Symbolic dynamics and bialgebras

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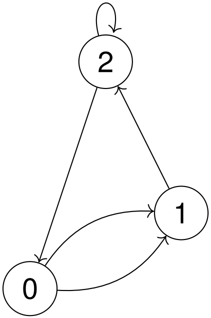
Plan

1. Symbolic Dynamics
2. Categories for symbolic dynamics
3. Applications
4. Conclusion

Edge shift

An edge shift is the set of all biinfinite walks (on edges) in a finite graph



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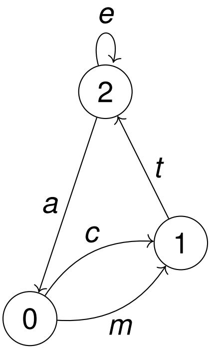

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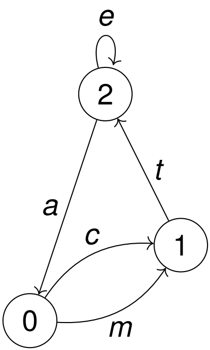

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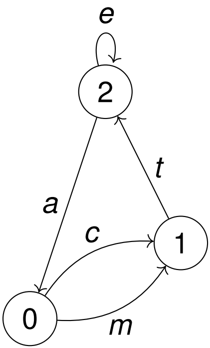
Conjugacy

Edge shifts are conjugate if they're isomorphic via local transformations



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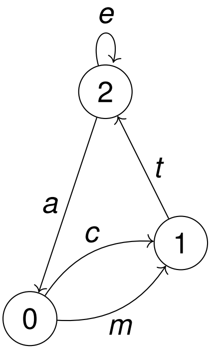

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$ac \to ac$

$am \to bm$

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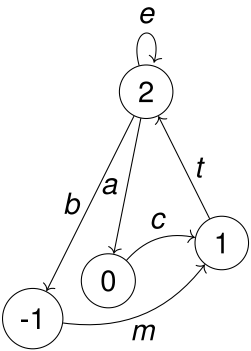

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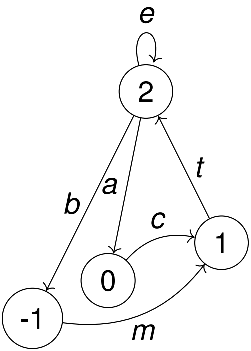

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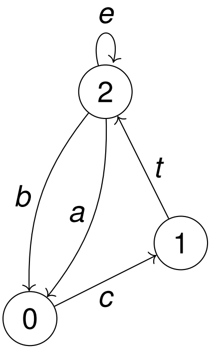

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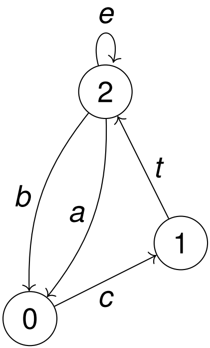

More generally, choose a vertex to duplicate

- Duplicate the inputs
- Redistribute the outputs

or vice-versa

SSE

This can be defined directly on adjacency matrices:

Definition

Two matrices M and N are 1-step equivalent if $M = RS$ and $N = SR$ for (nonnecessarily square) nonnegative integral matrices R, S

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Example:

$$M = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix} \quad N = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \end{pmatrix}$$

$$R = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad S = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

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Definition

Strong shift equivalence (SSE) is the transitive closure of 1-step equivalence.

Graphs G_1 and G_2 represent conjugate edge shifts iff their adjacency matrices are SSE.

History

Main open problem of symbolic dynamics: decide conjugacy/SSE

- Open since the 70s
- Decidable for matrices in $\mathbb{Z}$ (Krieger, 1980)
  - (almost the) same as conjugacy in $GL_n(\mathbb{Z})$
- Decidable for one-sided edge-shifts (Williams, 1973)
  - The rewriting system on graphs is confluent.

Invariants

Definition

An invariant is a function ψ s.t. $\psi(M) = \psi(N)$ if M and N are SSE.

- To prove that M and N are NOT SSE, just find an invariant that distinguishes them

Examples:

- Trace of the matrix
- More generally, number of cycles of size k
- The entropy (exponential growth of the number of paths of size n)
- ...

In this talk: A systematic way to find invariants

Plan

Goal

Build a category where arrows can represent matrices and equivalence classes of SSE.

Starting point

Folklore

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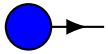
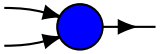
A commutative monoid ($\eta : 0 \to 1, \mu : 2 \to 1$)  

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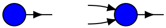
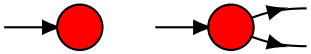
A commutative monoid ($\eta : 0 \to 1, \mu : 2 \to 1$) 

A cocommutative comonoid ($\epsilon : 1 \to 0, \Delta : 1 \to 2$) 

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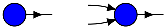
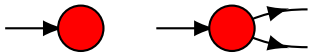
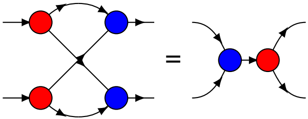
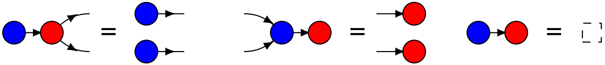
A commutative monoid ($\eta : 0 \to 1, \mu : 2 \to 1$) 

A cocommutative comonoid ($\epsilon : 1 \to 0, \Delta : 1 \to 2$) 

Bialgebra rules





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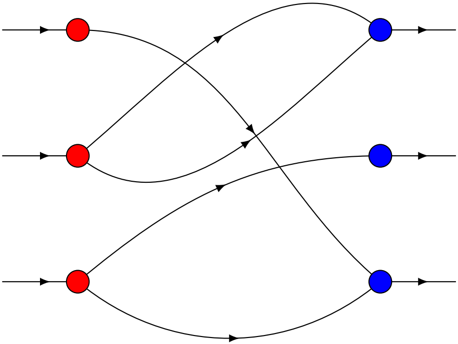
arrows $n \to m$ are exactly matrices of size $m \times n$.

$$M = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$



SSE

Technically, our matrices represent graph, so with inputs = outputs

SSE states that $RS \equiv SR$, how can we simulate that ?

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With a trace!

Trace

Consider the traced bialgebra prop (the previous prop + a trace).

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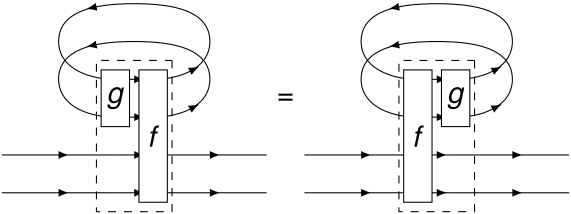
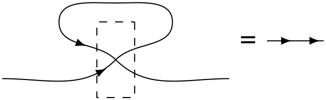
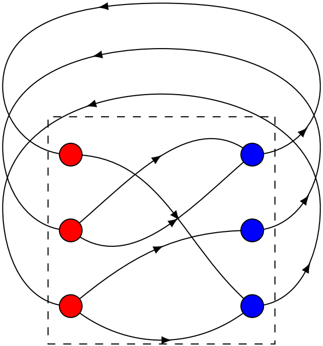


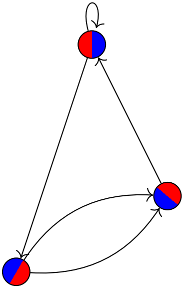


Trace

In this prop, we can associate a scalar $t(M)$ to each matrix M.

$$M = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$





Trace

In this prop, we can associate a scalar $t(M)$ to each matrix M.
Is it true that $t(M) = t(N)$ iff M and N are SSE ?

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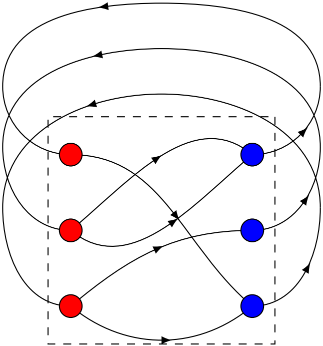
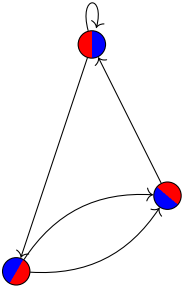
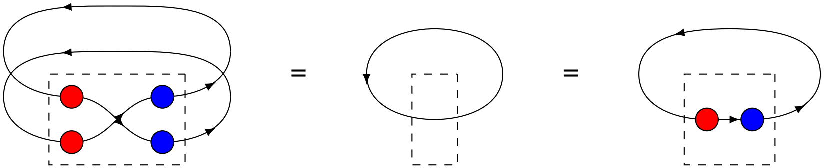




Flow equivalence

$M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $N = \begin{pmatrix} 1 \end{pmatrix}$ satisfy $t(M) = t(N)$ but are not SSE.



Flow equivalence

$t(M) = t(N)$ iff M and N are flow-equivalent, another equivalence notion coming from symbolic dynamics

(First observed by David Hillman, 1995)

We have lost a notion of time, that we need to recover

Solution

We add a generator that is a morphism for both the monoid and the comonoid.

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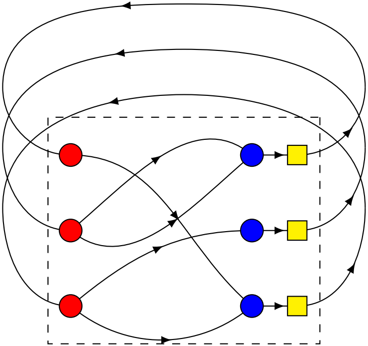

Note: this new prop represents matrices in $\mathbb{Z}_+[t]$ rather than $\mathbb{Z}_+$

Solution

We add a generator that is a morphism for both the monoid and the comonoid.

Theorem

In this prop, $t(M) = t(N)$ iff M and N are SSE.

Needs some new (easy) results in symbolic dynamics for the proof.

To recap: we need one bialgebra, a bialgebra morphism and a trace.

Plan

Applications

What do we gain from it ?

- If one knows a concrete representation of this prop, one can decide conjugacy
- One can find invariants !

Methodology

- Let $\mathfrak{C}$ be our category.
- Let $\mathfrak{D}$ be any traced category that contains a bialgebra
- By the universal property, there is a functor from $\mathfrak{C}$ to $\mathfrak{D}$
- Using this functor, we can associate to each matrix M an arrow $\psi(M)$ in $\mathfrak{D}$ s.t. $\psi(M) = \psi(N)$ if M and N are SSE.

Good news/Bad news

Good news: There are a lot of bialgebras in the wild

Bad news:

- Some of them are in categories that are not traced
- Some of them are Hopf algebras
  - Hopf algebras represent matrices with coefficients in $\mathbb{Z}$, not in $\mathbb{Z}_+$

Example from algebra 1/2

- If M is a finite abelian monoid, $\mathbb{K}[M]$ has a structure of a bialgebra
  - Product is the monoid product extended linearly
  - Coproduct is the copy extended linearly

Theorem

Let $\mathcal{M}$ be an additive monoid, and h an homomorphism. Then the number of solutions in $\mathcal{M}$ of the equation $h(Mx) = x$ is an invariant for SSE.

Example from algebra 2/2

- $\mathbb{K}[X]$ has a structure of a bialgebra
  - Product is the product of polynomials
  - Coproduct is defined by $\Delta(X) = 1 \otimes X + X \otimes 1$

Problem: no trace!

Example from algebra 2/2

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  - Product is the product of polynomials
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Solution: Replace $\mathbb{K}[X]$ by formal series in a complete semiring (like $\mathbb{R}_+ \cup \{+\infty\}$).
If we take the morphism $h(X) = tX$ we get:

Theorem

The quantity $f_M(t) = \frac{1}{det(I - tM)}$ is an invariant for SSE

This is the well-known Zeta function of a subshift.

Example from Category Theory

Let $\mathfrak{D}$ be a category with finite limits and colimits.

- Suppose one has a monoid X in a $\mathfrak{D}$ wih a morphism h.
- It extends to a bialgebra using the diagonal map as a comonoid.

Problem: What if there is no trace in $\mathfrak{D}$?

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Solution: Replace $\mathfrak{D}$ with $\mathrm{cospan}(\mathfrak{D})$ which is a compact category.

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Let M be a $n \times n$ matrix.

The object that is the coequalizer of $X^n \underset{h^n \circ M}{\overset{id^n}{\rightrightarrows}} X^n$ *is an invariant for SSE.*

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- Starting with the monoid $(\mathbb{Z}, +)$ in the category of abelian groups, we obtain the *Bowen-Franks* group (1977)
- Starting with the monoid $(\mathbb{Z}[t], +)$ in the category of $\mathbb{Z}[t]$-modules, we obtain the dimension group of Krieger (1977)

Plan

Where to go from here ?

- We have a systematic way to find invariants
- Investigate new bialgebras to find new invariants!