ZX-calculus

E. Jeandel

Loria (Nancy)

Plan

What

The ZX-Calculus is a graphical calculus designed by Coecke and Duncan (2008) with categorical foundations :

- Which represents quantum circuits and more
- With easy and interpretable rules

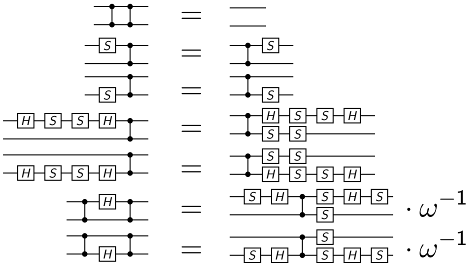
It can be seen as a carefully designed extension of quantum circuits allowing some specific gates which are not reversible.

Why

- Matrices are exponential in the size of the circuits
- Equational theory of circuits is not known yet (in april 2022)

(straight from the slides of Bian-Selinger)

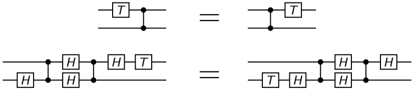
$$\omega^8 = 1$$
$$H^2 = 1$$
$$S^4 = 1$$
$$SHSHSH = \omega$$

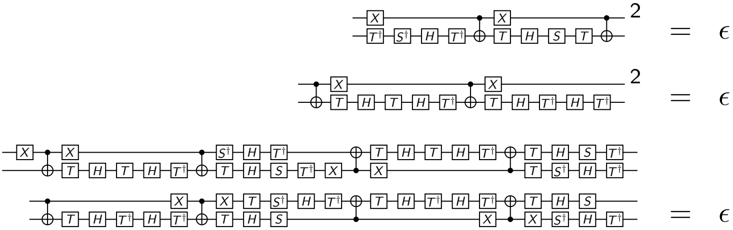


(straight from the slides of Bian-Selinger)

$$TT = S$$

$$(THSSH)^2 = \omega$$



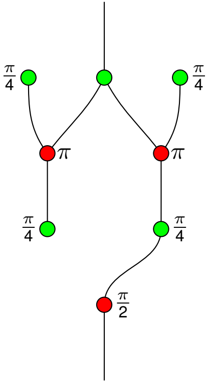


When

- To analyze quantum circuits, esp. on particular inputs
- To design quantum circuits (with some caveats)
- To optimize quantum circuits (with some caveats)
- For any computation on unitary matrices

When not

- For general linear algebra
  - Sums of matrices are not easy to handle
  - nonunitary matrices are hard to represent

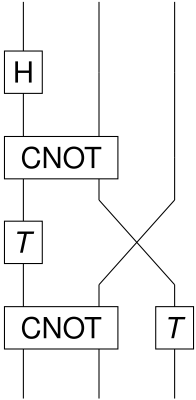


$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Plan

Circuits and props

Circuits (and diagrams) will have inputs on the top, and outputs at the bottom

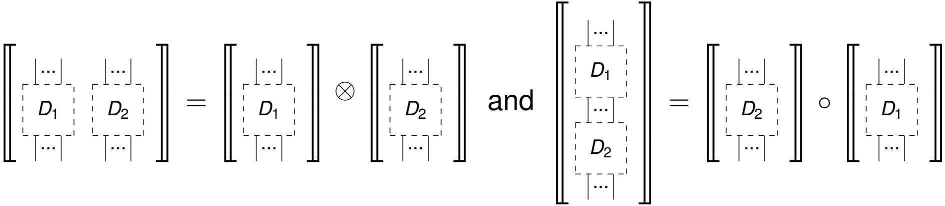


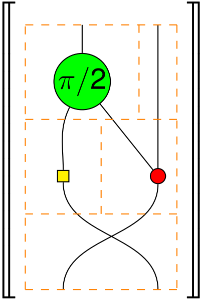
Circuits and props

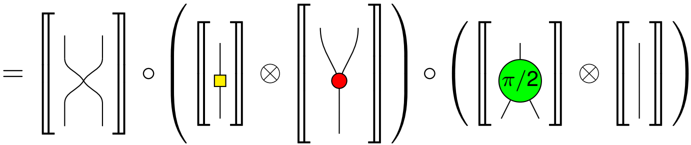
- In a circuit, each box corresponds to a matrix
- If two boxes are put in parallel, we do the tensor product of the matrices
- If two boxes are put sequentially, we do the matrix product of the matrices

Diagrams work the same

ZX-Calculus Compositions







$$= \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & i \\ 1 & 0 & 0 & -i \\ 0 & 1 & i & 0 \\ 0 & 1 & -i & 0 \end{pmatrix}$$

Cups and Caps

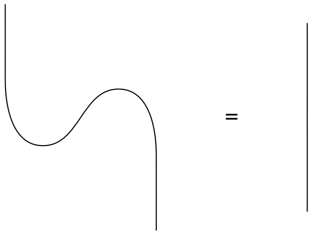
 $|00\rangle + |11\rangle$

 $\langle 00| + \langle 11|$

Cups and Caps

 $|00\rangle + |11\rangle$

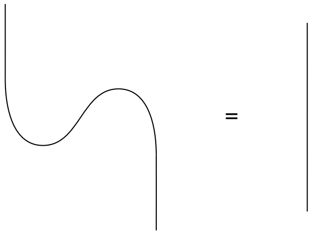
 $\langle 00| + \langle 11|$



Cups and Caps

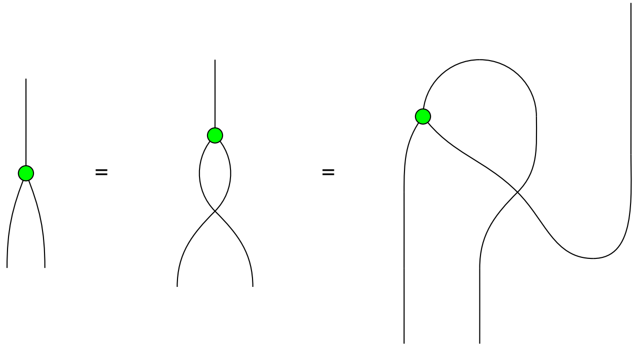
 $|00\rangle + |11\rangle$

 $\langle 00| + \langle 11|$



All generators in the language are chosen to be compatible with bending wires

Graphs

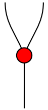


ZX-Calculus

X

First gate: X

Addition modulo 2.

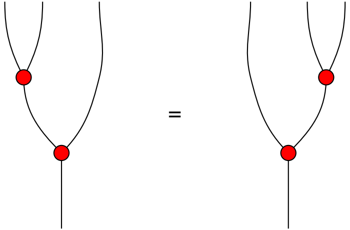


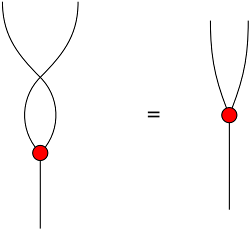
$$\begin{pmatrix}1 & 0 & 0 & 1\\ 0 & 1 & 1 & 0\end{pmatrix}$$

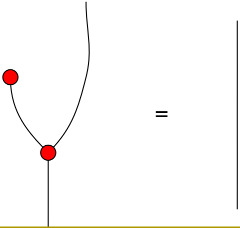
X

What equations does X satisfy ?

X is associative, commutative and has a neutral element



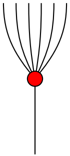




where  is $|0\rangle$.

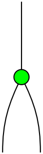
X

Due to the equations, one can generalize *X* as:



Second gate: Z

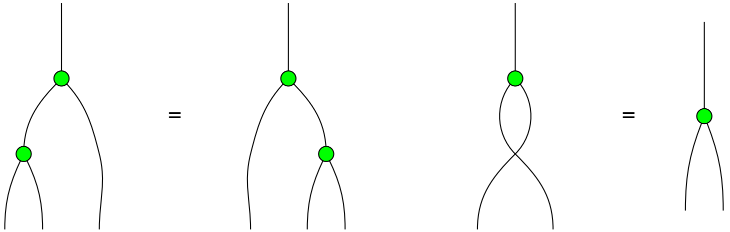
Copy

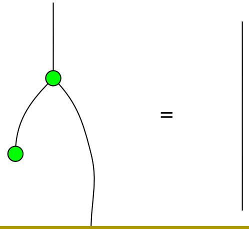


$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$$

What equations does Z satisfy ?

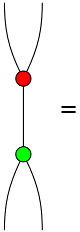
Z is (co)associative, (co)commutative and has a neutral element.



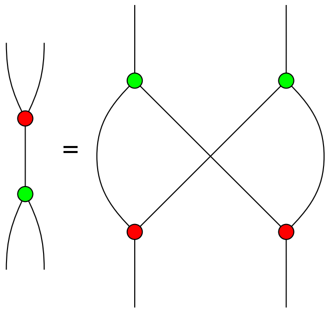


where  trashes* the qubits.

Equations



Equations



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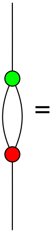

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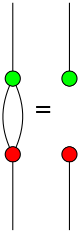

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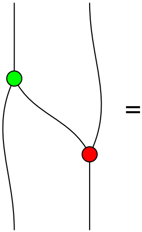

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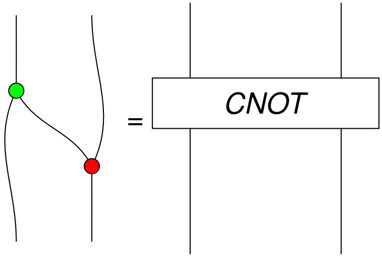
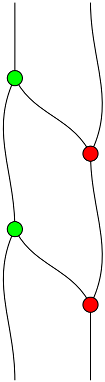


Equations

What happens if we do CNOT twice ?



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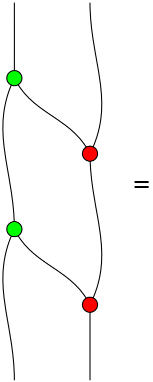

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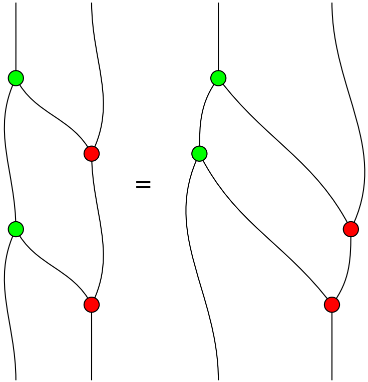

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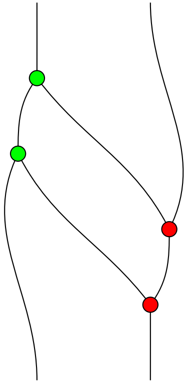

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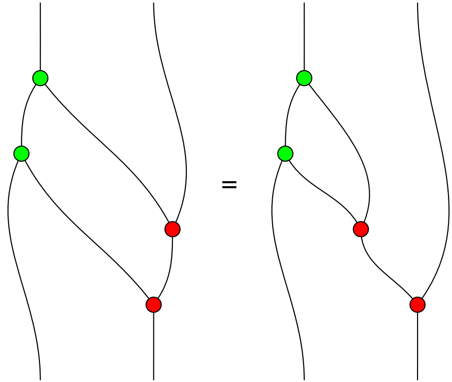

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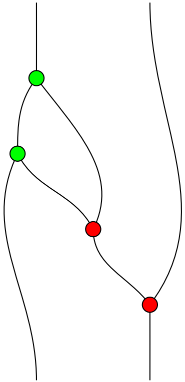

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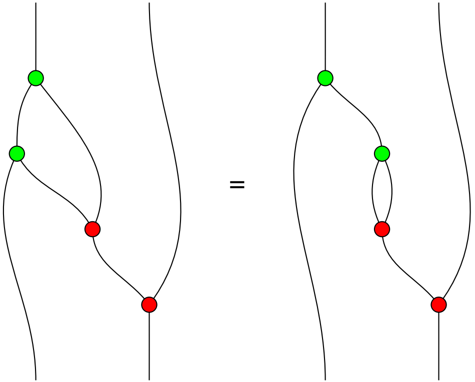

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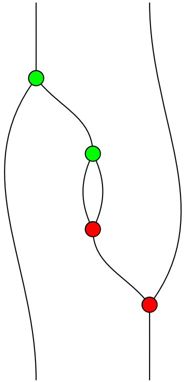

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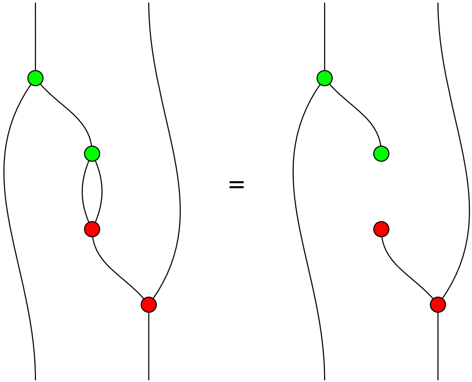

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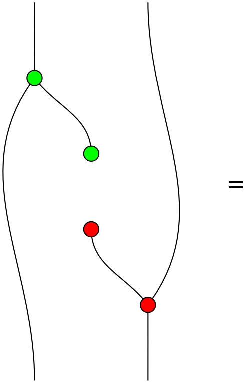

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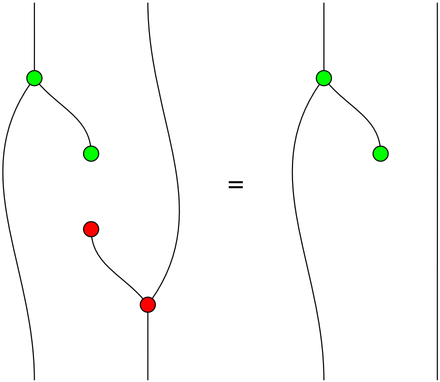

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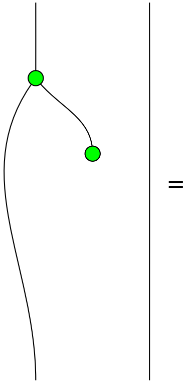

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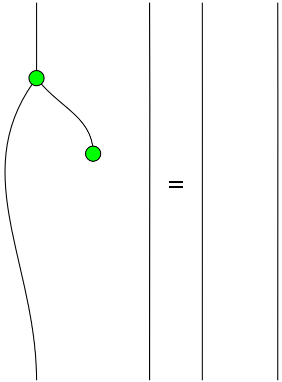

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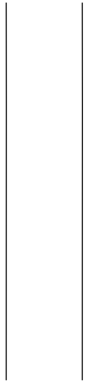
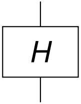
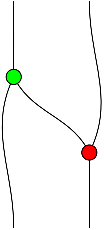
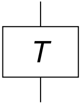


Quantum circuits

Quantum circuits are just like classical circuits but built on the following set of gates:







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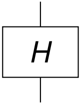
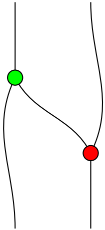
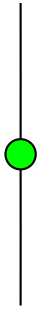




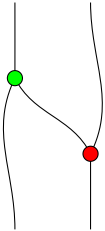


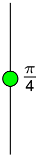
$$\begin{array}{rcr} |0\rangle & \mapsto & |0\rangle \\ |1\rangle & \mapsto & e^{i\alpha}\,|1\rangle \end{array}$$

Commutes with all green nodes.

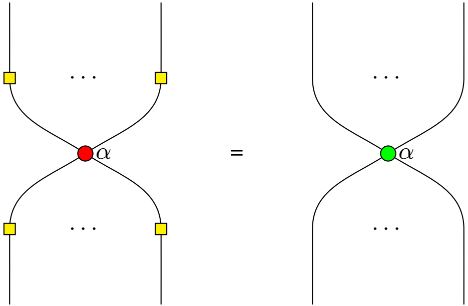
Quantum circuits

Quantum circuits are just like classical circuits but built on the following set of gates:



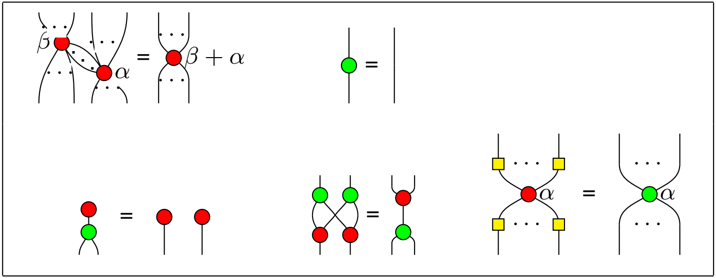


Spiders



Second summary

We have transformed our original gates into new nodes, that we better understand and that satisfy some equations, namely:



Almost all quantum algorithms and protocols can be understood using only these equations

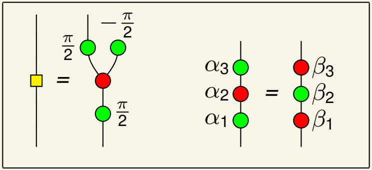
(and two more to follow in the next slide)

Is this all ?

Can we prove anything with this set of equations ?

Theorem (J.-Perdrix-Vilmart 2017, Vilmart 2019)

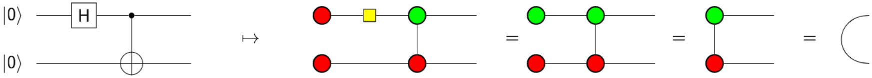
No. We need a few additional equations:



Plan

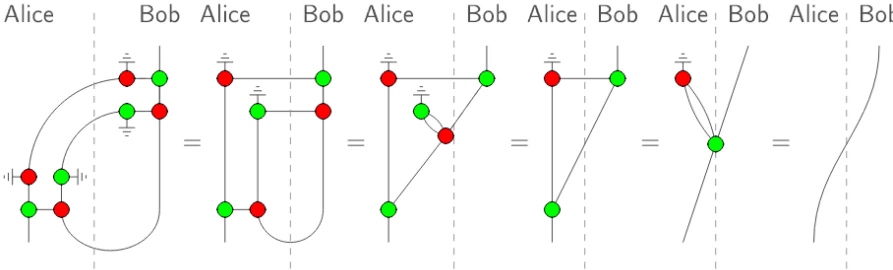
Circuit analysis, protocol analysis

slide from V. Zamdzhiev



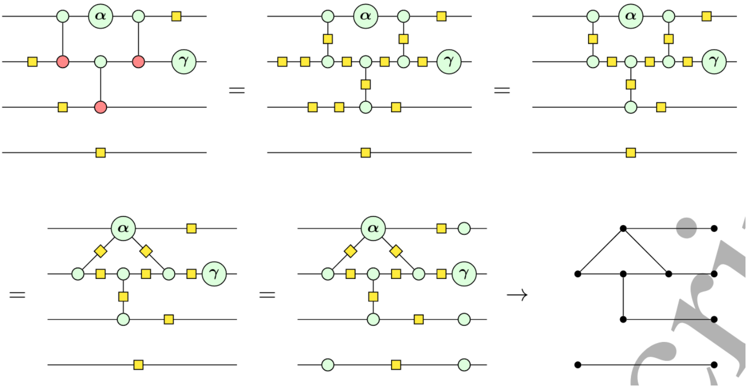
Circuit analysis, protocol analysis

slide from T. Carette

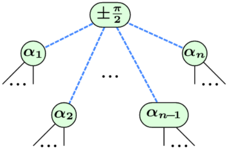


Circuit simulation

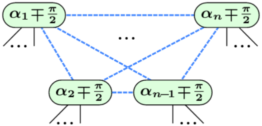
Kissinger, van de Wetering 2022



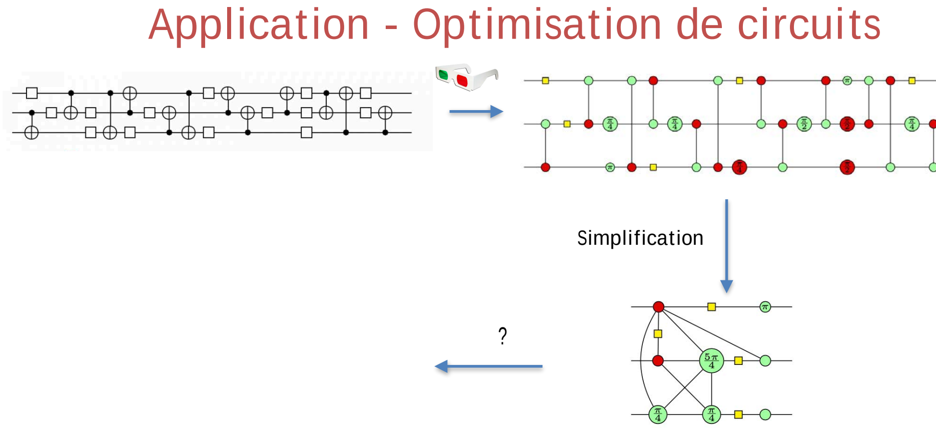
Kissinger, van de Wetering 2022



$$= e^{\pm i\pi/4}\sqrt{2}^{\frac{(n-1)(n-2)}{2}}$$

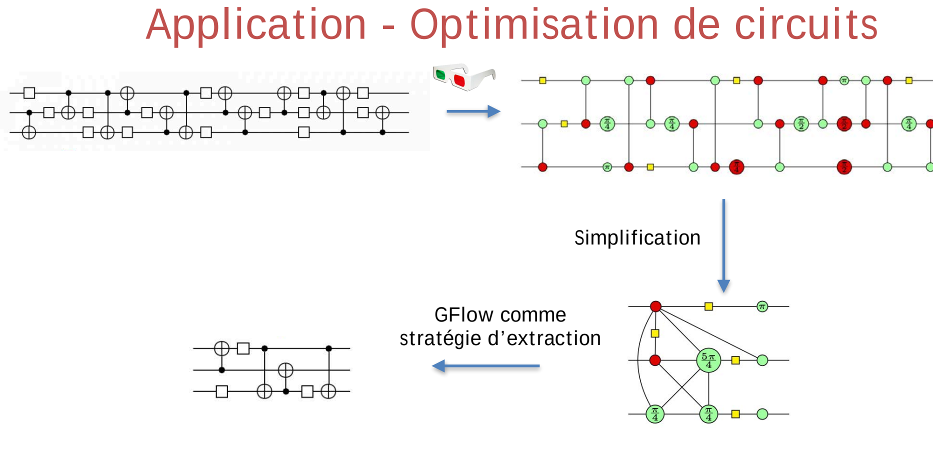


Duncan, Kissinger, Perdrix, van de Wetering 2020



Circuit simplification

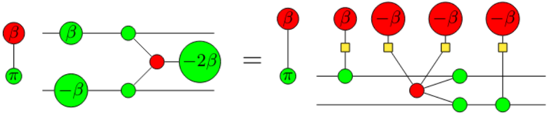
Duncan, Kissinger, Perdrix, van de Wetering 2020



Differentiation / Integration

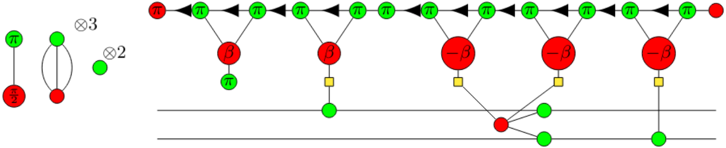
Jeandel-Perdrix-Veshchezerova-2022, $e^{\beta H}$ for $H = Z_1 + Z_2 - 2Z_1 Z_2$

$D_U(\beta) =$

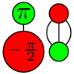


Using the formula (17) we find the derivative of $D_{U(\beta)}$:

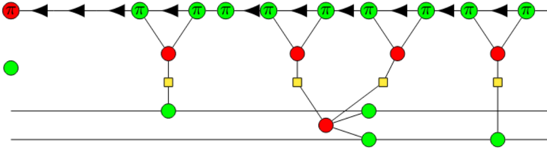
$\partial_{\mathrm{ZX}} D_{U(\beta)} =$



$h =$

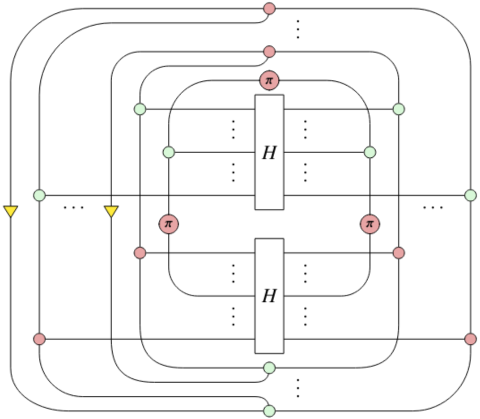


$\left[\partial_{\mathrm{ZX}} D_{U(\beta)}\right]_{\beta \to 0} =$



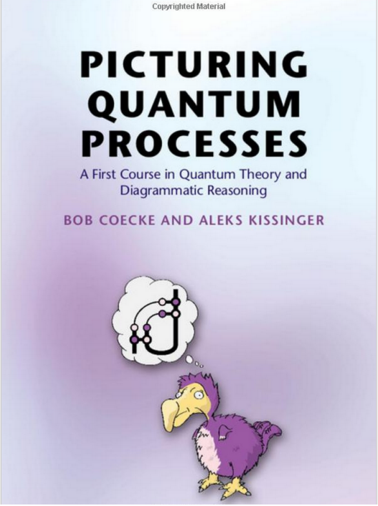
Wang-Yeung 2022

$$\mathbf{Var}\left(\frac{\partial \langle H \rangle}{\partial \theta_j}\right) =$$



Plan

Conclusion



New book: Quantum - in Pictures (Coecke-Gogioso, soon)