

ZX-calculus

E. Jeandel

Loria (Nancy)

Plan

- 1 Introduction
- 2 Description
- 3 Applications
- 4 Conclusion

What

The ZX-Calculus is a graphical calculus designed by Coecke and Duncan (2008) with categorical foundations :

- Which represents quantum circuits and more
- With easy and interpretable rules

It can be seen as a carefully designed extension of quantum circuits allowing some specific gates which are not reversible.

Why

- Matrices are exponential in the size of the circuits
- Equational theory of circuits is not known yet (in april 2022)

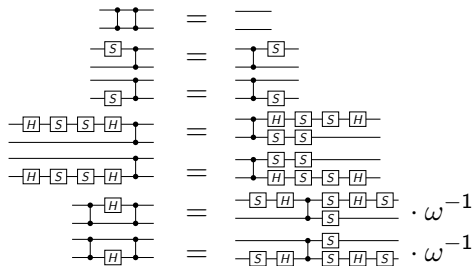
(straight from the slides of Bian-Selinger)

$$\omega^8 = 1$$

$$H^2 = 1$$

$$S^4 = 1$$

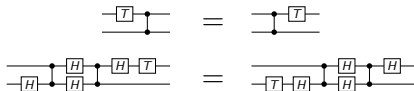
$$SHSHSH = \omega$$



(straight from the slides of Bian-Selinger)

$$TT = S$$

$$(THSSH)^2 = \omega$$



$$\begin{array}{c} \boxed{X} \\ \oplus \\ \boxed{T} \end{array} \text{---} \boxed{S^\dagger} \text{---} \boxed{H} \text{---} \boxed{T^\dagger} \oplus \begin{array}{c} \boxed{X} \\ \oplus \\ \boxed{T} \end{array} \text{---} \boxed{H} \text{---} \boxed{S} \text{---} \boxed{T} \oplus = \epsilon$$

$$\oplus \begin{array}{c} \boxed{X} \\ \oplus \\ \boxed{T} \end{array} \text{---} \boxed{H} \text{---} \boxed{T} \text{---} \boxed{H} \text{---} \boxed{T^\dagger} \oplus \begin{array}{c} \boxed{X} \\ \oplus \\ \boxed{T} \end{array} \text{---} \boxed{H} \text{---} \boxed{T^\dagger} \text{---} \boxed{H} \text{---} \boxed{T^\dagger} = \epsilon$$

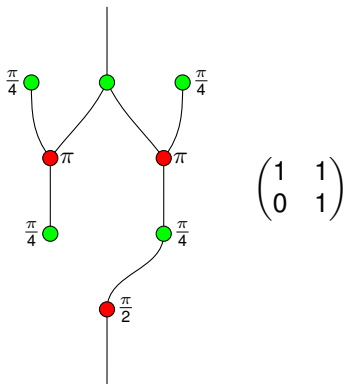
$$\begin{array}{c} \boxed{X} \\ \oplus \\ \boxed{T} \end{array} \oplus \begin{array}{c} \boxed{X} \\ \oplus \\ \boxed{T} \end{array} \text{---} \boxed{H} \text{---} \boxed{T} \text{---} \boxed{H} \text{---} \boxed{T^\dagger} \oplus \begin{array}{c} \boxed{S^\dagger} \text{---} \boxed{H} \text{---} \boxed{T^\dagger} \\ \oplus \\ \boxed{T} \end{array} \text{---} \boxed{H} \text{---} \boxed{S} \text{---} \boxed{T^\dagger} \oplus \begin{array}{c} \boxed{X} \\ \oplus \\ \boxed{T} \end{array} \text{---} \boxed{H} \text{---} \boxed{T} \text{---} \boxed{H} \text{---} \boxed{T^\dagger} \oplus \begin{array}{c} \boxed{X} \\ \oplus \\ \boxed{T} \end{array} \text{---} \boxed{H} \text{---} \boxed{S} \text{---} \boxed{T^\dagger} \\ \oplus \begin{array}{c} \boxed{X} \\ \oplus \\ \boxed{T} \end{array} \text{---} \boxed{H} \text{---} \boxed{T^\dagger} \text{---} \boxed{H} \text{---} \boxed{T^\dagger} \oplus \begin{array}{c} \boxed{X} \\ \oplus \\ \boxed{T} \end{array} \text{---} \boxed{H} \text{---} \boxed{T^\dagger} \text{---} \boxed{H} \text{---} \boxed{T^\dagger} \oplus \begin{array}{c} \boxed{X} \\ \oplus \\ \boxed{T} \end{array} \text{---} \boxed{H} \text{---} \boxed{S} \\ \oplus \begin{array}{c} \boxed{X} \\ \oplus \\ \boxed{T} \end{array} \text{---} \boxed{H} \text{---} \boxed{T^\dagger} \text{---} \boxed{H} \text{---} \boxed{T^\dagger} \oplus \begin{array}{c} \boxed{X} \\ \oplus \\ \boxed{T} \end{array} \text{---} \boxed{H} \text{---} \boxed{S} \text{---} \boxed{T^\dagger} = \epsilon$$

When

- To analyze quantum circuits, esp. on particular inputs
- To design quantum circuits (with some caveats)
- To optimize quantum circuits (with some caveats)
- For any computation on unitary matrices

When not

- For general linear algebra
 - Sums of matrices are not easy to handle
 - nonunitary matrices are hard to represent

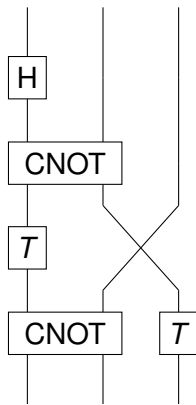


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Circuits and props

Circuits (and diagrams) will have inputs on the top, and outputs at the bottom



Circuits and props

- In a circuit, each box corresponds to a matrix
- If two boxes are put in parallel, we do the tensor product of the matrices
- If two boxes are put sequentially, we do the matrix product of the matrices

Diagrams work the same

ZX-Calculus Compositions

$$\left[\begin{array}{|c|} \hline \dots \\ \hline D_1 \\ \hline \dots \\ \hline \end{array} \right] \left[\begin{array}{|c|} \hline \dots \\ \hline D_2 \\ \hline \dots \\ \hline \end{array} \right] = \left[\begin{array}{|c|} \hline \dots \\ \hline D_1 \\ \hline \dots \\ \hline \end{array} \right] \otimes \left[\begin{array}{|c|} \hline \dots \\ \hline D_2 \\ \hline \dots \\ \hline \end{array} \right] \quad \text{and} \quad \left[\begin{array}{|c|} \hline \dots \\ \hline D_1 \\ \hline \dots \\ \hline D_2 \\ \hline \dots \\ \hline \end{array} \right] = \left[\begin{array}{|c|} \hline \dots \\ \hline D_2 \\ \hline \dots \\ \hline \end{array} \right] \circ \left[\begin{array}{|c|} \hline \dots \\ \hline D_1 \\ \hline \dots \\ \hline \end{array} \right]$$

$$\left[\begin{array}{|c|} \hline \dots \\ \hline \text{Diagram} \\ \hline \dots \\ \hline \end{array} \right] = \left[\begin{array}{|c|} \hline \dots \\ \hline \text{Crossing} \\ \hline \dots \\ \hline \end{array} \right] \circ \left(\left[\begin{array}{|c|} \hline \dots \\ \hline \text{Square} \\ \hline \dots \\ \hline \end{array} \right] \otimes \left[\begin{array}{|c|} \hline \dots \\ \hline \text{Dot} \\ \hline \dots \\ \hline \end{array} \right] \right) \circ \left(\left[\begin{array}{|c|} \hline \dots \\ \hline \text{Circle } \pi/2 \\ \hline \dots \\ \hline \end{array} \right] \otimes \left[\begin{array}{|c|} \hline \dots \\ \hline \text{Lines} \\ \hline \dots \\ \hline \end{array} \right] \right)$$

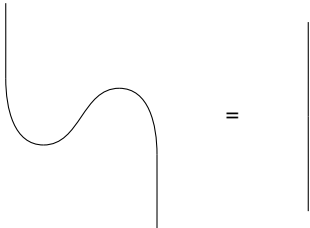
$$= \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & i \\ 1 & 0 & 0 & -i \\ 0 & 1 & i & 0 \\ 0 & 1 & -i & 0 \end{pmatrix}$$

Cups and Caps

$$\cap \quad |00\rangle + |11\rangle$$

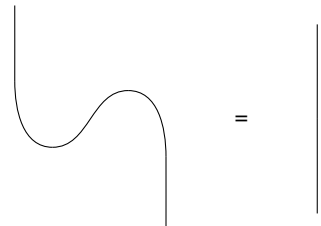
$$\cup \quad \langle 00| + \langle 11|$$

Cups and Caps

$$\cup \quad |00\rangle + |11\rangle \qquad \cup \quad \langle 00| + \langle 11|$$


The diagram shows an equality between two cup configurations. On the left, a cup shape is followed by the text $|00\rangle + |11\rangle$. Below this, a vertical line descends from the left side of the cup, curves to the right, then back to the left, and finally descends vertically to the right side of the cup. On the right, a cup shape is followed by the text $\langle 00| + \langle 11|$. Below this, a single vertical line descends from the top of the cup to the bottom.

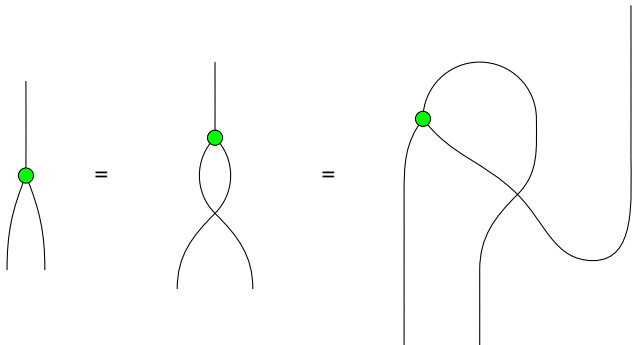
Cups and Caps

$$\cup \quad |00\rangle + |11\rangle \qquad \cup \quad \langle 00| + \langle 11|$$


The diagram shows an equation between two cup-shaped wires. On the left, a cup-shaped wire is connected to a wavy wire that starts vertically, curves to the left, then to the right, and ends vertically. On the right, a cup-shaped wire is connected to a straight vertical wire. An equals sign is placed between the two diagrams.

All generators in the language are chosen to be compatible with bending wires

Graphs



ZX-Calculus

First gate: X

Addition modulo 2.

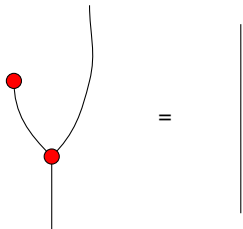
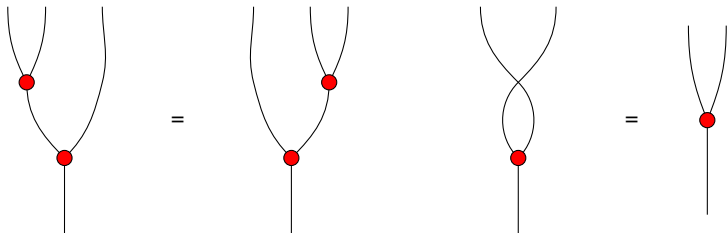


$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

X

What equations does X satisfy ?

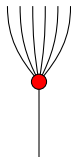
X is associative, commutative and has a neutral element



where $\begin{array}{c} \bullet \\ | \\ \hline \end{array}$ is $|0\rangle$.

X

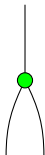
Due to the equations, one can generalize X as:



Z

Second gate: Z

Copy

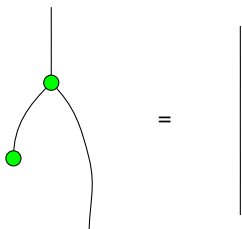
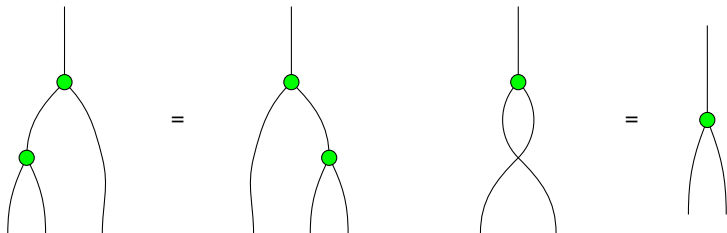



$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Z

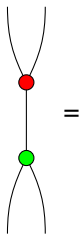
What equations does Z satisfy ?

Z is (co)associative, (co)commutative and has a neutral element.

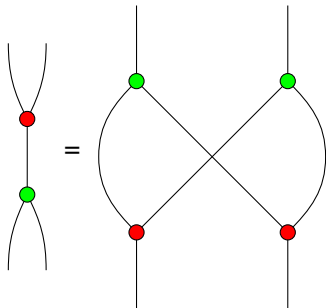


where  trashes* the qubits.

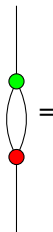
Equations



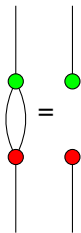
Equations



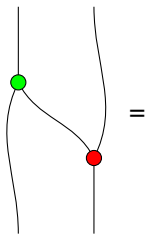
Equations



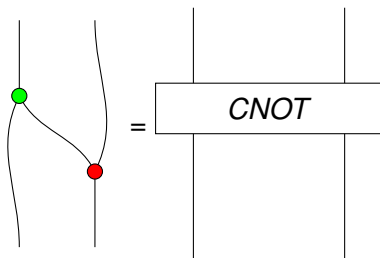
Equations



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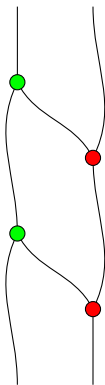


Equations



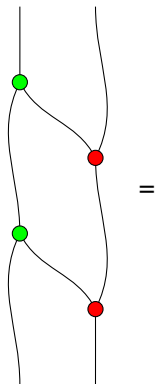
Equations

What happens if we do CNOT twice ?



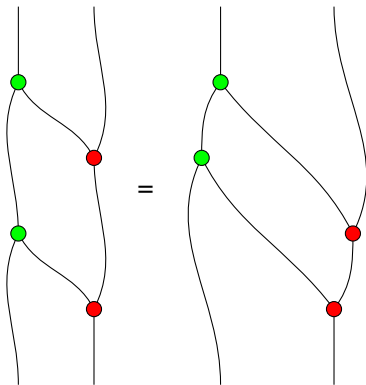
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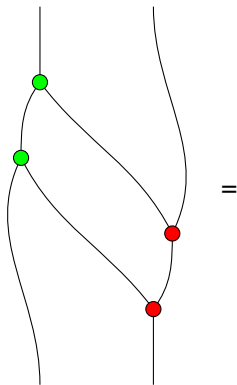
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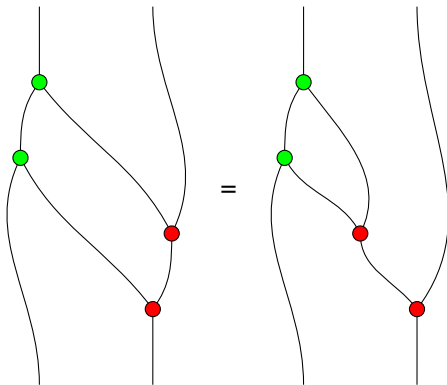
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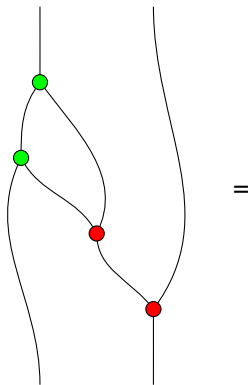
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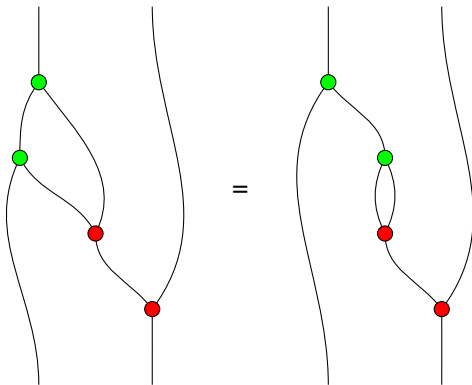
Equations

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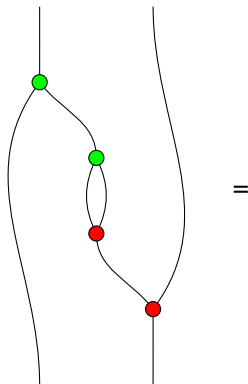
Equations

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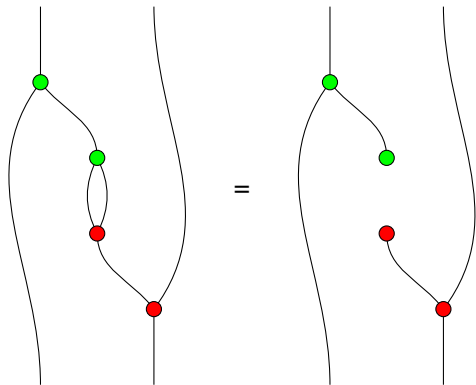
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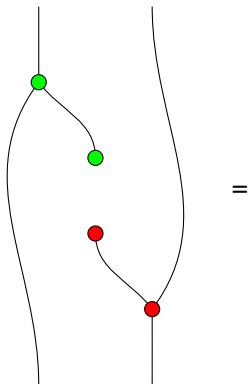
Equations

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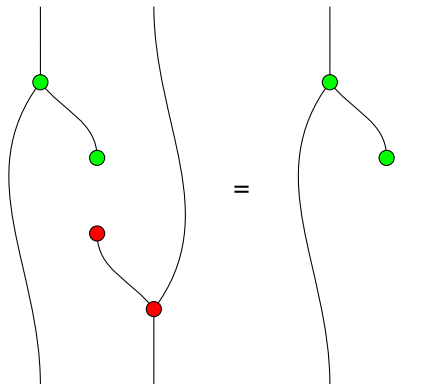
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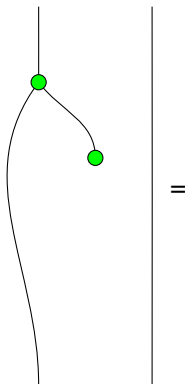
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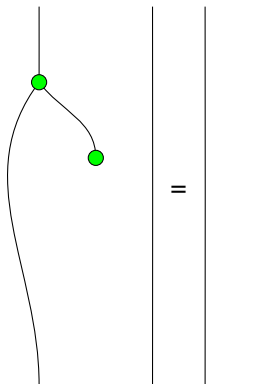
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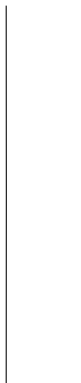
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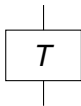
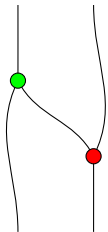
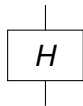
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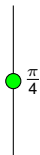
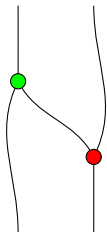
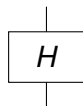
Quantum circuits

Quantum circuits are just like classical circuits but built on the following set of gates:



Quantum circuits

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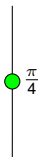
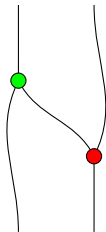


$$\begin{aligned} |0\rangle &\mapsto |0\rangle \\ |1\rangle &\mapsto e^{i\alpha} |1\rangle \end{aligned}$$

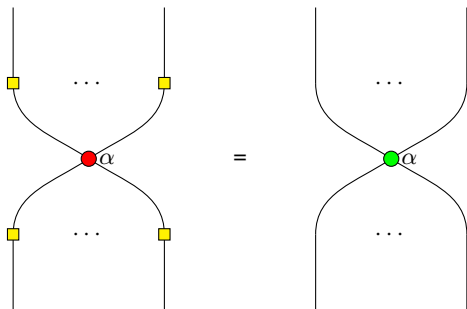
Commutates with all green nodes.

Quantum circuits

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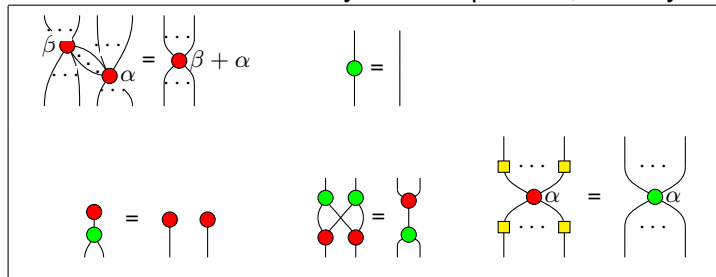


Spiders



Second summary

We have transformed our original gates into new nodes, that we better understand and that satisfy some equations, namely:



Almost all quantum algorithms and protocols can be understood using only these equations

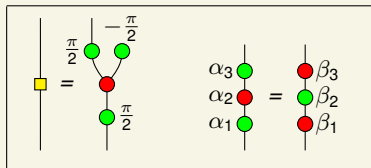
(and two more to follow in the next slide)

Is this all ?

Can we prove anything with this set of equations ?

Theorem (J.-Perdrix-Vilmart 2017, Vilmart 2019)

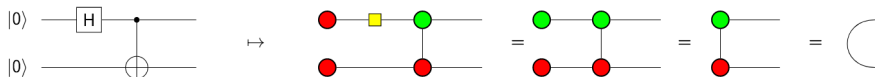
No. We need a few additional equations:



Plan

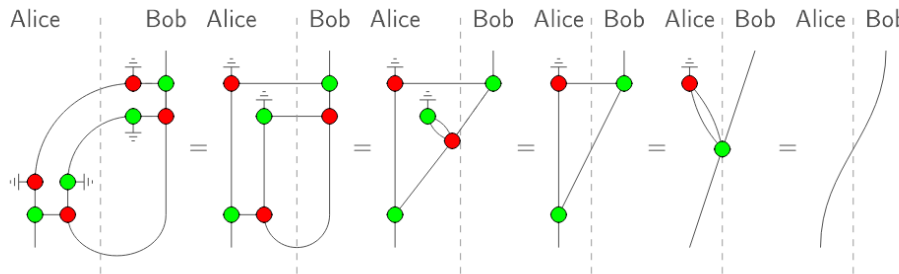
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slide from V. Zamdzhiev



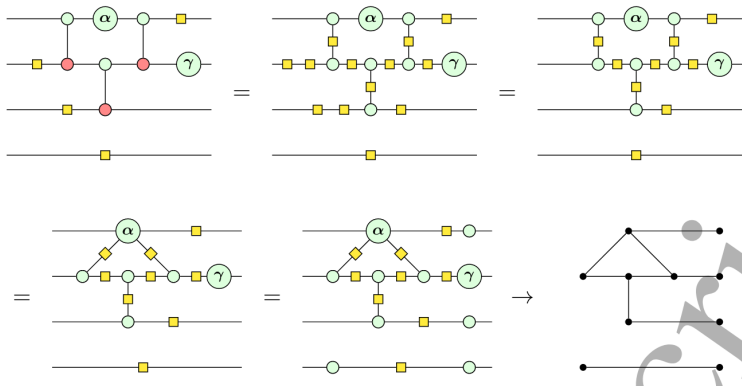
Circuit analysis, protocol analysis

slide from T. Carette



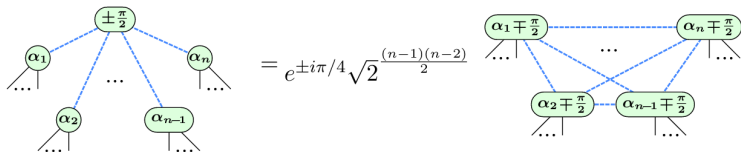
Circuit simulation

Kissinger, van de Wetering 2022



Circuit simulation

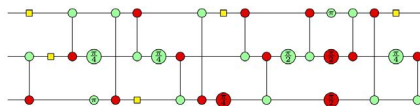
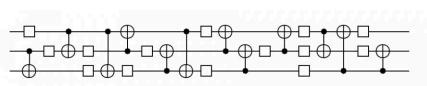
Kissinger, van de Wetering 2022



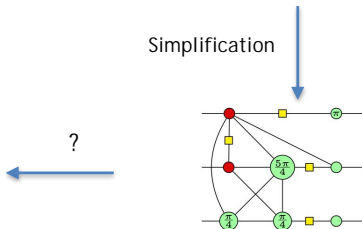
Circuit simplification

Duncan, Kissinger, Perdrix, van de Wetering 2020

Application - Optimisation de circuits



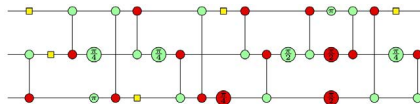
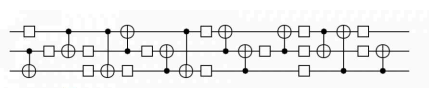
Simplification



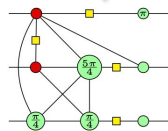
Circuit simplification

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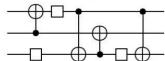
Application - Optimisation de circuits



Simplification



GFlow comme
stratégie d'extraction



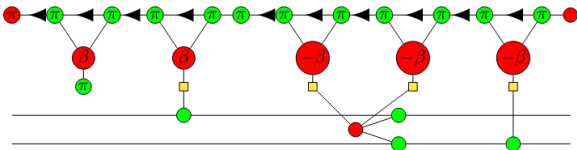
Differentiation / Integration

Jeandel-Perdrix-Veshchezerova-2022, $e^{\beta H}$ for $H = Z_1 + Z_2 - 2Z_1Z_2$

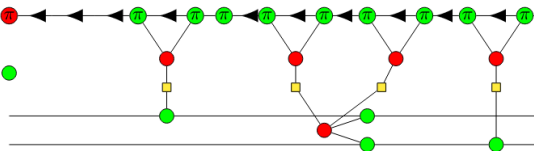
$$D_U(\beta) = \begin{array}{c} \text{red } \beta \\ \text{green } \pi \\ \text{green } \beta \\ \text{green } -\beta \\ \text{red } -2\beta \end{array} = \begin{array}{c} \text{red } \beta \\ \text{green } \pi \\ \text{green } \beta \\ \text{red } -\beta \\ \text{red } -\beta \\ \text{red } -\beta \end{array}$$

Using the formula (17) we find the derivative of $D_U(\beta)$:

$$\partial_{ZX} D_U(\beta) = \begin{array}{c} \text{green } \pi \\ \text{red } \beta \\ \text{green } \pi \\ \text{red } \beta \end{array} \otimes^3 \begin{array}{c} \text{green } \pi \\ \text{red } \beta \end{array} \otimes^2$$

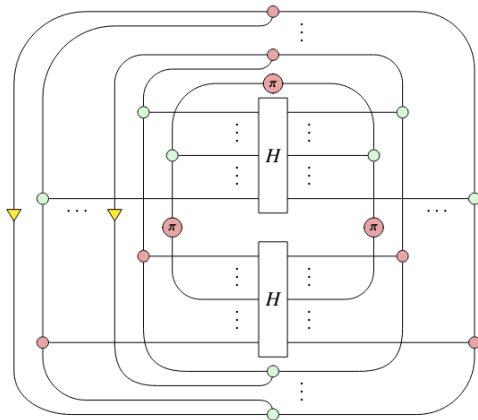


$$h = \begin{array}{c} \text{green } \pi \\ \text{red } \beta \\ \text{green } \pi \\ \text{red } \beta \end{array} [\partial_{ZX} D_U(\beta)]_{\beta \rightarrow 0} = \text{green } \pi$$



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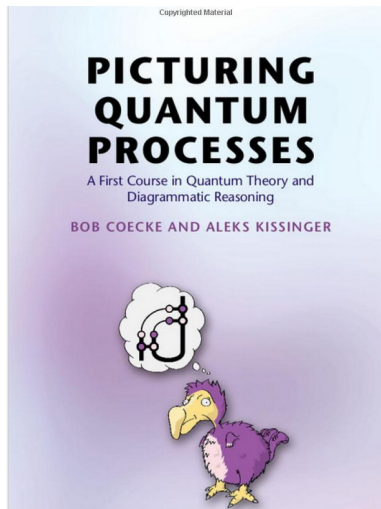
$$\text{Var} \left(\frac{\partial \langle H \rangle}{\partial \theta_j} \right) =$$



Plan

- 1 Introduction
- 2 Description
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- 4 Conclusion

Conclusion



New book: Quantum - in Pictures (Coecke-Gogioso, soon)