Differentiator-based velocity observer with sensor bias estimation: an inverted pendulum case study

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AnyWalker - an ultra-mobile chassis robot





Inverted pendulums







Reaction-wheel pendulum



System dynamics:

 $\begin{aligned} J_r \ddot{\theta}_r(t) + J_r \ddot{\theta}(t) &= kI(t), \\ (J+J_r)\ddot{\theta}(t) + J_r \ddot{\theta}_r(t) &= mlg\sin\theta(t), \end{aligned}$

where l(t) is the input current. <u>Measured signals</u>: $\theta(t)$, $\theta_r(t)$. <u>Unmeasured signals</u>: $\dot{\theta}(t)$, $\dot{\theta}_r(t)$ <u>Control goal</u>: to stabilize at the upper equilibrium.

Can be easily solved with a full-state feedback controller, if the velocities are available:

$$I(t) = -k_1\theta(t) - k_2\dot{\theta}(t) - k_3\dot{\theta}_r(t).$$

Velocity observers

- Model-based state observers
 - require good model knowledge,
 - estimate the full state vector (coupled),
 - provide good performance for a good model.
 - E.g., linearization-based designs, nonlinear observers.
- Model-free differentiators
 - no need for a model (series expansion),
 - decoupled, estimate for each DoF separately,
 - worse performance than model-based designs (with a good model).
 - E.g., linear (FIR) differentiators.
- Model-based differentiators
 - use the (partial) model knowledge,
 - decoupled,
 - better performance than model-free designs (with a good model).

E.g., sliding-mode (exact) differentiators, high-gain designs, homogeneous differentiators.

Problem statement



For our system, we use the model-based homogeneous observer (HOMD).

The bias problem

The bias in the measurements of $\theta(t)$ propagates through the model and yields the biased velocity estimation. \Downarrow The performance degrades and the stabilization controller can be compromised. \Downarrow To stabilize the system, a bias observer has to be designed and combined with the velocity observer.

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Let us consider the system

$$\ddot{y}(t)=f(v(t)),$$

where v(t) is measured and f(v) represents the *known* part of the dynamics.

The HOMD velocity observer is given by

$$\begin{split} \dot{x}_1(t) &= x_2(t) - k_1 [x_1(t) - y(t)]^{\alpha}, \\ \dot{x}_2(t) &= f(v(t)) - k_2 [x_1(t) - y(t)]^{2\alpha - 1}, \\ \hat{y}(t) &= x_2(t), \end{split}$$

where

$$[x]^{\alpha} := |x|^{\alpha} \operatorname{sgn}(x).$$

HOMD (2/2)

Define the estimation error $e_1 := x_1 - y$ and $e_2 = x_2 - \dot{y}$. Then the error dynamics is

$$\dot{e}_1(t)=e_2(t)-k_1\lceil e_1(t)
floor^lpha,$$

 $\dot{e}_2(t)=-k_2\lceil e_1(t)
floor^{2lpha-1}.$

Finite-time convergence

Proposition^a. Choose k_1 , k_2 such that the polynomial $s^2 + k_1s + k_2$ is Hurwitz and $\alpha \in \left(\frac{1}{2}, 1\right)$. Then the origin $e_1 = e_2 = 0$ is finite-time stable, *i.e.* there exists T = T(e(0)) > 0 such that e(t) is defined and unique on [0, T), bounded, and $\lim_{t\to T} e(t) = 0$. T is called the *settling-time function* of the system.

^aSee Perruquetti, W., Floquet, T., and Moulay, E. "Finite-time observers: Application to secure communi- cation", *IEEE Trans. on Automatic Control*, 2008.

Reaction-wheel pendulum velocity observer



The pendulum dynamics is given by

$$\ddot{ heta}(t) = -rac{k}{J}I(t) + rac{mlg}{J}\sin(heta(t)),$$

and the HOMD-based velocity observer is

$$\begin{split} \dot{\hat{x}}_{p,1}(t) &= \hat{x}_{p,2}(t) - k_1 \lceil e_{p,1}(t) \rfloor^{\alpha}, \\ \dot{\hat{x}}_{p,2}(t) &= -\frac{k}{J} I(t) + \frac{m l g}{J} \sin(\theta(t)) - k_2 \lceil e_{p,1}(t) \rfloor^{2\alpha - 1}, \\ \hat{\dot{\theta}}(t) &= \hat{x}_{p,2}(t), \end{split}$$

where $e_{p,1} := x_{p,1} - \theta$.

It has the same error dynamics and provides the finite-time convergence.

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Biased measurements



Consider now the case when the measurements are biased, and the available signal is

$$y(t)=\theta(t)+d,$$

where d is a constant.

Assumption

We assume that the constant displacement d is sufficiently small and the following approximation holds:

$$\sin(y) = \sin(\theta + d) \approx \sin(\theta) + \cos(\theta)d.$$

What happens when we apply the same observer?

Biased estimation

The previous error dynamics was:

$$\dot{e}_{p,1}(t) = e_{p,2}(t) - k_1 \lceil e_{p,1}(t)
floor^{lpha}, \ \dot{e}_{p,2}(t) = -k_2 \lceil e_{p,1}(t)
floor^{2lpha-1}.$$

The new error dynamics is

$$\dot{e}_{p,1}(t) = e_{p,2}(t) - k_1 \lceil e_{p,1}(t) - d
floor^{lpha},$$

 $\dot{e}_{p,2}(t) = a_1 \cos(heta(t))d - k_2 \lceil e_{p,1}(t) - d
floor^{2lpha-1},$

where $a_1 := mlg/J$. The origin $e_{p,1} = e_{p,2} = 0$ is not an equilibrium any more!

The new equilibrium (for fixed θ) is now

$$e_{
ho,2}^0:=k_1\left(rac{a_1\cos(heta)}{k_2}|d|
ight)^{rac{lpha}{2lpha-1}}\mathrm{sgn}(d).$$

Bias propagation

What is the result of the bias propagation?

The measurement bias d propagates through the model of the dynamics and yields a bias in the velocity estimation. If this biased estimate is applied for control as

$$I(t) = -K \begin{bmatrix} y(t) & \widehat{\dot{ heta}}(t) & \widehat{\dot{ heta}}_r(t) \end{bmatrix}^{ op}$$

then the closed-loop system equilibrium is

$$\begin{bmatrix} \theta \\ \dot{\theta} \\ \dot{\theta}_r \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ e_{p,2}^0 \frac{k_3 - k_2}{k_3} - d \frac{k_1}{k_3} \end{bmatrix}$$

We have to estimate the bias.

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How to estimate the bias if the velocities are available? Define $z := \begin{bmatrix} \theta & \dot{\theta} & d \end{bmatrix}^{\top}$, then $y = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} z$. Let \hat{z} be an estimate of z and define $e := \hat{z} - z$.

Then the dynamics of z can be written as

$$\dot{z} = \begin{bmatrix} z_2 \\ -\frac{k}{J}I + a_1\sin(z_1) \\ 0 \end{bmatrix} \approx \begin{bmatrix} z_2 \\ a_1\cos(y - \hat{z}_3)e_3 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \beta_z \\ 0 \end{bmatrix},$$

where the signal $\beta_z := -\frac{k}{J}I + a_1 \sin(y - \hat{z}_3)$ is available. If the velocity z_2 is measured, then the Luenberger-like observer is

$$\begin{split} \dot{\mathbf{v}} &= \beta_z, \\ \hat{d} &= \hat{z}_3 = L_0(\mathbf{v} - z_2), \end{split}$$

where $L_0 > 0$. Then

$$\dot{e}_3=-L_0a_1\cos(y-\hat{z}_3)e_3,$$

and $e_3 \rightarrow 0$ under reasonable assumptions.

The coupled dynamics



The coupled dynamics:

$$\begin{aligned} \dot{\hat{z}}_1 &= \hat{z}_2 - k_1 \lceil \hat{z}_1 + \hat{z}_3 - y \rfloor^{\alpha}, \\ \dot{\hat{z}}_2 &= \beta_z - k_2 \lceil \hat{z}_1 + \hat{z}_3 - y \rfloor^{2\alpha - 1}, \\ \dot{v} &= \beta_z, \\ \hat{z}_3 &= L_0 (v - \hat{z}_2). \end{aligned}$$
Recall that $z = \begin{bmatrix} \theta & \dot{\theta} & d \end{bmatrix}^{\top}.$

The error dynamics

$$\begin{aligned} \dot{e}_1 &= e_2 - k_1 \lceil e_1 + e_3 \rfloor^{\alpha}, \\ \dot{e}_2 &= -a_1 \cos(z_1) e_3 - k_2 \lceil e_1 + e_3 \rfloor^{2\alpha - 1}, \\ \dot{e}_3 &= L_0 k_2 \lceil e_1 + e_3 \rfloor^{2\alpha - 1}. \end{aligned}$$

When is this system stable?

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Stability analysis (1/3)

Define $s := \begin{bmatrix} e_1 + e_3 & e_2 & e_3 \end{bmatrix}^{\top}$. Then the observers dynamics: $\dot{s}_1 = s_2 - k_1 \lceil s_1 \rfloor^{\alpha} + L_0 k_2 \lceil s_1 \rfloor^{2\alpha - 1},$ $\dot{s}_2 = -a_1 \cos(z_1) s_3 - k_2 \lceil s_1 \rfloor^{2\alpha - 1},$ $\dot{s}_3 = L_0 k_2 \lceil s_1 \rfloor^{2\alpha - 1}.$

Under the reasonable assumption $cos(z_1) > c_0$ for some $c_0 > 0$, the only equilibrium is the origin s = e = 0.

Stability analysis (2/3)

The system can be approximated by

$$\dot{s} = A_0(z_1)s - A_1 \begin{bmatrix} 1 - \alpha \\ 0 \\ 0 \end{bmatrix} \psi(s_1), \quad \psi(s_1) := \begin{cases} \ln(s_1^2) s_1 \text{ for } s_1 \neq 0, \\ 0 \text{ for } s_1 = 0 \end{cases}$$

where

$$A_0(z_1) := \begin{bmatrix} -k_1 + L_0 k_2 & 1 & 0 \\ -k_2 & 0 & -a_1 \cos(z_1) \\ L_0 k_2 & 0 & 0 \end{bmatrix}, \ A_1 := \begin{bmatrix} L_0 k_2 - \frac{1}{2} k_1 & 0 & 0 \\ -k_2 & 0 & 0 \\ L_0 k_2 & 0 & 0 \end{bmatrix}.$$

Define A_m and A_M as the values of $A_0(z_1)$ for $\cos(z_1) = c_0$ and $\cos(z_1) = 1$. Suppose there exists $P = P^{\top} > 0$ such that for some $\gamma > 0$ and $\mu \in \mathbb{R}$:

$$Q := -\left(PA_1 + A_1^{\top}P\right) \ge 0,$$

$$PA_m + A_m^{\top}P + \mu Q + \gamma P \le 0,$$

$$PA_M + A_M^{\top}P + \mu Q + \gamma P \le 0.$$

Stability analysis (3/3)

Consider $V = s^{\top} P s$. Then $\dot{V} \leq -\gamma s^{\top} P s$ for $s_1 = 0$ and otherwise

$$\dot{V} \leq -\gamma s^{\top} P s - \left(\mu - \ln \left(s_1^2 \right) (1 - \alpha) \right) s^{\top} Q s.$$

There can be found $C = C(\mu, \alpha, P)$ such that for the set $\Omega := \{s : s^\top P s < C\}$ it holds $\dot{V} < 0$.

Stability of the coupled observers

For the considered observers, choose parameters such that the LMIs are feasible. Then there exist $\varepsilon > 0$ and a compact set Ω , such that for $\alpha \in (1 - \varepsilon, 1]$ and all initial conditions $s(0) \in \Omega$ it holds $s \to 0$ and $\lim_{t\to\infty} |\hat{z} - z| = 0$.

An alternative solution¹

Apply the static feedback control

$$I(t) = -K \begin{bmatrix} y(t) & \hat{\dot{\theta}}(t) & \hat{\dot{\theta}}_r(t) \end{bmatrix}^{\top}$$

Then the closed-loop system equilibrium is

$$\begin{bmatrix} \theta \\ \dot{\theta} \\ \dot{\theta}_r \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_r^0 \end{bmatrix}$$

for some constant $\dot{\theta}_r^0$, e.g., $\dot{\theta}_r^0 = e_{p,2}^0 \frac{k_3 - k_2}{k_3} - d\frac{k_1}{k_3}$.

As we approach the equilibrium, $\theta(t) \rightarrow 0 \Rightarrow y(t) \rightarrow d$. The idea is to use y(t) to estimate d:

$$\hat{d}(t) = \frac{\gamma}{p+\gamma} y(t).$$

¹M. Gajamohan, M. Merz, I. Thommen, and R. D'Andrea, "The cubli: A cube that can jump up and balance," *Conference on Intelligent Robots and Systems*, 2012.



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The hardware



Description	Symbol	Value
Mass of the pendulum, kg	m _p	0.58
Pivot – pendulum's center of mass distance, m	l _p	0.10
Mass of the reaction wheel, kg	m _r	0.35
Pivot – reaction wheel axis distance, m	l _r	0.22
Resolution of the pendulum angle measurement, rad	_	$6.28 \cdot 10^{-4}$
Sampling frequency, Hz	_	500

The closed-loop stabilization without bias estimation, d = -0.08.



The closed-loop stabilization with the low-pass bias observer, d = -0.08.



The proposed HOMD and linear bias observer, d = -0.08.



(f) Bias estimate \hat{d} and estimate of the pendulum position $\hat{\theta} = y - \hat{d}$



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Conclusion

- The measurement bias can propagate through model-based differentiators.
- A bias observer has to be designed.
- One particular case has been studied in this research, that is the combination of the HOMD differentiator with the linear bias observer.

Our ongoing research is to apply it to the AnyWalker robot.

The takeaway message

In applications, when we combine well-known elements, they can yield an interesting interaction.