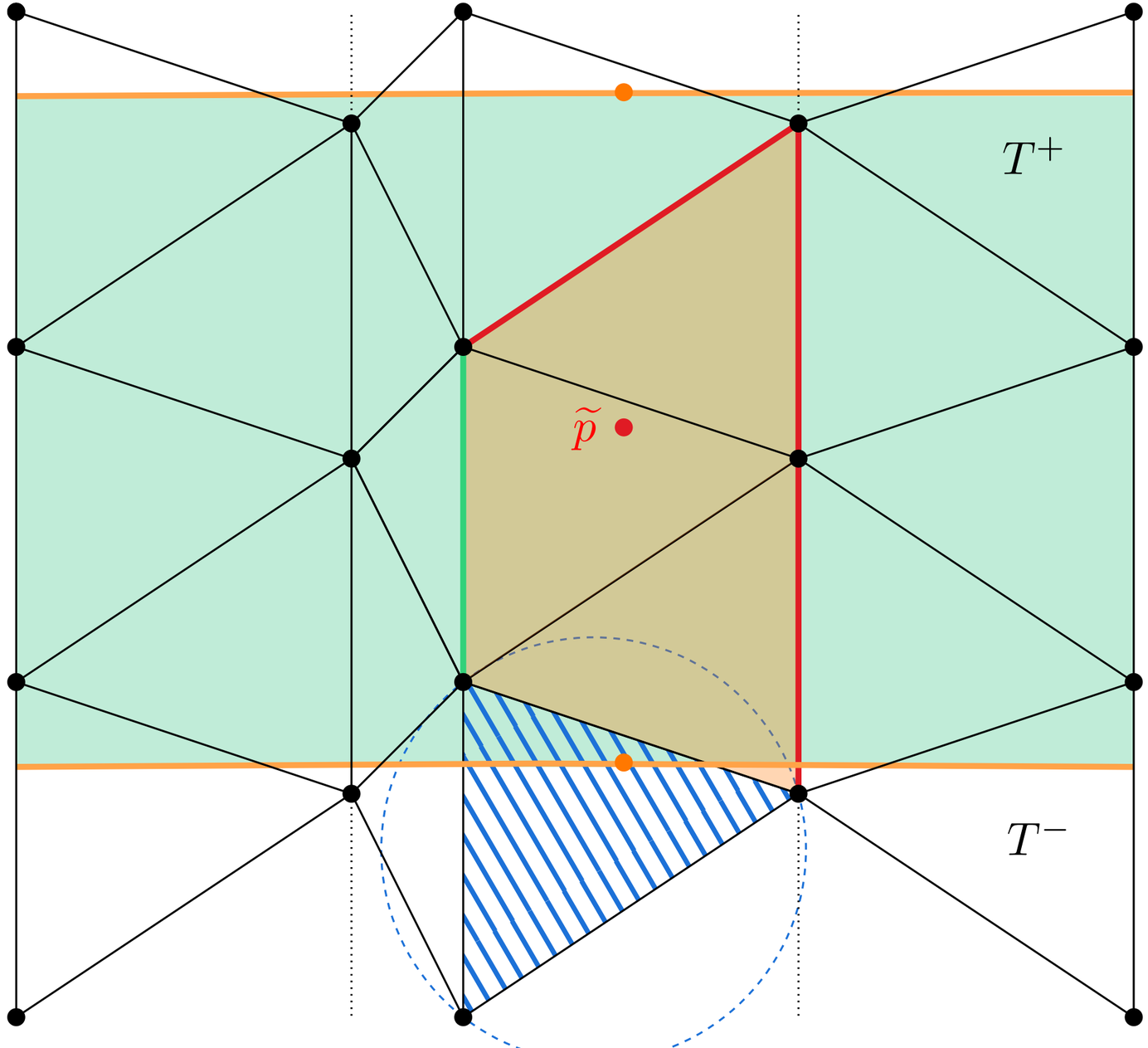

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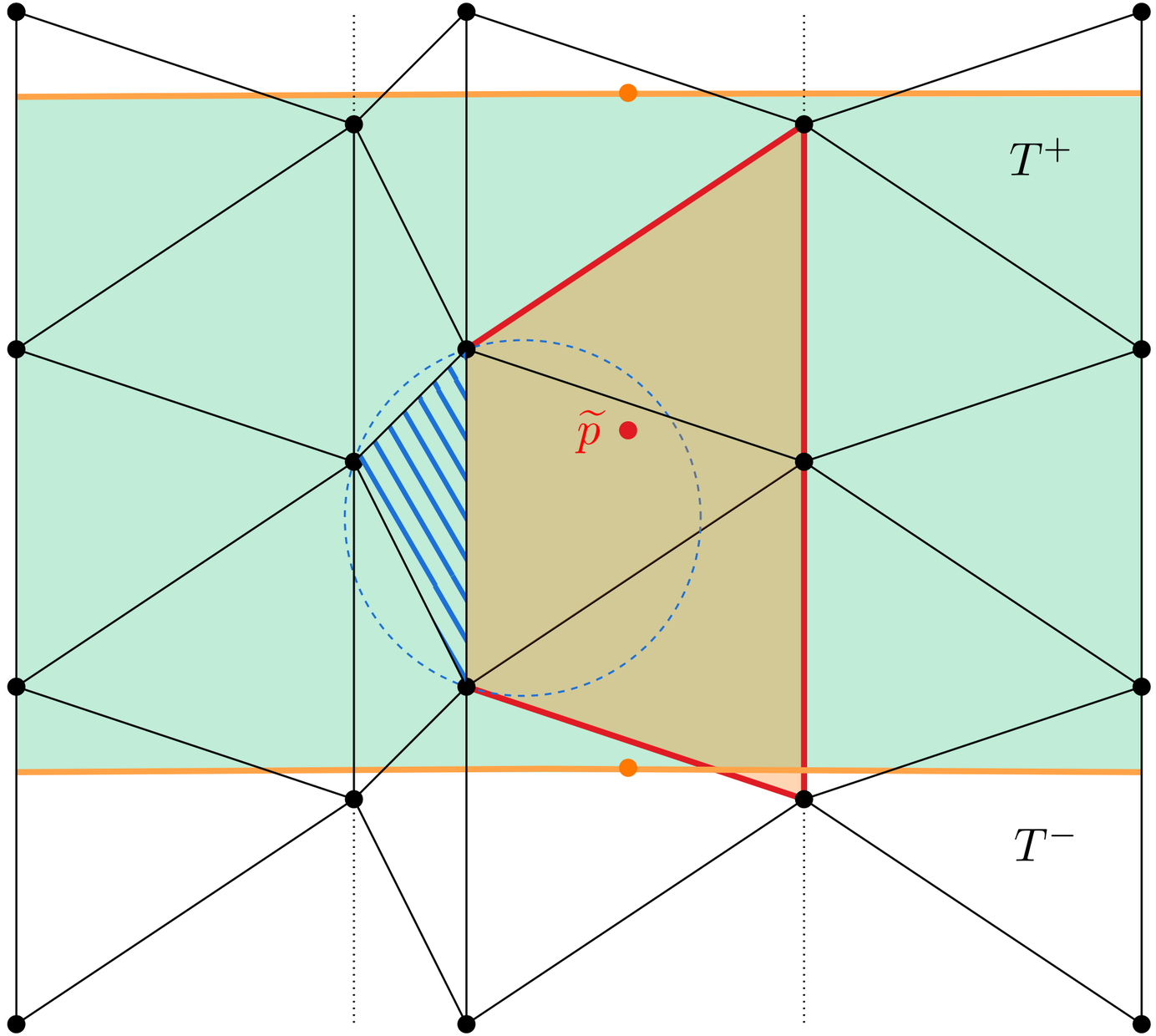

Processing in a banana

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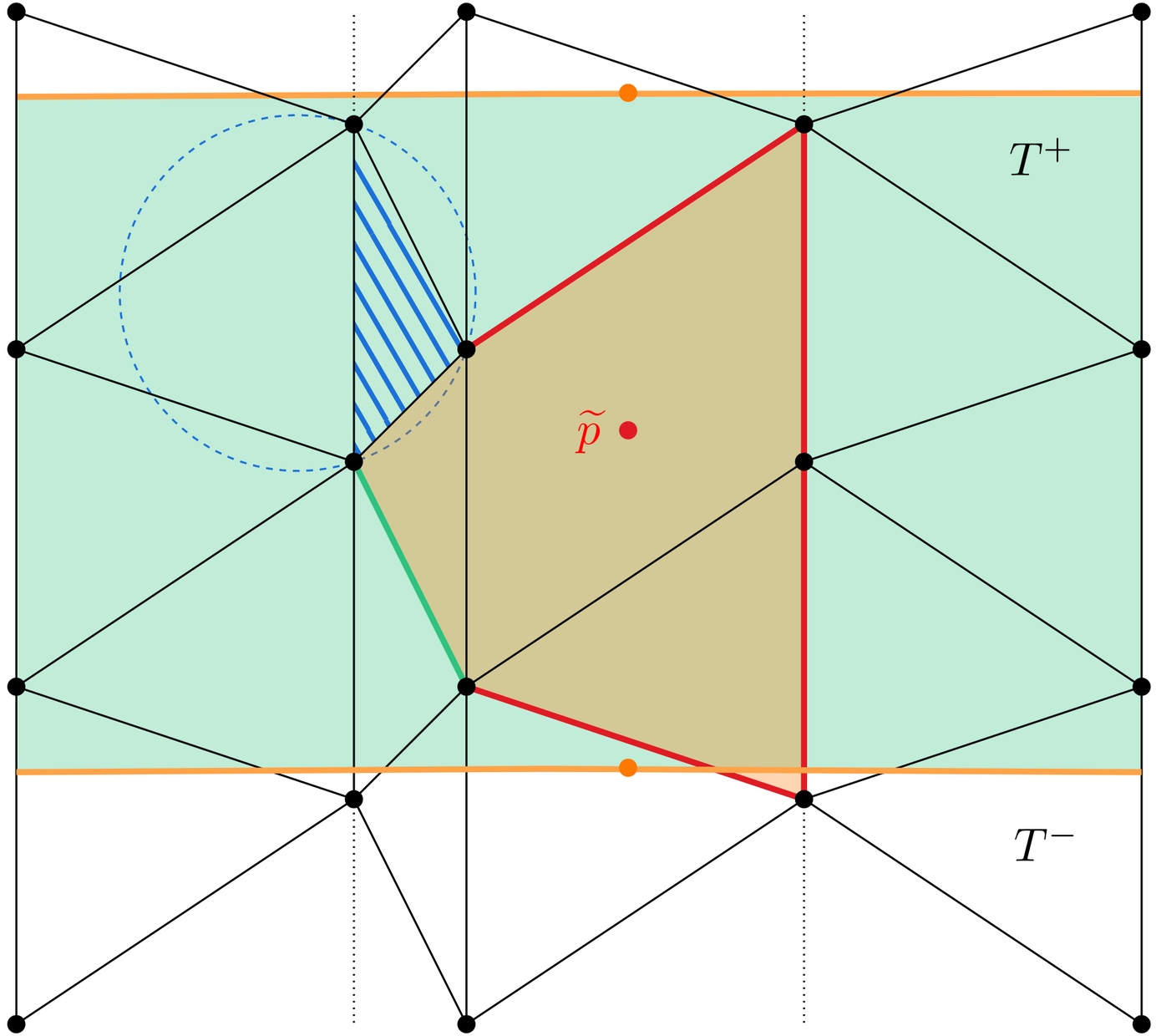
Insertion in the 4ε-thin part



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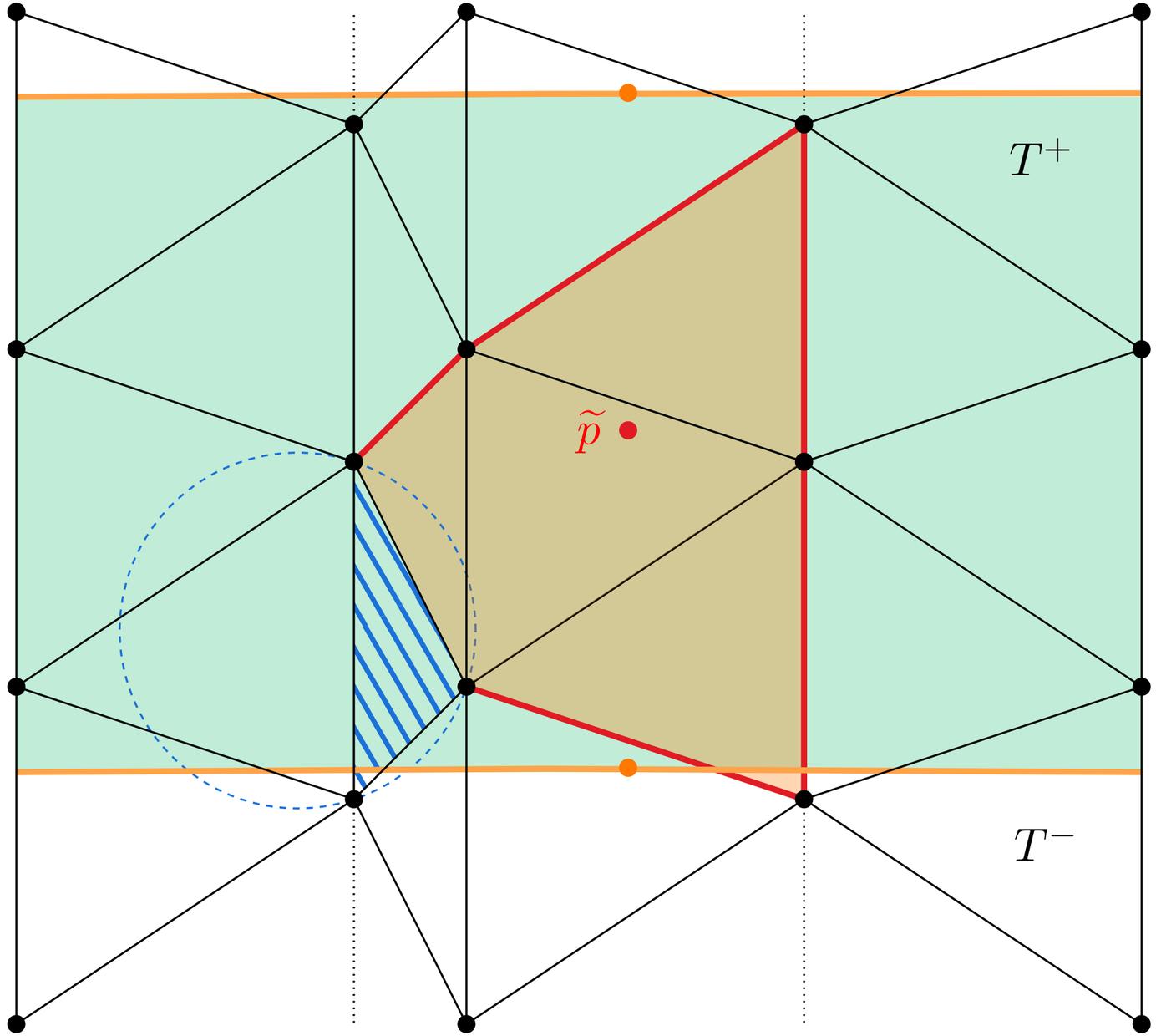

Processing in a banana

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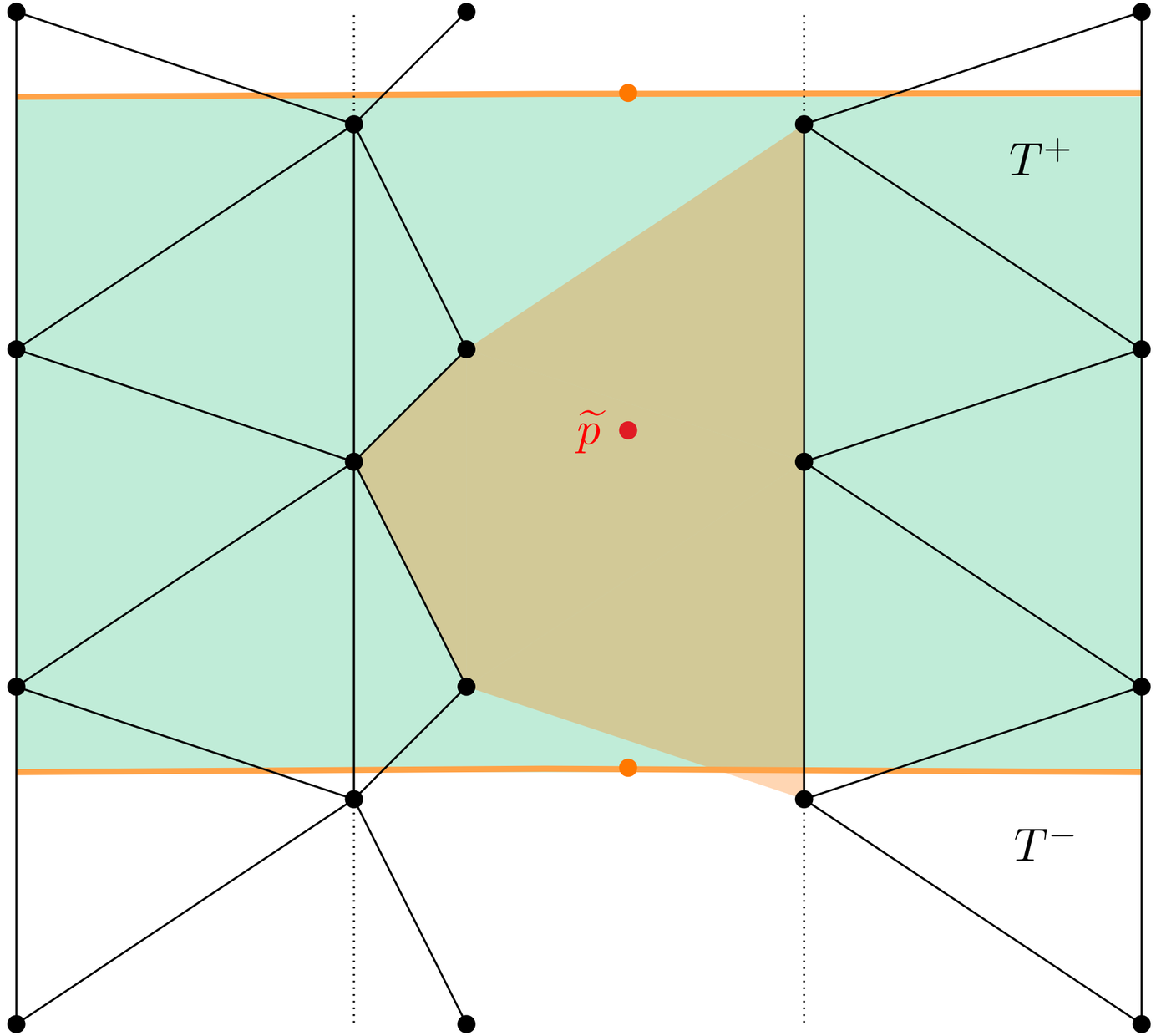
→ The side in commun with the conflict zone belongs to the border.

Insertion in the 4ε-thin part

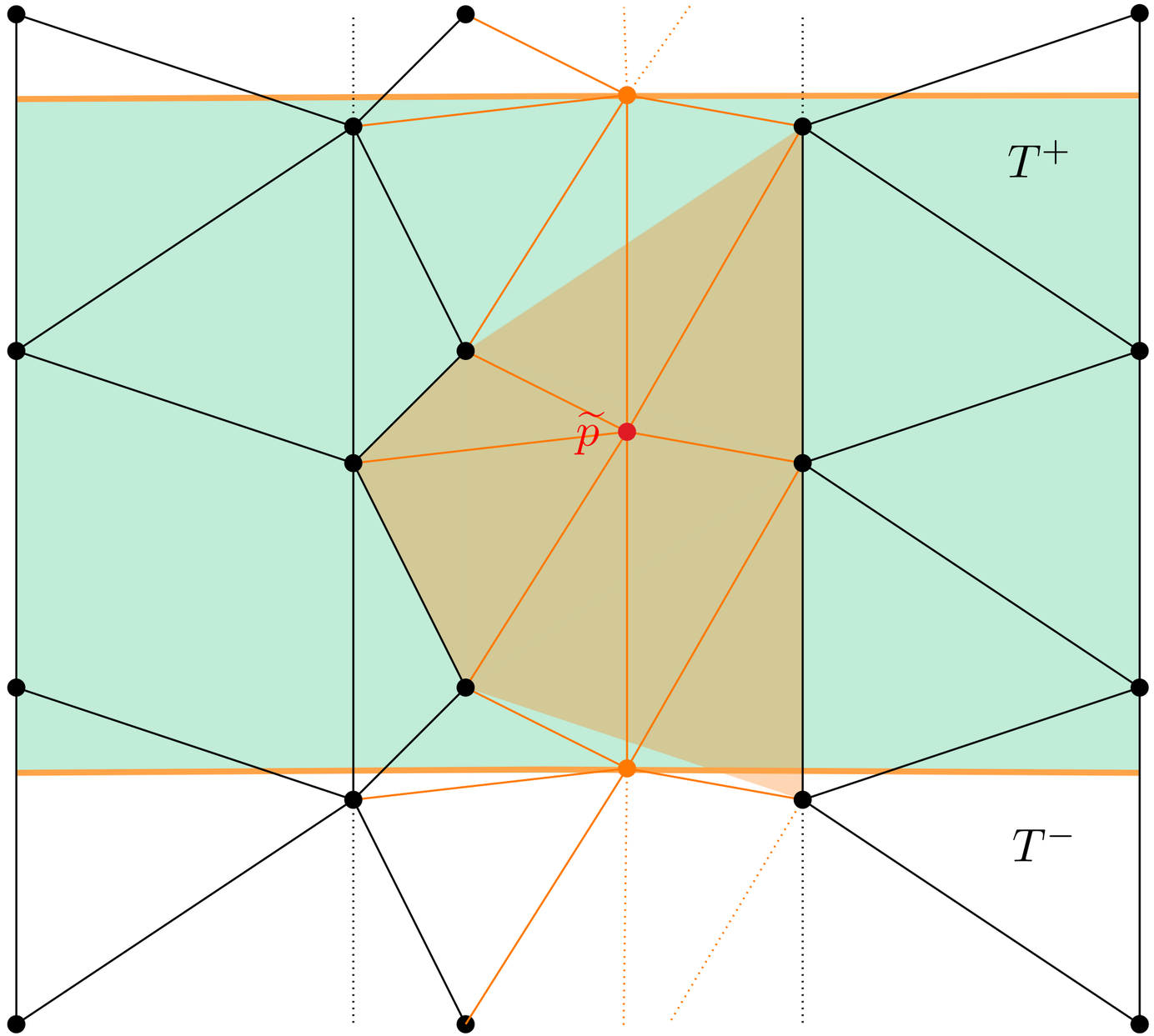


Processing in a banana

→ End of the BFS, we find the restricted conflict zone.

→ Remove all triangles (and their copies) of the restricted conflict zone.

Insertion in the 4ε-thin part



Processing in a banana

→ Connect the border's vertices to $\widetilde{p}$.

→ Check if new triangles are in conflict with two nearest copies of $\widetilde{p}$ and adapt the triangulation.

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